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On some new paranormed sequence spaces defined by the matrix $(\hat{D})$ $(\hat{r},0,0,\hat{s})$

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Abstract:

In this paper, we introduce some new paranormed sequence spaces and study some topological properties. Further, we determine α, β and γ-duals of the new sequence spaces and finally, we establish the necessary and sufficient conditions for characterization of matrix mappings.

Keywords: Paranorm sequence space; α-dual; β-dual; γ-dual; Matrix characterization.

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1. Introduction

The idea to construct a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by Altay and Basar[3, 4], Malkowsky and Savas[17], Basar et al.[7], Kirisci and Basar[11], Ng and Lee[18], Sönmez[21] and many more. Moreover, Altay and Basar [1, 2], Malkowsky[16] and Aydin and Basar[6] have employed on to construct new paranormed sequence spaces by means of the domain of some infinite matrices. The domain of generalized difference matrix $B(r, s)$ on some Maddox's spaces was studied by Aydin and Altay [5]. More recently, domain of the double sequential band matrix $B(\tilde{r}, \tilde{s})$ on some Maddox's spaces was studied by Özger and Basar[19].

2. Preliminaries

Throughout the paper we denote w, ℓ_∞, c, c_0 and ℓ_p be the space of all, bounded, convergent, null and p-absolutely summable sequences respectively. Also, bs, cs and ℓ_1 denote the spaces of all bounded, convergent and absolutely convergent series respectively.

Let X and Y be two sequence spaces and $B = (b_{nk})$ be an infinite matrix of real or complex numbers b_{nk}, where $n, k \in \mathbf{N} = \{1, 2, - - -\}$. Then, we say that B defines a matrix mapping from X into Y, denoted by $B : X \to Y$, if for every sequence $x = (x_n) \in X$, the sequence $Bx = (Bx)_n$ is in Y where,

$$(Bx)_n = \sum_{k=1}^{\infty} b_{nk} x_k, (n \in \mathbf{N} \text{and} x \in X), \tag{2.1}$$

provided the right hand side converges for every $n \in \mathbf{N}$ and $x \in X$.

If μ is a normed sequence space, we write $D_\mu(B)$ for $x \in w$ for which the sum in (2.1) converges in the norm of μ. We write $(\lambda, \mu) = \{B : \lambda \subset D_\mu(B)\}$ for the space of those matrices which send the whole of the sequence space λ into the sequence space μ in this sense.

The sequence space $\lambda_B = \{x = (x_k) \in w : Bx \in \lambda\}$ is called the domain of an infinite matrix B in a sequence space λ. One can easily verify that the sequence spaces λ_B and λ are linearly isomorphic when B is triangle.

A paranormed space (X, g) is a topological linear space in which the topology is given by paranorm g, a real sub-additive function on X such that $g(\theta) = 0,\ \ g(x) = g(-x)$ and scalar multiplication is continuous means that $\lambda_n \to \lambda, x_n \to x$ imply $\lambda_n x_n \to \lambda x$, for scalars λ and vectors x.

We consider (p_k) is a bounded sequence of positive real numbers with $supp_k = H$ and $M = max\{1, H\}$. Throughout we assume $p_k^{-1} + (\acute{p}_k)^{-1} = 1$ provided $0 < inf p_k \le H < \infty$.

Throughout $\mathbf{C}$ denotes the complex field.

Maddox [13] define the following sequence spaces:

$\ell_\infty(p) = \{x = (x_k) \in w : sup_{k\in\mathbf{N}}|x_k|^{p_k} < \infty\}$,

$c(p) = \{x = (x_k) \in w : \exists\ l \in \mathbf{C}, \ni lim_{k\to\infty}|x_k - l|^{p_k} = 0\}$,

$c_0(p) = \{x = (x_k) \in w :\ni lim_{k\to\infty}|x_k|^{p_k} = 0\}$,

$\ell(p) = \{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\}, (0 < p_k < \infty)$.

