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Super antimagic total labeling under duplication operations

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Abstract:

For a graph G the duplication operation of a vertex v by a new edge $e = uw$ results in a new graph G' such that $N(u) = \{v, w\}$ and $N(w) = \{v, u\}$. The duplication operation of an edge $e = uv$ by a new vertex w results in a new graph G'' such that $N(w) = \{u, v\}$. In this article we have discussed that the properties of a graph, with minimum degree 2 of any vertex, to be super vertex-antimagic total and to be super edge-antimagic total are invariant under the above duplication operations. Also, we have discussed on the existence of the so-called k super vertex-antimagic total vertex modifications and k super edge-antimagic total edge modifications for graphs.

Keywords: Super edge-antimagic total graph; Super vertex-antimagic total graph; Duplication operations; Prism; Antiprism; Crossed prism; Cycle and complete graphs.

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1. Introduction

Let G be a graph consisting of the set V of vertices and the set E of edges. A 1-1 mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is called a vertex-antimagic total labeling if the weights $wt_f(v)$ of all the vertices v of G are distinct, where the $wt_f(v) = f(v) + \sum_{v \in e} f(e)$. When the labels of vertices are smallest among all the labels, that is $1, 2, \ldots, |V|$, the vertex-antimagic total labeling is called a super vertex-antimagic total labeling(SVAT-labeling). The graph G is called super vertex-antimagic total if it has super vertex-antimagic total labeling.

Similarly, a 1-1 mapping f from $V \cup E$ to the set $\{1, 2, \ldots, |V| + |E|\}$ is an edge-antimagic total labeling if the weights $wt_f(e)$ of all the edge e of G are distinct, where for $e = uv$ the $wt_f(e) = f(e) + f(u) + f(v)$. When in such a labeling the labels of vertices are $1, 2, \ldots, |V|$, the edge-antimagic total labeling is called a super edge-antimagic total labeling(SEAT-Labeling). The graph G is called super edge-antimagic total graph if it has super edge-antimagic total labeling.

The idea of duplication of vertex by edge operation on graphs was introduce by Vaidya and Barasara in 2011, [6]. Duplication of a vertex v by a new edge $e = uw \notin G$ results in a new graph G' such that $N(u) = \{v, w\}$ and $N(w) = \{v, u\}$. Again in 2011, Vaidya and Dani [7] introduced the duplication of edge by vertex. In a graph G duplication operation of an edge $e = uv$ by a new vertex $w \notin G$ results in a new graph G'' such that $N(w) = \{u, v\}$.

In mathematics whenever new notions and operations are introduced the mathematicians apply these notion and operations on the previous definitions and relate their properties. Over the year the SCAT-labeling and SEAT-labeling gain the attention of a large number of scientists. Some useful and interesting results on SCAT and SEAT graphs can be found in [1, 2, 3, 4, 5]. For more information and study see the "A Dynamic Survey of Graph Labeling" of 2016 by Joseph A. Gallian.

2. Main results

All the graphs considered in this and next section have vertices of minimum degree 2.

Theorem 1. Let G be a graph with α be the SVAT-labeling and moreover;

$$\begin{aligned} \deg(v_1) &\leq \deg(v_2) \leq \cdots \leq \deg(v_n = v) \\ wt_\alpha(v_1) &< wt_\alpha(v_2) < \cdots < wt_\alpha(v_n = v). \end{aligned}$$

Let G' be the graph obtained by duplication of vertex v by an edge $e = v'v''$, then G' is also SVAT graph.

Proof. Let $|V(G)| = n$ and α be the SVAT-labeling of G with,

$$\begin{aligned} \deg(v_1) &\leq \deg(v_2) \leq \cdots \leq \deg(v_n = v) \\ wt_\alpha(v_1) &< wt_\alpha(v_2) < \cdots < wt_\alpha(v_n = v). \end{aligned}$$

We define the labeling $\beta : V(G') \cup E(G') \to \{1, 2, 3, \ldots, |E(G')| + |V(G')|\}$ as under.

$$\beta(v_i) = \begin{cases} 1, & \text{if } v_i = v'; \\ 2, & \text{if } v_i = v''; \\ \alpha(v_i) + 2, & \text{if } v_i \in V(G). \end{cases}$$

