Entire forgotten topological index of graphs

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Abstract:

The Forgotten topological index or F-index is defined as the sum of cubes of the degrees of vertices of a graph. For this classical F-index, we ignore the intermolecular forces that exist between the atoms and bonds and consider only the intermolecular forces between the atoms of a molecule. In this paper, we introduce a new graph invariant called "Entire Forgotten topological index" or "Entire F-index", that includes incidence of edges and vertices in addition to the adjacency of the vertices. We obtain some important properties of the index and also establish formulae of this newly defined index for some operations of graphs.

Keywords: Degree of vertex; Zagreb indices; F-index; Operations of graphs.

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1. Introduction

Let G be a graph and the set of vertices and edges of G be denoted as $V(G)$ and $E(G)$ respectively. Throughout the paper we consider only simple finite graphs. The degree of a vertex $u \in V(G)$ is denoted by $d_G(u)$, if there is no confusion we simply write it as $d(u)$. Two vertices u and v are called adjacent if there is an edge connecting them. The connecting edge is usually denoted by uv . The degree of an edge uv is defined as the number of edges connected to the edge, i.e. the number of edges connected to both u and v except the edge uv . Hence $d(uv) = d(u) + d(v) - 2$ [18]. The complement of a graph G is denoted as $\overline{G}$. By $G \in \Omega(n, m)$ we mean that G is a graph with n vertices and m edges. The notations and terminologies used but not clearly stated in this article may be found in [26].

Topological indices are the numerical values which are associated with a graph structure. These graph invariants are utilized for modeling information of molecules in structural chemistry and biology. Over the years many topological indices are proposed and studied based on degree, distance and other parameters of graph. Some of them may be found in [7, 11]. Historically Zagreb indices can be considered as the first degree-based topological indices, which came into picture during the study of total π -electron energy of alternant hydrocarbons by Gutman and Trinajstić in 1972 [13]. Various studies related to different aspects of the Zagreb indices are reported, for detail see the papers [6, 8, 12, 16, 22, 23, 27] and the references therein. In 2018, Alwardi, A., et al., studied the entire Zagreb indices of graphs [2].

The objective of this paper is twofold. First to introduce a new topological index. We call it “Entire Forgotten Topological index” or “Entire F -index”. Second to study important properties of the index and its coindices. Also we establish explicit expressions of the index for some operations of graphs in terms of vertices and edges and in terms of some well known topological indices such as Zagreb indices ($M_1(G)$ and $M_2(G)$) [13], Hyper-Zagreb index ($HM(G)$) [24], First generalized Zagreb index ($M_\alpha(G)$) [17], F -index ($F(G)$) [9] etc.

The F -index has been well studied in last couple of years, some of them may be found in [1, 15, 19, 20]. Because it can reinforce the physico-chemical flexibility of Zagreb indices [25]. De, N., et al., [20] also introduced and studied the F -coindex in connection with some operations of graphs.

In 2013, the Zagreb indices were re-defined as first, second and third Zagreb indices by Ranjini et al. [21]. The Re-defined Third Zagreb Index $ReZG_3(G)$ of a graph G is defined as

$$ReZG_3(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)(d_G(u) + d_G(v)).$$

This index can also be seen as a special case of the generalized inverse sum indeg index $ISI_{(\alpha, \beta)}(G)$ of a graph G proposed by *Buragohain* et al. in [4].

Analogously, the Re-defined Third Zagreb coindex of a graph G can be defined as

$$\overline{ReZG_3}(G) = \sum_{uv \notin E(G)} d_G(u)d_G(v)(d_G(u) + d_G(v)).$$

2. The Entire F -index and Coindices

Motivated by the work of *Alwardi* et al. [2], here we define the entire Forgotten Topological Index and then obtain some of its fundamental properties. The chemical reason for defining this new index is the limitation of the classical F -index to capture the interactions (forces) between the edges and vertices in addition to the interactions between vertices.

Definition 2.1. Let G be a graph. Then the Entire Forgotten Topological Index is defined as

$$F^e(G) = \sum_{u \in V(G) \cup E(G)} d_G^3(u).$$

Now we propose the following theorem that relates the new topological index with some other well known topological indices.