If $p \in \ell_\infty$ then (Maddox[14], Theorem 6]) $c_0(p)$ and $c(p)$ are complete paranormed sequence spaces paranormed by $g_1(x) = sup_{k\in\mathbf{N}}|x_k|^{\frac{p_k}{M}}$. Also $\ell(p)$ is a complete paranormed sequence space paranormed by g_1 if and only if $inf p_k > 0$. Further, $\ell(p)$ is complete paranormed sequence space paranormed by $g_2(x) = (\sum_k |x_k|^{p_k})^{\frac{1}{M}}$.

Let, $\hat{r} = (r_k)$ and $\hat{s} = (s_k)$ are convergent sequences whose entries either constants or distinct non-zero numbers then we define the matrix $\hat{D}(\hat{r}, 0, 0, \hat{s})$ as follows: $\hat{D}(\hat{r}, 0, 0, \hat{s}) = [d_{nk}(r, s)]$ where,

$$d_{nk}(r,s) = \begin{cases} r_k, & (k = n) \\ s_k, & (k = n-3) \\ 0, & \text{otherwise.} \end{cases}$$

for all $k, n \in \mathbf{N}$.

3. Some new paranormed sequence spaces and their topological properties

We define the sequence spaces $\ell_\infty(\hat{D},p), c(\hat{D},p)$ and $c_0(\hat{D},p)$ as the set of sequences whose transforms are in the spaces $\ell_\infty(p), c(p)$ and $c_0(p)$ respectively, that is-

$$\ell_\infty(\hat{D},p) = \{x = (x_k) : sup_{k\in\mathbf{N}}|r_k x_k + s_{k-3}x_{k-3}|^{p_k} < \infty\},$$

$$c(\hat{D},p) = \{x = (x_k) : \exists\ l \in \mathbf{C}, \ni lim_{k\to\infty}|r_k x_k + s_{k-3}x_{k-3} - l|^{p_k} = 0\},$$

$$c_0(\hat{D},p) = \{x = (x_k) : lim_{k\to\infty}|r_k x_k + s_{k-3}x_{k-3}|^{p_k} = 0\}.$$

Theorem 3.1 *The sequence spaces $\ell_\infty(\hat{D},p), c(\hat{D},p)$ and $c_0(\hat{D},p)$ are the complete linear metric spaces paranormed by g, defined by $g(x) = sup_{k\in\mathbf{N}}|r_k x_k + s_{k-3}x_{k-3}|^{p_k}$.*

Proof: We proof the result only for the space $c_0(\hat{D},p)$ to avoid the repetition of the similar statements for other given spaces. One can easily prove that $c_0(\hat{D},p)$ is a linear space with co-ordinate wise addition and scalar multiplication since $\hat{D}(\hat{r},0,0,\hat{s})$ is a triangle matrix and $c_0(p)$ is a linear space.

It is clear that $g(\theta) = 0,\ g(x) \geq 0$ for all $x \in c_0(\hat{D},p)$ and $g(-x) = g(x)$. Consider any sequence $\{x^n\}$ of points of $c_0(\hat{D},p)$ such that $g(x^n - x) \to 0$ as $n \to \infty$ and (β_n) is a sequence of scalars with $\beta_n \to \beta$ as $n \to \infty$.

Now, $\{g(x^n)\}$ is bounded sequence since $g(x^n) \leq g(x) + g(x^n - x)$ holds by subadditivity of the function g. Again, we have,

$$\begin{aligned}(\beta_n x^n - \beta x) &= \sup_{k\in\mathbf{N}} |r_k(\beta_n x_k^n - \beta x_k) + s_{k-3}(\beta_n x_{k-3}^n - \beta x_{k-3})|^{\frac{p_k}{M}} \\ &\leq |\beta_n - \beta| g(x^n) + |\beta| g(x^n - x)\end{aligned}$$

which tends to zero as $n \to \infty$.

Thus, scalar multiplication is continuous and hence $c_0(\hat{D},p)$ is a paranormed sequence space.

Next, let $\{x^i\}$ be a Cauchy sequence in $c_0(\hat{D},p)$, where $x^i = \{x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, - - -\}$.
Then, by the definition of Cauchy sequence, for a given $\varepsilon > 0$, there exists

a positive integer n_0 depending on ε such that $g(x^i - x^j) < \varepsilon$ for all $i, j \geq 0$.