The edge labels are $\beta(v'v'') = n + 3$, $\beta(v_n v') = n + 4$, $\beta(v_n v'') = n + 5$ and $\beta(e) = \alpha(e) + 5$ if $e \in E(G)$, where $v = v_n$. It is clear from the definition of β that all the small labels from $1, 2, \ldots, n+2$ are assigned to the vertices of G' first. Now we shall show that the weight of the vertices of G' under β are distinct. The weights of v' and v'' are $wt_\beta(v') = 1 + n + 3 + n + 4 = 2n + 8$, $wt_\beta(v'') = 2 + n + 3 + n + 5 = 2n + 10$. $wt_\beta(v_1) = \beta(v_1) + \sum_{v_1 \in e} \beta(e) \geq 3 + \sum_{v_1 \in e}(\alpha(e) + 5) > 3 + 2n + 10 > 2n + 10$, since each $\alpha(e) > n$. So we have $wt_\beta(v') < wt_\beta(v'') < wt_\beta(v_1)$. Now for any $1 \leq i \leq n - 1$

$\mathrm{wt}_\beta(v_{i+1}) = \beta(v_{i+1}) + \sum_{v_{i+1} \in\ e} \beta(e)$
$= \alpha(v_{i+1}) + 2 + \sum_{v_{i+1} \in\ e}(\alpha(e) + 5)$
$> 2 + \alpha(v_i) + \sum_{v_i \in\ e}(\alpha(e) + 5)$
$= wt_\beta(v_i)$.

So, we get an ascending chain of weights of vertices of G' under β as follows:

$$wt_\beta(v') < wt_\beta(v'') < wt_\beta(v_1) < \ldots < wt_\beta(v_n).$$

Thus β is a SVAT-labeling and G' is a SVAT graph. $\square$

Corollary 1. Let G be an r-regular graph with α be the SVAT-labeling and v be the vertex with maximum weight under α. Let G' be the graph obtained by duplication of vertex v by an edge $e = v'v''$. Then G' is also SVAT graph.

Proof. Let $|V(G)| = n$ and α be the SVAT-labeling of G. Without loss of generality we can assume that the ascending chain of the weights of the vertices under α is of the form:

$$wt_\alpha(v_1) < wt_\alpha(v_2) < \cdots < wt_\alpha(v_n = v).$$

Thus the result follows from the above theorem. $\square$
So in particular the complete graph K_n and the cycle graph C_n, for $n \geq 3$ having SVAT-labeling remain invariant under the vertex duplication operation.

Theorem 2. Let G be a graph with α be the SEAT-labeling and e be the edge with minimum weight. Let G'' be the graph obtained by duplication of the edge e by a vertex v'. Then G'' is also SEAT total.

Proof. Let $|V(G)| = n$, $|E(G)| = m$ and α be the SEAT-labeling of G. Let the weight of the edge e is smallest from all the weights of edges of G under α. Also without loss of generality we can assume that the ascending chain of the weights of the edges of G under α is of the form:

$$wt_\alpha(e = e_1) < wt_\alpha(e_2) < \cdots < wt_\alpha(e_m).$$

We define the labeling $\beta : V(G'') \cup E(G'') \to \{1, 2, 3, \ldots, |E(G'')| + |V(G'')|\}$ as under.

$$\beta(v_i) = \begin{cases} 1, & \text{if } v_i = v'; \\ \alpha(v_i) + 1, & \text{if } v_i \in V(G). \end{cases}$$

Let $e = e_1 = v_1v_2$ and $\alpha(v_1) < \alpha(v_2)$, then we define edge labels as $\beta(v_1v') = n+2$, $\beta(v_2v') = n+3$ and $\beta(e) = \alpha(e)+3$ if $e \in E(G)$. It is clear from the definition of β that all the small labels from $1, 2, \ldots, n+1$ are firstly assigned to the vertices of G''. Now we shall show that the weight of the edges of G'' under β are distinct. The weights of v_1v' and v_2v' edges are computed and compared as $wt_\beta(v_1v') = \beta(v_1)+\beta(v')+\beta(v_1v') = \alpha(v_1)+n+3 < \alpha(v_2)+n+4 = wt_\beta(v_2v')$. Moreover, the $wt_\beta(e_1) = \beta(v_1)+\beta(v_2)+\beta(e_1) = \alpha(v_1) + \alpha(v_2) + \alpha(e_1) + 5 > \alpha(v_2) + \alpha(v') + n + 5 = wt_\beta(v_2v')$, since each $\alpha(e) > n$. So we have $wt_\beta(v_1v') < wt_\beta(v_2v') < wt_\beta(e_1)$. Now for any $1 \leq i \leq m-1$

$\mathrm{wt}_\beta(e_{i+1}) = \beta(e_{i+1}) + \sum_{v\in\ e_{i+1}} \beta(v)$
$= \alpha(e_{i+1}) + 3 + \sum_{v\in\ e_{i+1}} (\alpha(v) + 1)$
$= 5 + \alpha(e_{i+1}) + \sum_{v\in\ e_{i+1}} \alpha(v)$
$= 5 + wt_\alpha(e_{i+1})$
$> 5 + wt_\alpha(e_i) = wt_\beta(e_i)$.