Theorem 2.2. Let $G \in \Omega(n, m)$. Then

$$F^e(G) = 12M_1(G) - 12M_2(G) - 5F(G) + 3ReZG_3(G) + M_4(G) - 8m.$$

Proof: As $F(G) = \sum_{u \in V(G)} d_G^3(u) = \sum_{uv \in E(G)} (d_G^2(u) + d_G^2(v))$, consequently

$$\begin{aligned} F^e(G) &= \sum_{u \in V(G) \cup E(G)} d_G^3(u) = F(G) + \sum_{uv \in E(G)} d^3(uv) \\ &= \sum_{uv \in E(G)} (d^2(u) + d^2(v)) + \sum_{uv \in E(G)} (d(u) + d(v) - 2)^3 \\ &= \sum_{uv \in E(G)} [(d^3(u) + d^3(v)) - 5(d^2(u) + d^2(v)) + 3d(u)d(v)(d(u) + d(v)) \\ &\quad - 12d(u)d(v) + 12(d(u) + d(v)) - 8], \\ &= 12M_1(G) - 12M_2(G) - 5F(G) + 3ReZG_3(G) + M_4(G) - 8m, \end{aligned}$$

which completes the proof.

Now we will give the expression for Entire F-index of the graph $\overline{G}$. Before continuing we reiterate the following theorem and proposition, as the theorem and proposition will be frequently used in establishing further results.

Theorem 2.3. [10] For the graph $G \in \Omega(n, m)$

1. $M_1(\overline{G}) = n(n-1)^2 + M_1(G) - 4m(n-1)$,
2. $\overline{M}_1(G) = 2m(n-1) - M_1(G)$,
3. $M_2(\overline{G}) = \frac{2n-3}{2}M_1(G) - M_2(G) + \frac{n(n-1)^3}{2} - 3m(n-1)^2 + 2m^2$,
4. $\overline{M}_2(G) = 2m^2 - \frac{1}{2}M_1(G) - M_2(G)$.

Proposition 2.4. [20] If $G \in \Omega(n, m)$ then

1. $F(\overline{G}) = 2(n-1)^2(\overline{m} - 2m) + 3(n-1)M_1(G) - F(G)$,
2. $\overline{F}(G) = (n-1)M_1(G) - F(G)$.

Lemma 2.5. If $G \in \Omega(n, m)$ then $\overline{ReZG}_3(G) = 2mM_1(G) - F(G) - ReZG_3(G)$.

Proof: By the definition of Re-defined Third Zagreb index and coindex,

$$\begin{aligned} ReZG_3(G) + \overline{ReZG}_3(G) &= \sum_{uv \in E(G)} d_G(u)d_G(v)(d_G(u) + d_G(v)) \\ &\quad + \sum_{uv \notin E(G)} d_G(u)d_G(v)(d_G(u) + d_G(v)) \\ &= \sum_{u,v \in V(G)} d_G(u)d_G(v)(d_G(u) + d_G(v)) \\ &= \sum_{u \in V(G)} d_G^2(u) \sum_{u \in V(G)} d_G(u) - \sum_{u \in V(G)} d_G^3(u) \\ &= 2mM_1(G) - F(G). \end{aligned}$$

Theorem 2.6. Let $G \in \Omega(n, m)$. Then

$$\begin{aligned} F^\epsilon(\overline{G}) &= [18(n-1)^2 - 33n - 6m + 51]M_1(G) + 12(2-n)M_2(G) + (15-7n)F(G) \\ &\quad + 3ReZG_3(G) + M_4(G) + 4n(n-1)^4 - 11n(n-1)^3 + 12n(n-1)^2 - 4n(n-1) \\ &\quad - 32m(n-1)^3 + 36m(n-1)^2 - 48m(n-1) - 24m^2 + 24(n-1)m^2 + 38m \end{aligned}$$

Proof: From Theorem 2.2,

$$(2.1) F^\epsilon(\overline{G}) = 12M_1(\overline{G}) - 12M_2(\overline{G}) - 5F(\overline{G}) + 3ReZG_3(\overline{G}) + M_4(\overline{G}) - 8\overline{m},$$

where $m + \overline{m} = \frac{n(n-1)}{2}$. Now,

$$\begin{aligned}
 ReZG_3(\overline{G}) &= \sum_{uv \in E(\overline{G})} d_{\overline{G}}(u) d_{\overline{G}}(v) [d_{\overline{G}}(u) + d_{\overline{G}}(v)] \\
 &= \sum_{uv \notin E(G)} (n-1-d_G(u))(n-1-d_G(v)) [2(n-1)-d_G(u)-d_G(v)] \\
 &= \sum_{uv \notin E(G)} [2(n-1)^3 - 3(n-1)^2 \{d_G(u) + d_G(v)\} \\
 &\quad + (n-1) \{d_G^2(u) + d_G^2(v)\} \\
 &\quad + 4(n-1)d_G(u)d_G(v) - d_G(u)d_G(v)(d_G(u) + d_G(v))] \\
 &= n(n-1)^4 - 2m(n-1)^3 - 3(n-1)^2 \overline{M}_1(G) + (n-1)\overline{F}(G) \\
 &\quad + 4(n-1)\overline{M}_2(G) - \overline{ReZG}_3(G).
 \end{aligned}
 \tag{2.2}$$