Now, using the definition of g for each fixed $k \in \mathbf{N}$,

$$|\{\hat{D}(\hat{r}, 0, 0, \hat{s})x^i\}_k - \{\hat{D}(\hat{r}, 0, 0, \hat{s})x^j\}_k|$$

$$\leq sup_{k\in\mathbf{N}}|\{\hat{D}(\hat{r}, 0, 0, \hat{s})x^i\}_k - \{\hat{D}(\hat{r}, 0, 0, \hat{s})x^j\}_k|^{\frac{p_k}{M}} < \varepsilon \tag{3.1}$$

for every $i, j \in n_0$, which leads us to the fact that

$$\{\{\hat{D}(\hat{r}, 0, 0, \hat{s})x^1\}_k, \{\hat{D}(\hat{r}, 0, 0, \hat{s})x^2\}_k, \{\hat{D}(\hat{r}, 0, 0, \hat{s})x^3\}_k, ---\}$$

is a Cauchy sequence of complex numbers and is convergent for each $k \in \mathbf{N}$.

Suppose, for each fixed k,

$$\{\hat{D}(\hat{r}, 0, 0, \hat{s})x^i\}_k \to (\hat{D}(\hat{r}, 0, 0, \hat{s})x)_k$$

as $i \to \infty$.

Now, define the sequence

$$\{(\hat{D}(\hat{r}, 0, 0, \hat{s})x)_1, (\hat{D}(\hat{r}, 0, 0, \hat{s})x)_2, (\hat{D}(\hat{r}, 0, 0, \hat{s})x)_3, ---\}$$

From, the relation (3.1) with $j \to \infty$ we have,

$$sup_{k\in\mathbf{N}}|\{\hat{D}(\hat{r}, 0, 0, \hat{s})x^i\}_k - \{\hat{D}(\hat{r}, 0, 0, \hat{s})x\}_k|^{\frac{p_k}{M}} < \varepsilon, \tag{3.2}$$

for every fixed $k \in \mathbf{N}$.

Now, $x^i = \{x_k^i\} \in c_0(\hat{D}, p)$ implies

$$|\{\hat{D}(\hat{r}, 0, 0, \hat{s})x^i\}_k|^{\frac{p_k}{M}} < \varepsilon,$$

for each $k \in \mathbf{N}$.

Thus, by using relation (3.2) we have

$$|\{\hat{D}(\hat{r}, 0, 0, \hat{s})x\}_k|^{\frac{p_k}{M}}$$
$$\leq |\{\hat{D}(\hat{r}, 0, 0, \hat{s})x\}_k - \{\hat{D}(\hat{r}, 0, 0, \hat{s})x^i\}_k|^{\frac{p_k}{M}} + \{\hat{D}(\hat{r}, 0, 0, \hat{s})x^i\}_k|^{\frac{p_k}{M}} < 2\varepsilon$$

Hence, the sequence $\{\hat{D}(\hat{r}, 0, 0, \hat{s})x\}$ belongs to $c_0(p)$. Since, $\{x^i\}$ is an arbitrary Cauchy sequence, the space $c_0(\hat{D}, p)$ is complete. This completes the proof.

It is well known that the matrix domain λ_B of a sequence space λ has a basis whenever $B = (b_{nk})$ is a triangle. (Jarrah and Malkowsky [10], Remark 2.4), we have

Proposition 3.2 *Let* $\gamma_k = \{\hat{D}(\hat{r}, 0, 0, \hat{s})x\}$ *and* $0 < p_k \leq M < \infty$ *for all* $k \in \mathbf{N}$. *Define the sequences* $f = (f_n)$ *and* $d^{(k)} = \{d_n^{(k)}(\hat{r}, \hat{s})\}_{n \in \mathbf{N}}$ *of the elements of the space* $c_0(\hat{D}, p)$ *by*

$$d_n^{(k)}(\hat{r}, \hat{s}) = \begin{cases} \text{For fixed } n, k \text{ and for } j \in \mathbf{N} \\ d_{n+3j}^{(k)} = \frac{(-1)^{n-k} s_k s_{k+3} - - - s_{n-3}}{r_k r_{k+3} - - - r_n} & n \geq k, \text{taking } \ s_{-t} = 1 \\ 0, \quad \text{otherwise.} \end{cases}$$