So, we get an ascending chain of weights of edges of G'' under β as follows;

$$wt_\beta(v_1v') < wt_\beta(v_2v') < wt_\beta(e_1) < \ldots < wt_\beta(e_m).$$

Thus β is a SEAT-labeling and G'' is a SEAT graph. $\square$

Corollary 2. Let G be the complete SEAT-graph K_n. Let K_n'' be the graph obtained by duplication of the arbitrary edge e by a vertex v', then K_n'' is also SEAT total.

Proof. Let $|V(G)| = n$, $|E(G)| = m = \frac{n(n-1)}{2}$ and α be the SEAT-labeling of G. Since in the complete graph K_n there exists rotational automorphism such that any two vertices can be map to each other. Therefore, we can define α in such a way that the weight of the edge e is smallest from all the weights of edges of G under α. Remaining part of the proof is obvious by Theorem 2. $\square$

Corollary 3. Let G be the cycle SEAT-graph C_n. Let C_n'' be the graph obtained by duplication of the arbitrary edge e by a vertex v', then C_n'' is also SEAT total.

Proof. The proof is now clear from Corollary 2 and Theorem 2. $\square$

Corollary 4. Let G be the prism SEAT-graph D_n. Let D_n'' be the graph obtained by duplication of the arbitrary edge e by a vertex v', then D_n'' is also SEAT total.

Proof. The proof is now clear from Corollary 2 and Theorem 2. $\square$

Corollary 5. Let G be the antiprism SEAT-graph A_n. Let A_n'' be the graph obtained by duplication of the arbitrary edge e by a vertex v', then A_n'' is also SEAT total.

Proof. The proof is obvious from Corollary 2 and Theorem 2. $\square$

Corollary 6. Let G be the crossed prism SEAT-graph CP_n, for even $n \geq 4$. Let CP_n'' be the graph obtained by duplication of the arbitrary edge e by a vertex v', then CP_n'' is also SEAT total.

Proof. The proof is trivial from above discussion. $\square$

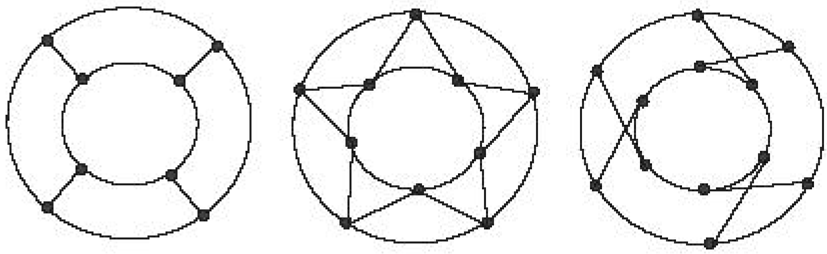

Figure 2.1: (a) prism graph D_4, (b) antiprism graph A_5, (a) crossed prism graph CP_6

3. On the existence of k SVAT-vertex modifications

We say that the graph G have SVAT-vertex modification if there exists a vertex v having an edge $e = v'v''$ modification such that the new produced graph G' is SVAT and we called G' to be SVAT-vertex modification of G. A graph is said to have k SVAT-vertex modifications if there exists vertex v having edges $e_1 = v_1'v_1'', e_2 = v_2'v_2'', \ldots, e_k = v_k'v_k''$ modifications such that the new produced graph is SVAT.

Theorem 3. Let G be a (p, q) graph with $\deg(v_1) \leq \deg(v_2) \leq \ldots \leq \deg(v_p)$ and let α be a SVAT-labeling of G with $wt_\alpha(v_1) < wt_\alpha(v_2) < \ldots < wt_\alpha(v_p)$. Then the following are equivalent;

(1) The graph G has one SVAT-vertex modification.

(2) The graph G has $k \geq 1$ number of SVAT-vertex modifications.

Proof. (2) $\Rightarrow$ (1) is trivial.
(1) $\Rightarrow$ (2) Suppose that the graph G has one SVAT-vertex modification and G' be the SVAT-vertex modification. Since G' is itself SVAT graph therefore it is enough to show that if G is SVAT graph then it has a SVAT vertex modification. So, let $|V(G)| = n$ and α be the SVAT-labeling of G.