Now using the results of Theorem 2.3, Proposition 2.4 and Lemma 2.5 in the above equation (2.2), we obtain

$$\begin{aligned}
 ReZG_3(\overline{G}) &= 2[2(n-1)^2 - (n-1) - m]M_1(G) - 4(n-1)M_2(G) + (2-n)F(G) \\
 &\quad + ReZG_3(G) + n(n-1)^4 - 8m(n-1)^3 + 8(n-1)m^2
 \end{aligned}
 \tag{2.3}$$

Further,

$$\begin{aligned}
 M_4(\overline{G}) &= \sum_{u \in V(\overline{G})} d_{\overline{G}}^4(u) \\
 &= \sum_{u \in V(G)} (n-1-d_G(u))^4 \\
 &= n(n-1)^4 - 8m(n-1)^3 + 6(n-1)^2 M_1(G) - 4(n-1)F(G) + M_4(G).
 \end{aligned}
 \tag{2.4}$$

Using equations (2.3), (2.4) and the results of Theorem 2.3, Proposition 2.4 and Lemma 2.5 in expression (2.1), yield the result.

Again we can define the Entire F -coindex as

Definition 2.7. Let G be a graph. Then the Entire F -coindex is

$$\overline{F}^e(G) = \sum_{u \text{ is neither adjacent nor incident to } v} (d_G^2(u) + d_G^2(v)).$$

Now to establish an explicit expression for F -coindex, we need the following lemma.

Lemma 2.8. Let $G \in \Omega(n, m)$ and $L(G)$ be the line graph of G . Then

1. $M_1(L(G)) = F(G) + 2M_2(G) - 4M_1(G) + 4m$,
2. $F(L(G)) = 12M_1(G) - 12M_2(G) - 6F(G) + 3ReZG_3(G) + M_4(G) - 8m$.

Proof: (i) Note that,

$$\begin{aligned}
 M_1(L(G)) &= \sum_{uv \in E(G)} d_G^2(uv) \\
 &= \sum_{uv \in E(G)} (d_G(u) + d_G(v) - 2)^2 \\
 &= \sum_{uv \in E(G)} [d_G^2(u) + d_G^2(v) + 4 + 2\{d_G(u)d_G(v) - 2d_G(u) - 2d_G(v)\}] \\
 &= F(G) + 4m + 2M_2(G) - 4M_1(G).
 \end{aligned}$$

(ii) From the definition of F -index and degree of the edge uv it is clear that

$$\begin{aligned}
F(L(G)) &= \sum_{uv \in E(G)} (d_G(u) + d_G(v) - 2)^3 \\
&= \sum_{uv \in E(G)} [d_G^3(u) + d_G^3(v) + 3d_G(u)d_G(v)(d_G(u) + d_G(v)) - 6(d_G^2(u) + d_G^2(v)) \\
&\quad - 12d_G(u)d_G(v) + 12(d_G(u) + d_G(v)) - 8],
\end{aligned}$$

from which the result follows.

Theorem 2.9. *If $G \in \Omega(n, m)$ then*

$$\begin{aligned}
\overline{F}^\varepsilon(G) &= 2(2n - 2m - 3)M_1(G) + 2(m + 3)M_2(G) + (m + 1)F(G) - 3ReZG_3(G) \\
&\quad - M_4(G) + 4m(3m - 1) - 16m(n - 1) + 4n(n - 1)
\end{aligned}$$

Proof: Precisely,

$$\begin{aligned}
(2.5) \quad \overline{F}^\varepsilon(G) &= \overline{F}(G) + \overline{F}(L(G)) + \sum_{u \text{ is not incident to } e} (d^2(e) + d^2(u)) \\
&= \overline{F}(G) + \overline{F}(L(G)) + \sum_{u \in V(G)} \sum_{v \notin N(u)} (d^2(u) + d^2(uv)),
\end{aligned}$$

where $N(u)$ is the neighborhood of u , i.e. the set of adjacent vertices to u . Now as $\overline{F}(G) = (n - 1)M_1(G) - F(G)$, by Proposition 2.4, therefore $\overline{F}(L(G)) = (m - 1)M_1(L(G)) - F(L(G))$. Lemma 2.8 gives

$$\begin{aligned}
(2.6) \quad \overline{F}(L(G)) &= (m + 5)F(G) + 2(m + 5)M_2(G) - 4(m + 2)M_1(G) \\
&\quad - 3ReZG_3(G) - M_4(G) + 4m(m + 1).
\end{aligned}$$

