Weak implicative filters in quasi-ordered residuated systems

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Abstract:

The concept of residuated relational systems ordered under a quasiorder relation was introduced in 2018 by S. Bonzio and I. Chajda as a structure $A = \langle A, \cdot, \rightarrow, 1, R \rangle$, where $(A, \cdot)$ is a commutative monoid with the identity 1 as the top element in this ordered monoid under a quasi-order R . The author introduced and analyzed the concepts of filters and implicative filters in this type of algebraic structures. In this article, the concept of weak implicative filters in a quasi-ordered residuated system is introduced as a continuation of previous researches. Also, some conditions for a filter of such system to be a weak implicative filter are listed.

Keywords: Quasi-ordered residuated relational system; Filter; Implicative filter; Weak implicative filter.

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1. Introduction

Let $(A, \cdot, 1)$ be a commutative semigroup with the identity 1. Suppose that on the carrier A there exists another operation $\rightarrow$ and one relation R that with multiplication in A have a link $(x \cdot y, z) \in R \iff (x, y \rightarrow z) \in R$ for each $x, y, z \in A$. A relational system designed in this way, when R is a quasi-ordered relation on A , is in the focus of this paper.

The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda [2] in 2018. Previously, this concept was discussed in [1]. R. D. Maddux suggests that text [7] written by A. Tarski in 1941 is probably one of the first articles which relates to 'The calculus of relations' ([4], page 438). The approach outlined in [7] is worked out in more detail in [8]. According to R. D. Maddux already mentioned, the first definition of relation algebras appears in [3] (cited by [4], page 441). The approach outlined in [7] is worked out in more detail in [8]. In addition, according to R. D. Maddux, the first definition of relation algebras appears in [3].

This paper continues the investigations of quasi-ordered residuated systems and of their filters which were started in article [5]. Implicative filters in quasi-ordered residuated systems have been introduced and analyzed in the article [6]. In particular, the concept of weak implicative filters in a quasi-ordered residuated system is introduced as a continuation of the previous researches. Also, some conditions for a filter of such system to be a weak implicative filter are listed.

2. Preliminaries

2.1. Concept of quasi-ordered residuated systems

In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of 'residual relational systems'.

Definition 2.1. ([2], Definition 2.1) *A residuated relational system is a structure $\mathbf{A} = \langle A, \cdot, \rightarrow, 1, R \rangle$, where $\langle A, \cdot, \rightarrow, 1 \rangle$ is an algebra of type $\langle 2, 2, 0 \rangle$ and R is a binary relation on A and satisfying the following properties:*

- (1) $(A, \cdot, 1)$ is a commutative monoid;
- (2) $(\forall x \in A)((x, 1) \in R)$;
- (3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \iff (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

Example 2.1. Let $A = \{1, 2, 3, 4\}$ and operations $\cdot$ and $\rightarrow$ defined on A as follows:

$\cdot$	$\begin{array}{c cccc} 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 2 & 4 \\ 3 & 3 & 2 & 2 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{array}$	and	$\rightarrow$	$\begin{array}{c cccc} 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 1 & 3 \\ 4 & 1 & 1 & 1 & 1 \end{array}$
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Then $\mathbf{A} = \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated systems where the relation $\preceq$ is defined as follows $4 \preceq 3 \preceq 2 \preceq 1$.

The basic properties for residuated relational systems are subsumed in the following

Theorem 2.1 ([2], Proposition 2.1). Let $\mathbf{A} = \langle A, \cdot, \rightarrow, 1, R \rangle$ be a residuated relational system. Then

- (4) $(\forall x, y \in A)(x \rightarrow y = 1 \implies (x, y) \in R)$,
- (5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$,
- (6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$,
- (7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \implies (z \cdot x, y) \in R)$,
- (8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a quasi-order relation $\preceq$ on a set A is a binary relation which is reflexive and transitive (Some authors use the term pre-order relation).