We have an ascending chain of the weights of the vertices of G under α of the form:

$$wt_\alpha(v_1) < wt_\alpha(v_2) < \cdots < wt_\alpha(v_p).$$

Take the modification of the vertex v_p by an edge $v'_p v''_p$ to produce the new graph G'. We define the labeling $\beta : V(G') \cup E(G') \to \{1, 2, 3, \ldots, |E(G')| + |V(G')|\}$ as under.

$$\beta(v_i) = \begin{cases} 1, & \text{if } v_i = v'_p; \\ 2, & \text{if } v_i = v''_p; \\ \alpha(v_i) + 2, & \text{if } v_i \in V(G). \end{cases}$$

The edge labels are $\beta(v'_p v''_p) = p + 3$, $\beta(v_p v'_p) = p + 4$, $\beta(v_p v''_p) = p + 5$ and $\beta(e) = \alpha(e) + 5$ if $e \in E(G)$.

Then, it is clear from the proof of Theorem 1 that we have $wt_\beta(v') < wt_\beta(v'') < wt_\beta(v_1)$. Now for any $1 \leq i \leq q - 1$

$\mathrm{wt}_\beta(v_{i+1}) = \beta(v_{i+1}) + \sum_{v_{i+1} \in\ e} \beta(e)$

$= 2 + 5\deg(v_{i+1}) + wt_\alpha(v_{i+1})$

$> 2 + 5\deg(v_i) + wt_\alpha(v_i) = wt_\beta(v_i)$. Thus β is a SVAT-labeling and G' is SVAT graph. $\square$

4. On the existence of k SEAT-edge modifications

We say that a graph G have SEAT-edge modification if there exists an edge e having a vertex v' modification such that the new produced graph G'' is SEAT and we called G'' to be SEAT-edge modification of G. A graph is said to have k SEAT-edge modifications if there exists an edge e having vertices $v'_1, v'_2, \ldots, v'_k$ modifications such that the new produced graph is SEAT.

Theorem 4. Let G be a (p, q) graph with a SEAT-labeling α. Then the following are equivalent;

(1) The graph G has one super edge-antimagic total edge modification.

(2) The graph G has $k \geq 1$ number of super edge-antimagic total edge modifications.

Proof. (2) $\Rightarrow$ (1) is trivial.
(1) $\Rightarrow$ (2) Suppose that the graph G has one SEAT-edge modification and G'' be the SEAT-edge modification. Since G'' is itself SEAT-graph

therefore it is enough to show that if G is SEAT graph then it has a SEAT-edge modification. So, let $|V(G)| = n$ and α be the SEAT-labeling of G. Without loss of generality we can assume that that the ascending chain of the weights of the edges of G under α is of the form:

$$wt_\alpha(e_1) < wt_\alpha(e_2) < \cdots < wt_\alpha(e_n).$$

Take the modification of the edge e_1 by a vertex v' to produce the new graph G''. We define the labeling $\beta : V(G'') \cup E(G'') \to \{1, 2, 3, \ldots, |E(G'')| + |V(G'')|\}$ as under.

$$\beta(v_i) = \begin{cases} 1, & \text{if } v_i = v'; \\ \alpha(v_i) + 1, & \text{if } v_i \in V(G). \end{cases}$$

Let $e = e_1 = v_1v_2$ and $\alpha(v_1) < \alpha(v_2)$, then we define edge labels as $\beta(v_1v') = p + 2$, $\beta(v_2v') = p + 3$ and $\beta(e) = \alpha(e) + 3$ if $e \in E(G)$. It is clear from the definition of β that all the small labels from $1, 2, \ldots, p + 1$ are firstly assigned to the vertices of G''. Now we shall show that the weight of the edges of G'' under β are distinct. It is clear from the proof of Theorem 2 that we have an ascending chain of weight of edges of G'' under β as follows;

$$wt_\beta(v_1v') < wt_\beta(v_2v') < wt_\beta(e_1) < \ldots < wt_\beta(e_m).$$

Thus β is a SEAT-labeling and G'' is SEAT graph. $\square$

5. Open Problems

Problem 1: Find the similar kind of results for arbitrary duplication of vertex of a graph having SVAT-labeling

Problem 2: Find the similar kind of results for arbitrary duplication of edge of a graph having SEAT-labeling

Problem 3: In particular, will a bipartite graph having SVAT-labeling and SEAT-labeling remain invariant under duplication operations?

Conclusion

In this paper we have investigated the super vertex-antimagic total labeling and super edge-antimagic total labeling of graphs under the described vertex and edge modification operations. In this regard, we have obtained some fruitful results. In the end we have given some open problems.

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