Again,

$$\begin{aligned}
&\sum_{u \in V(G)} \sum_{v \notin N(u)} (d^2(u) + d^2(uv)) \\
&= \sum_{u \in V(G)} \sum_{v \notin N(u)} d^2(u) + 2 \sum_{uv \in E(\overline{G})} (d^2(u) + d^2(v)) \\
&\quad - 8 \sum_{uv \in E(\overline{G})} (d(u) + d(v)) + 4 \sum_{uv \in E(\overline{G})} d(u)d(v) + 4n(n - 1) - 8m \\
&= \sum_{u \in V(G)} (n - 1 - d_G(u))d_G^2(u) + 2\overline{F}(G) - 4\overline{M}_1(G) \\
&\quad + 4\overline{M}_2(G) + 4n(n - 1) - 8m \\
&= (n - 1)M_1(G) - F(G) + 2\overline{F}(G) - 8\overline{M}_1(G) + 4\overline{M}_2(G) + 4n(n - 1) - 8m
\end{aligned}$$

(2.7)

Using equations (2.6) and (2.7) in the equation (2.5) and using the results of Theorem 2.3, Proposition 2.4 we have the result.

3. Entire F-index of Some Products of Graphs

In this section we obtain the formulas of the Entire F -index for two operations of graphs viz., Cartesian product and Indu-Bala product.

3.1. Cartesian product

The cartesian product $G_1 \times G_2$ [5] of the graphs G_1 and G_2 is the graph whose vertex set is $V(G_1) \times V(G_2)$ and $(u_1, u_2)(v_1, v_2)$ is an edge of $G_1 \times G_2$ if $[u_1 = v_1 \text{ and } u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \text{ and } u_1v_1 \in E(G_1)]$.

We first propose two lemmas, which will be helpful in establishing the formula for the Entire F -index of Cartesian product of graphs.

Lemma 3.1. Let $G_i \in \Omega(n_i, m_i)$, $i = 1, 2$. Then

$$M_4(G_1 \times G_2) = n_2 M_4(G_1) + n_1 M_4(G_2) + 8(m_2 F(G_1) + m_1 F(G_2)) + 6M_1(G_1)M_1(G_2).$$

Proof: The result follows from the fact that

$$\begin{aligned} M_4(G_1 \times G_2) &= \sum_{(u,v) \in V(G_1 \times G_2)} d_{G_1 \times G_2}^4(u, v) \\ &= \sum_{u \in V(G_1)} \sum_{v \in V(G_2)} (d_{G_1}(u) + d_{G_2}(v))^4. \end{aligned}$$

Lemma 3.2. If $G_i \in \Omega(n_i, m_i)$, $i = 1, 2$ then

$$\begin{aligned} ReZG_3(G_1 \times G_2) &= \sum_{(a,b)(c,d) \in E(G_1 \times G_2)} d_{G_1 \times G_2}(a, b) d_{G_1 \times G_2}(c, d) (d_{G_1 \times G_2}(a, b) \\ &\quad + d_{G_1 \times G_2}(c, d)) \\ &= \sum_{(a,b)(c,d) \in E(G_1 \times G_2)} (d_{G_1}(a) + d_{G_2}(b))(d_{G_1}(c) + d_{G_2}(d)) \\ &\quad ((d_{G_1}(a) + d_{G_2}(b)) + (d_{G_1}(c) + d_{G_2}(d))) \\ &= \sum_{u \in V(G_1)} \sum_{bd \in E(G_2)} (d_{G_1}(u) + d_{G_2}(b))(d_{G_1}(u) + d_{G_2}(d)) \\ &\quad (2d_{G_1}(u) + d_{G_2}(b) + d_{G_2}(d)) + \sum_{v \in V(G_2)} \sum_{ac \in E(G_1)} (d_{G_1}(a) \\ &\quad + d_{G_2}(v))(d_{G_1}(c) + d_{G_2}(v))(2d_{G_2}(v) + d_{G_1}(a) + d_{G_1}(c)) \\ &= I_1 + I_2, \end{aligned}$$