Definition 2.2. ([2], Definition 3.1) A quasi-ordered residuated system is a residuated relational system $\mathbf{A} = \langle A, \cdot, \rightarrow, 1, \preceq \rangle$, where $\preceq$ is a quasi-order relation in the monoid $(A, \cdot)$

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([2], Proposition 3.1). Let A be a quasi-ordered residuated system. Then

- (9) $(\forall x, y, z \in A)(x \preceq y \implies (x \cdot z \preceq y \cdot z \wedge z \cdot x \preceq z \cdot y))$;
- (10) $(\forall x, y, z \in A)(x \preceq y \implies (y \rightarrow z \preceq x \rightarrow z \wedge z \rightarrow x \preceq z \rightarrow y))$;
- (11) $(\forall x, y \in A)(x \cdot y \preceq x \wedge x \cdot y \preceq y)$.

Estimating that this topic is interesting ([1, 2, 5]), it is certain that there is interest in the development of the concept of some substructures and processes in these systems.

2.2. Concepts of filters

In the article [5], in order to determine the concept of filters in quasi-ordered residuated systems, the relationships between the following conditions are analyzed:

- (F0) $1 \in F$;
- (F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F)))$;
- (F2) $(\forall u, v \in A)((u \in F \wedge u \preceq v) \implies v \in F)$; and
- (F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$.

It is shown ([5], Proposition 3.2) that $(F2) \implies (F1)$. In addition, it is shown ([5], Proposition 3.4) that for every nonempty subset of F of system $\mathbf{A}$ is valid $(F2) \implies (F0)$. Additionally, it has been shown to be valid

Proposition 2.2 ([5], Proposition 3.6). *Let F be a submonoid of the monoid $(A, \cdot)$ in a quasi-ordered residuated system $\mathbf{A} = \langle (A, \cdot, \rightarrow, 1, \preceq) \rangle$. Then $F(2) \implies F(3)$.*

Based on our previous analysis of the interrelationship between conditions (F1), F(2) and (F3) in a quasi-ordered residual system, we introduced the concept of filters by the following definition.

Definition 2.3. ([5], Definition 3.1) *For a non-empty subset F of a quasi-ordered residuated system $\mathbf{A}$ we say that it is a filter of $\mathbf{A}$ if it satisfies conditions (F2) and (F3).*

Example 2.2. *Let $A = \{1, 2, 3, 4\}$ and operations $\cdot$ and $\rightarrow$ defined on A as follows:*

$\cdot$	1	2	3	4		$\rightarrow$	1	2	3	4
1	1	2	3	4		1	1	2	3	4
2	2	2	2	4	and	2	1	1	1	1
3	3	2	2	4		3	1	2	1	4
4	4	4	4	4		4	1	2	3	1

Then $\mathbf{A} = \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated systems where the relation $\preceq$ is defined as follows

$$\preceq = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 1), (3, 1), (2, 1), (2, 3), (2, 4)\}.$$

Then the subsets $\{1\}$, $\{1, 3\}$ and $\{1, 3, 4\}$ are filters in $\mathbf{A}$.

The concept of implicative filters in this algebraic structure was introduced and analyzed in [6].

Definition 2.4. ([6], Definition 3.1) For a non-empty subset F of a quasi-ordered residuated system $\mathbf{A}$ we say that the implicative filter in $\mathbf{A}$ if (F2) and the following condition (IF) $(\forall u, v, z \in A)((u \rightarrow (v \rightarrow z)) \in F \wedge u \rightarrow v \in F) \implies u \rightarrow z \in F$ are valid.

Example 2.3. In Example 2.2, filter $\{1, 3\}$ is an implicative filter in $\mathbf{A}$.

3. The concept of weak implicative filters

It is known ([6], Theorem 3.1) that any implicative filter in a quasi-ordered residuated system is a filter in that system. As, in the general case, the opposite does not have to be true, our intention in this paper is to analyze the possibility of the existence of some kind of filter between the previous two and which would be different from both of them.

In this section, we introduce the concept of weak implicative filters in quasi-ordered residuated systems and analyze it.