(3.1)

where

$$\begin{aligned} I_1 &= \sum_{u \in V(G_1)} \sum_{bd \in E(G_2)} (d_{G_1}(u) + d_{G_2}(b))(d_{G_1}(u) + d_{G_2}(d)) \\ &\quad (2d_{G_1}(u) + d_{G_2}(b) + d_{G_2}(d)) \\ &= \sum_{u \in V(G_1)} \sum_{bd \in E(G_2)} (2d_{G_1}^3(u) + 3d_{G_1}^2(u)(d_{G_2}(b) + d_{G_2}(d)) \\ &\quad + d_{G_1}(u)(d_{G_2}(b) + d_{G_2}(d))^2 + 2d_{G_1}(u)d_{G_2}(b)d_{G_2}(d) + d_{G_2}(b)d_{G_2}(d) \\ &\quad (d_{G_2}(b) + d_{G_2}(d))) \\ &= 2 \sum_{u \in V(G_1)} \sum_{bd \in E(G_2)} d_{G_1}^3(u) + 3 \sum_{u \in V(G_1)} d_{G_1}^2(u) \sum_{bd \in E(G_2)} \\ &\quad (d_{G_2}(b) + d_{G_2}(d)) + \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{bd \in E(G_2)} (d_{G_2}(b) + d_{G_2}(d))^2 \\ &\quad + 2 \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{bd \in E(G_2)} d_{G_2}(b)d_{G_2}(d) \\ &\quad + \sum_{u \in V(G_1)} \sum_{bd \in E(G_2)} d_{G_2}(b)d_{G_2}(d)(d_{G_2}(b) + d_{G_2}(d)) \\ &= 2m_2 F(G_1) + 3M_1(G_1)M_1(G_2) + 2m_1 HM(G_2) + 4m_1 M_2(G_2) \\ &\quad + n_1 ReZG_3(G_2). \end{aligned}$$

Similarly,

$$I_2 = 2m_1 F(G_2) + 3M_1(G_2)M_1(G_1) + 2m_2 HM(G_1) + 4m_2 M_2(G_1) + n_2 ReZG_3(G_1).$$

Putting the values of I_1 and I_2 in (3.1), we have the result.

Theorem 3.3. Let $G_i \in \Omega(n_i, m_i)$, $i = 1, 2$. Then

$$\begin{aligned} F^\varepsilon(G_1 \times G_2) &= \sum_{i,j=1, i \neq j}^2 [6(2n_i - 11m_i)M_1(G_j) + 12(m_i - n_i)M_2(G_j) \\ &\quad + (14m_i - 5n_i)F(G_j) + 3n_i \text{ReZ}G_3(G_j) + 6m_i \text{HM}(G_j) \\ &\quad + n_i M_4(G_j)] + 24M_1(G_1)M_1(G_2) + 96m_1m_2 - 8(m_1n_2 + n_1m_2) \end{aligned}$$

Proof: From theorem 2.2, we can write

$$\begin{aligned} F^\varepsilon(G_1 \times G_2) &= 12M_1(G_1 \times G_2) - 12M_2(G_1 \times G_2) - 5F(G_1 \times G_2) \\ (3.2) \quad &\quad + 3\text{ReZ}G_3(G_1 \times G_2) + M_4(G_1 \times G_2) - 8(m_1n_2 + n_1m_2). \end{aligned}$$

Now we can replace the values of $F(G_1 \times G_2)$, $M_1(G_1 \times G_2)$, and $M_2(G_1 \times G_2)$ in (3.2) using the results from Theorem 3 in [19], Theorem 1 and Theorem 4 in [16] respectively. Then using Lemma 3.1 and Lemma 3.2 we have the theorem.

3.2. Indu-Bala product

Before defining the Indu-Bala product we first define the join of two graphs. The join $G_1 \vee G_2$ [16] of the graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \phi$ and $E(G_1) \cap E(G_2) = \phi$ is the graph which is obtained from G_1 and G_2 by joining each vertex in $V(G_1)$ to all vertices in $V(G_2)$. The Indu-Bala product [14] $G_1 \nabla G_2$ of two graphs G_1 and G_2 is obtained from two disjoint copies of the join $G_1 \vee G_2$ of G_1 and G_2 by joining the corresponding vertices in the two copies of G_2 .

Figure 1: An example of Indu-Bala product.

Lemma 3.4. Let $G_i \in \Omega(n_i, m_i)$, $i = 1, 2$. Then

$$M_4(G_1 \nabla G_2) = 2[M_4(G_1) + M_4(G_2) + 4(n_2 F(G_1) + (n_1 + 1)F(G_2)) + 6(n_2^2 M_1(G_1) + (n_1 + 1)^2 M_1(G_2))] + \lambda_1$$
where $\lambda_1 = 2[8(m_1 n_2^3 + m_2(n_1 + 1)^3) + n_1 n_2^4 + n_2(n_1 + 1)^4]$.

Proof: Clearly,

$$\begin{aligned} M_4(G_1 \nabla G_2) &= 2 \left[\sum_{u \in V(G_1)} d_{G_1 \nabla G_2}^4(u) + \sum_{u \in V(G_2)} d_{G_1 \nabla G_2}^4(u) \right] \\ &= 2 \left[\sum_{u \in V(G_1)} (d_{G_1}(u) + n_2)^4 + \sum_{u \in V(G_2)} (d_{G_2}(u) + n_1 + 1)^4 \right]. \end{aligned}$$

The result follows from the following fact that

$$\begin{aligned} \sum_{u \in V(G_1)} (d_{G_1}(u) + n_2)^4 &= \sum_{u \in V(G_1)} (d_{G_1}^4(u) + 4n_2 d_{G_1}^3(u) + 6n_2^2 d_{G_1}^2(u) + 4n_2^3 d_{G_1}(u) + n_2^4) \\ &= M_4(G_1) + 4n_2 F(G_1) + 6n_2^2 M_1(G_1) + 8m_1 n_2^3 + n_1 n_2^4 \end{aligned}$$

and similarly,

$$\begin{aligned} \sum_{u \in V(G_2)} (d_{G_2}(u) + n_1 + 1)^4 &= M_4(G_2) + 4(n_1 + 1)F(G_2) + 6(n_1 + 1)^2 M_1(G_2) + 8m_2(n_1 + 1)^3 \\ &\quad + n_2(n_1 + 1)^4. \end{aligned}$$

Lemma 3.5. If $G_i \in \Omega(n_i, m_i)$, $i = 1, 2$ then

$$\begin{aligned} ReZG_3(G_1 \nabla G_2) &= 2[(3n_2^2 + 2m_2 + n_1 n_2 + n_2)M_1(G_1) + (3(n_1 + 1)^2 + 3(n_1 + 1) + 2m_1 \\ &\quad + n_1 n_2)M_1(G_2) + 4n_2 M_2(G_1) + 4(n_1 + 1)M_2(G_2) + ReZG_3(G_1) \\ &\quad + ReZG_3(G_2) + n_2 F(G_1) + (n_1 + 2)F(G_2)] + \lambda_2, \end{aligned}$$

where $\lambda_2 = 2[2n_2^3 m_1 + 2m_2(n_1 + 1)^3 + 6m_2(n_1 + 1)^2 + n_2(n_1 + 1)^3 + 8(n_1 + n_2 + 1)m_1 m_2 + 2m_1 n_2(n_1 + 1)(n_1 + 1 + 2n_2) + 2n_1 n_2 m_2(2n_1 + n_2 + 2) + n_1 n_2^2(n_1 + 1)(n_1 + n_2 + 1)]$.

Proof: Partition the edge set $E(G_1 \nabla G_2)$ into four sets E_1, E_2, E_3 and E_4 where $E_1 = \{uv | u, v \in V(G_1)\}$, $E_2 = \{uv | u \in V(G_1), v \in V(G_2)\}$, $E_3 = \{uv | u \neq v, u, v \in V(G_2)\}$ and $E_4 = \{uu' | u, u' \in V(G_1 \nabla G_2), u, u' \notin V(G_1) \text{ and } u' \text{ is the corresponding vertex of } u \text{ of } G_2 \text{ in the two copies of } G_1 \vee G_2\}$. Precisely, $|E_1| = 2m_1$, $|E_2| = 2n_1 n_2$, $|E_3| = 2m_2$ and $|E_4| = n_2$. Again if $uu' \in E_4$ then $d_{G_1 \nabla G_2}(u) = d_{G_1 \nabla G_2}(u')$. Now as $d_{G_1 \nabla G_2}(u) = d_{G_1}(u) + n_2$ or $d_{G_2}(u) + n_1 + 1$ according as $u \in V(G_1)$ or $u \in V(G_2)$ respectively, therefore

$$\begin{aligned} ReZG_3(G_1 \nabla G_2) &= \sum_{uv \in E_1} [d_{G_1}(u) + n_2][d_{G_1}(v) + n_2][d_{G_1}(u) + d_{G_1}(v) + 2n_2] \\ &\quad + \sum_{uv \in E_2} [d_{G_1}(u) + n_2][d_{G_2}(v) + n_1 + 1] \\ &\quad [d_{G_1}(u) + d_{G_2}(v) + n_1 + n_2 + 1] \\ &\quad + \sum_{uv \in E_3} [d_{G_2}(u) + n_1 + 1][d_{G_2}(v) + n_1 + 1] \\ &\quad [d_{G_2}(u) + d_{G_2}(v) + 2n_1 + 2] + \sum_{uu' \in E_4} [d_{G_2}(u) + n_1 + 1] \\ &\quad [d_{G_2}(u') + n_1 + 1][d_{G_2}(u) + d_{G_2}(u') + 2n_1 + 2] \\ &= \sum 1 + \sum 2 + \sum 3 + \sum 4 \text{ (say)} \end{aligned} \tag{3.3}$$