Definition 3.1. For a non-empty subset F of a quasi-ordered residuated system $\mathbf{A}$ we say that the weak implicative filter in $\mathbf{A}$ if (F2) and the following condition (WIF) $(\forall u, v, z \in A)((u \rightarrow (v \rightarrow z)) \in F \wedge u \rightarrow v \in F) \implies u \rightarrow (u \rightarrow z) \in F$ are valid.

In what follows, we need the following lemma

Lemma 3.1 ([6], Lemma 3.1). Let a subset F of a quasi-ordered residuated system $\mathbf{A}$ satisfies the condition (F2). Then the following holds (12) $(\forall u \in A)(u \in F \iff 1 \rightarrow u \in F)$.

Let us show that every weak implicative filter in $\mathbf{A}$ is a filter of $\mathbf{A}$.

Theorem 3.1. Any weak implicative filter in a quasi-ordered residuated system $\mathbf{A}$ is a filter of $\mathbf{A}$.

Proof. Let F be a weak implicative filter in a quasi-ordered residuated system $\mathbf{A}$. To prove that F is a filter of $\mathbf{A}$, it suffices to prove that F satisfies condition (F3). Let $u, v \in A$ be arbitrary elements such that $u \rightarrow v \in F$ and $u \in F$. Then $1 \rightarrow (u \rightarrow v) \in F$ and $1 \rightarrow u \in F$ by (12). Thus $1 \rightarrow (1 \rightarrow v) \in F$ by (WIF). Hence $v \in F$ by (12). So, the set F is a filter of $\mathbf{A}$. $\square$

Theorem 3.2. *Every implicative filter of a quasi-ordered residuated system $\mathbf{A}$ is a weak implicative filter of $\mathbf{A}$.*

Proof. Let F be an implicative filter of a quasi-ordered residuated system $\mathbf{A}$. The F is a filter of $\mathbf{A}$ by Proposition 3.1 in [6]. Let $u, v, z \in A$ be such that $u \rightarrow (v \rightarrow z) \in F$ and $u \rightarrow v \in F$. Then $u \rightarrow z \in F$ since F is an implicative filter of $\mathbf{A}$. From the claim (11) of Proposition 2.1 ($\forall u, z \in A)(z \preceq u \rightarrow z)$), we obtain $(\forall u, z \in A)(u \rightarrow z \preceq u \rightarrow (u \rightarrow z))$ by the claim (10). Thus, from here and from $u \rightarrow z \in F$ we got $u \rightarrow (u \rightarrow z) \in F$ with reference to (F2), since F is a filter of $\mathbf{A}$. Therefore, F is a weak implicative filter of $\mathbf{A}$. $\square$

In the following theorem, we demonstrate one equivalent condition when a weak implicative filter of a quasi-ordered residuated system $\mathbf{A}$ to become an implicative filter of $\mathbf{A}$.

Theorem 3.3. *Let F be a weak implicative filter of a quasi-ordered residuated system $\mathbf{A}$. Then F is an implicative filter of $\mathbf{A}$ if and only if the following condition holds*

$$(13) (\forall u, z \in A)(u \rightarrow (u \rightarrow z) \in F \iff u \rightarrow z \in F).$$

Proof. Let F be a weak implicative filter of $\mathbf{A}$. Then F is a filter of $\mathbf{A}$ by Theorem 3.1.

Assume that F is an implicative filter of $\mathbf{A}$. Suppose that for the variables $u, z \in A$, the following $u \rightarrow (u \rightarrow z) \in F$ holds. As $u \rightarrow u \in F$ holds, we have

$$(u \rightarrow (u \rightarrow z) \in F \wedge u \rightarrow u \in F) \implies u \rightarrow z \in F$$

according to (IF). On the other hand, from (11), in the form $z \preceq u \rightarrow z$, we have $u \rightarrow z \preceq u \rightarrow (u \rightarrow z)$ by (10). Now, from this and $u \rightarrow z \in F$ it follows $u \rightarrow (u \rightarrow z) \in F$ by (F2). Thus, the implication $(WIF) \wedge (WF) \implies (13)$ is valid.