Now,

$$\begin{aligned}
 \sum 1 &= \sum_{uv \in E_1} [d_{G_1}(u)d_{G_1}(v)\{d_{G_1}(u) + d_{G_1}(v)\} + 4n_2d_{G_1}(u)d_{G_1}(v) \\
 (3.4) \quad &+ n_2\{d_{G_1}^2(u) + d_{G_1}^2(v)\} + 3n_2\{d_{G_1}(u) + d_{G_2}(v)\} + 2n_2^3] \\
 &= 2[ReZG_3(G_1) + 4n_2M_2(G_1) + n_2F(G_1) + 3n_2^2M_1(G_1) + 2n_2^3m_1],
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \sum 3 &= 2[ReZG_3(G_2) + 4(n_1 + 1)M_2(G_2) + (n_1 + 1)F(G_2) + 3(n_1 + 1)^2M_1(G_2) \\
 (3.5) \quad &+ 2m_2(n_1 + 1)^3]
 \end{aligned}$$

Again,

$$\begin{aligned}
 \sum 2 &= \sum_{uv \in E_2} [d_{G_1}^2(u)d_{G_2}(v) + d_{G_1}(u)d_{G_2}^2(v) + 2(n_1 + n_2 + 1)d_{G_1}(u)d_{G_2}(v) \\
 &+ \{(n_1 + 1)d_{G_1}^2(u) + n_2d_{G_2}^2(v)\} + \{(n_1 + 1)(n_1 + 2n_2 + 1)d_{G_1}(u) \\
 &+ n_2(2n_1 + n_2 + 2)d_{G_2}(v)\} + n_2(n_1 + 1)(n_1 + n_2 + 1)] \\
 &= 2[\sum_{u \in V(G_1)} d_{G_1}^2(u) \sum_{v \in V(G_2)} d_{G_2}(v) + \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{v \in V(G_2)} d_{G_2}^2(v) \\
 &2(n_1 + n_2 + 1) \sum_{u \in V(G_1)} d_{G_1}(u) \sum_{v \in V(G_2)} d_{G_2}(v) + n_2(n_1 + 1) \sum_{u \in V(G_1)} d_{G_1}^2(u) \\
 &+ n_1n_2 \sum_{v \in V(G_2)} d_{G_2}^2(v) + n_2(n_1 + 1)(n_1 + 2n_2 + 1) \sum_{u \in V(G_1)} d_{G_1}(u) \\
 &+ n_1n_2(2n_1 + n_2 + 2) \sum_{v \in V(G_2)} d_{G_2}(v) + n_1n_2^2(n_1 + 1)(n_1 + n_2 + 1)] \\
 &= 2[(2m_2 + n_1n_2 + n_2)M_1(G_1) + (2m_1 + n_1n_2)M_1(G_2) + 8m_1m_2(n_1 + n_2 + 1) \\
 &+ 2m_1n_2(n_1 + 1)(n_1 + 2n_2 + 1) + 2n_1n_2m_2(2n_1 + n_2 + 2) \\
 &+ n_1n_2^2(n_1 + 1)(n_1 + n_2 + 1)], \\
 (3.6)
 \end{aligned}$$

Now, $d_{G_1 \nabla G_2}(u) = d_{G_1 \nabla G_2}(u')$ for $uu' \in E_4$, so as $d_{G_2}(u) = d_{G_2}(u')$ gives

$$\begin{aligned}
 \sum 4 &= 2 \sum_{uu' \in E_4} [d_{G_2}(u) + n_1 + 1]^3 \\
 (3.7) \quad &= 2[F(G_2) + 3(n_1 + 1)M_1(G_2) + 6m_2(n_1 + 1)^2 + n_2(n_1 + 1)^3]
 \end{aligned}$$

Using equations (3.4), (3.5), (3.6) and (3.7) in equation (3.3) yield the result.

Theorem 3.6. *Let $G_i \in \Omega(n_i, m_i)$, $i = 1, 2$. Then*

$$\begin{aligned}
 F^\varepsilon(G_1 \nabla G_2) &= 6(5n_2^2 - 8n_2 + n_1n_2 + 2m_2 + 4)M_1(G_1) + 6(5n_1^2 \\
 &+ 4n_1 + 2m_1 + n_1n_2 + 1)M_1(G_2) + 24(n_2 - 1)M_2(G_1) \\
 &+ 24n_1M_2(G_2) + 2(7n_2 - 5)F(G_1) + 2(7n_1 + 5)F(G_2) \\
 &+ 6(ReZG_3(G_1) + ReZG_3(G_2)) + 2(M_4(G_1) + M_4(G_2)) \\
 &+ \lambda_1 + 3\lambda_2 + 12\lambda_3 - 12\lambda_4 - 5\lambda_5 - \lambda_6
 \end{aligned}$$

where λ_1 and λ_2 are same as those in Lemma 3.4 and Lemma 3.5 respectively, and $\lambda_3 = 2[n_1n_2^2 + n_2(n_1 + 1)^2 + 4(m_1n_2 + n_1m_2 + m_2)]$, $\lambda_4 = 2m_2(n_1 + 1)^2 + 2m_1n_2^2 + 8m_1m_2 + 4m_1n_2(n_1 + 1) + 4m_2n_1n_2 + 2n_2^2n_1(n_1 + 1) + n_2(n_1 + 1)^2 + 4m_2(n_1 + 1)$, $\lambda_5 = 2[n_1n_2^3 + 6n_2^2m_1 + n_2(n_1 + 1)^3 + 6m_2(n_1 + 1)^2]$ and $\lambda_6 = 8(2m_1 + 2m_2 + 2n_1n_2 + n_2)$.

Proof: Since $|E(G_1 \nabla G_2)| = 2(m_1 + m_2 + n_1 n_2) + n_2$ [3], from theorem 2.2,

$$\begin{aligned}
 F^e(G_1 \nabla G_2) &= 12M_1(G_1 \nabla G_2) - 12M_2(G_1 \nabla G_2) - 5F(G_1 \nabla G_2) \\
 (3.8) \quad &+ 3ReZG_3(G_1 \nabla G_2) + M_4(G_1 \nabla G_2) - 8(2m_1 + 2m_2 \\
 &+ 2n_1 n_2 + n_2)
 \end{aligned}$$

Now from Corollary 3.2, Theorem 3.4 and Corollary 3.3 of [3] we have $M_1(G_1 \nabla G_2) = 2[M_1(G_1) + M_1(G_2)] + \lambda_3$, where $\lambda_3 = 2[n_1 n_2^2 + n_2(n_1 + 1)^2 + 4(m_1 n_2 + n_1 m_2 + m_2)]$, $M_2(G_1 \nabla G_2) = 2n_2 M_1(G_1) + (2n_1 + 3)M_1(G_2) + 2M_2(G_1) + 2M_2(G_2) + \lambda_4$, where $\lambda_4 = 2m_2(n_1 + 1)^2 + 2m_1 n_2^2 + 8m_1 m_2 + 4m_1 n_2(n_1 + 1) + 4m_2 n_1 n_2 + 2n_2^2 n_1(n_1 + 1) + n_2(n_1 + 1)^2 + 4m_2(n_1 + 1)$ and $F(G_1 \nabla G_2) = 2[3n_2 M_1(G_1) + 3(n_1 + 1)M_1(G_2) + F(G_1) + F(G_2)] + \lambda_5$, where $\lambda_5 = 2[n_1 n_2^3 + 6n_2^2 m_1 + n_2(n_1 + 1)^3 + 6m_2(n_1 + 1)^2]$ respectively. Now using these values of $M_1(G_1 \nabla G_2)$, $M_2(G_1 \nabla G_2)$, $F(G_1 \nabla G_2)$ and Lemma 3.4 and Lemma 3.5 in equation (3.8) we obtain the desired result.

Summary and Conclusions

In this paper we define a modified version of forgotten topological index, which can also capture the intermolecular forces that exist between the atoms and bonds in addition to the intermolecular forces between the atoms of a molecule. We call it ‘Entire forgotten topological index’ or ‘Entire F -index’. We also propose its coindex. Some fundamental relations of these indices with some popular topological indices are obtained. The formulas of F -index are obtained in connections with two operations of graphs. We can similarly obtain the formulae for other popular operations of graphs like Join, Composition, Corona product etc. with *mutatis mutandis*. In this communication we consider the indices as graph invariants only. But it will be very interesting to study the indices for their efficiency in predicting various physio-chemical properties of a molecule, which may be a topic of further study.

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