Conversely, assume that F satisfies the condition (13). We will prove that F is an implicative filter of $\mathbf{A}$. For this purpose, let $u, v, z \in A$ be such that $u \rightarrow (v \rightarrow z) \in F$ and $u \rightarrow v \in F$. Since F is a weak implicative filter, we get that $u \rightarrow (u \rightarrow z) \in F$. Then by the assumed condition (13), it yields that $u \rightarrow z \in F$. So, F is an implicative filter of $\mathbf{A}$. Therefore, the implication $(WIF) \wedge (13) \implies (IF)$ is valid. $\square$

Example 3.1. Let $A = \{1, a, b, c, d\}$ and operations $\cdot$ and $\rightarrow$ defined on A as follows:

$\cdot$	1	a	b	c	d		$\rightarrow$	1	a	b	c	d
1	1	a	b	c	d		1	1	a	b	c	d
a	a	a	a	a	b	and	a	1	1	b	c	b
b	b	a	b	b	d		b	1	a	1	b	a
c	c	a	b	c	d		c	1	a	1	1	a
d	d	b	d	d	d		d	1	1	1	b	1

Then $\mathbf{A} = \langle A, \cdot, \rightarrow, 1 \rangle$ is a quasi-ordered residuated systems where the relation ' $\preceq$ ' is defined as follows

$$\preceq = \{(1, 1), (a, 1), (b, 1), (c, 1), (d, 1), (a, a), (b, b), (c, c), (d, d), (d, a), (d, b), (c, b)\}.$$

Then the subsets $\{1, a\}$ is a weak implicative filter in $\mathbf{A}$.

In what follows, we need the following lemma

Lemma 3.2 ([6], **Lemma 3.4**). *Let a subset F of a quasi-ordered residuated system $\mathbf{A}$ satisfy the condition (F2). Then the following holds*

$$(14) (\forall u, v, z \in A)(u \rightarrow (v \rightarrow z) \in F \iff v \rightarrow (u \rightarrow z) \in F).$$

Theorem 3.4. *Let $\mathbf{A}$ be a quasi-ordered residuated system and let F be a filter of $\mathbf{A}$. Then F is a weak implicative filter of $\mathbf{A}$ if it satisfies the following condition:*

$$(15) (\forall y, v, z \in A)(u \rightarrow (v \rightarrow z) \in F \implies u \rightarrow ((u \rightarrow v) \rightarrow (u \rightarrow z)) \in F).$$

Proof. Let F be a filter of $\mathbf{A}$. Let $u, v, z \in A$ be such that $u \rightarrow (v \rightarrow z) \in F$ and $u \rightarrow v \in F$. Then $u \rightarrow ((u \rightarrow v) \rightarrow (u \rightarrow z)) \in F$ by (15). Thus $(u \rightarrow v) \rightarrow (u \rightarrow (u \rightarrow z)) \in F$ by (14). Now, from this and from $u \rightarrow v \in F$ it follows $u \rightarrow (u \rightarrow z) \in F$ by (F3). Hence F is a weak implicative filter of $\mathbf{A}$. $\square$

Theorem 3.5. *Let the quasi-ordered residuated system $\mathbf{A}$ satisfy the additional condition*

$$(16) (\forall u, v, z \in A)(v \rightarrow z \preceq (u \rightarrow v) \rightarrow (u \rightarrow z)).$$

Then every filter of $\mathbf{A}$ is a weak implicative filter of $\mathbf{A}$.

Proof. Let $u, v, z \in A$ be such that $u \rightarrow (v \rightarrow z) \in F$ and $u \rightarrow v \in F$. Then $v \rightarrow (u \rightarrow z) \in F$ by (14). Since $\mathbf{A}$ satisfies the additional condition (16), we have $v \rightarrow (u \rightarrow z) \preceq (u \rightarrow v) \rightarrow (u \rightarrow (u \rightarrow z))$. Hence $(u \rightarrow v) \rightarrow (u \rightarrow (u \rightarrow z)) \in F$ according to (F2). From here we get $u \rightarrow (v \rightarrow z) \in F$ according to (F3). Therefore F is a weak implicative filter of $\mathbf{A}$. $\square$

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