$f_n = \sum_{k=1}^{n} d_n^{(k)}(\hat{r}, \hat{s})$. *Then, the following statements hold*:
(i) The sequence $\{d^{(k)}\}$ *is a basis for* $c_0(\hat{D}, p)$ *and any* $x \in c_0(\hat{D}, p)$ *has a unique representation of the form* $x = \sum_k \gamma_k d^{(k)}$.
(ii) The set $\{f, d^{(k)}\}$ *is a basis for* $c(\hat{D}, p)$ *and any* $x \in c(\hat{D}, p)$ *has a unique representation of the form* $x = \eta f + \sum_k (\gamma_k - \eta) d^{(k)}$. *where* $\eta = lim_{k \to \infty} \{\hat{D}(\hat{r}, 0, 0, \hat{s})x\}_k$.

4. Duals of the sequence spaces

The idea of dual sequence space was introduced by Köthe and Toeplitz [12]. Then, Maddox, [15], generalized this notion to -X valued sequence classes where X is a Banach space. Further, Chandra Tripathy[8] studied on generalized Köthe-Toeplitz duals of some sequence spaces.

The set $S(\lambda, \mu)$ is defined by

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz \in \mu, \forall x = (x_k) \in \lambda\} \tag{4.1}$$

is called the multiplier space of the spaces λ and μ. One can easily observe for a sequence space γ with $\lambda \supset \gamma \supset \mu$ that the inclusions $S(\lambda, \mu) \subset S(\gamma, \mu)$ and $S(\lambda, \mu) \subset S(\lambda, \gamma)$ hold. With the notation (4.1), the α, β and γ duals of a sequence space λ, which are respectively denoted by $\lambda^\alpha, \lambda^\beta$ and λ^γ are defined by $\lambda^\alpha = S(\lambda, \ell_1), \lambda^\beta = S(\lambda, cs)$ and $\lambda^\gamma = S(\lambda, bs)$.

Define the sequence $y = (y_k)$ which will be frequently used by $\hat{D}(\hat{r}, 0, 0, \hat{s})$-transform of a sequence $x = (x_k)$, that is
$y_k = \{\hat{D}(\hat{r}, 0, 0, \hat{s})x\}_k = r_k x_k + s_{k-3} x_{k-3}$ for $k \in \mathbf{N}$, taking

$$x_{-3} = x_{-2} = x_{-1} = 0 \tag{4.2}$$

Throughout we denote N_m be the collection of those integers which are greater than or equal to $m \in \mathbf{N}$ and also $\wp$ denote the collection of all finite subsets of $\mathbf{N}$.

Theorem 4.1 *Define the sets* $T_i(p)$, $i \in \{1, 2, 3, ---, 7\}$ as follows:

(i) $T_1(p) = \bigcup_{S>1}\{a = (a_k) \in w : sup_{K\in\wp} \sum_n |\sum_{k\in K} d_n^{(k)}(\hat{r}, \hat{s}) a_n| S^{\frac{-1}{p_k}} < \infty\}$,

(ii) $T_2(p) = \bigcup_{S>1}\{a = (a_k) \in w : \sum_k |\sum_{i=k}^{n} \frac{(-1)^i s_k s_{k+3} --- s_{i-3}}{r_k r_{k+3} --- r_i} a_i| S^{\frac{-1}{p_k}} < \infty\}$,

(iii) $T_3(p) = \bigcup_{S>1}\{a = (a_k) \in w : (\sum_{i=k}^{n} \frac{(-1)^i s_k s_{k+3} --- s_{i-3}}{r_k r_{k+3} --- r_i} a_i S^{\frac{-1}{p_k}}) \in \ell_\infty\}$,

(iv) $T_4(p) = \bigcup_{S>1}\{a = (a_k) \in w : sup_{n\in\mathbf{N}} \sum_k |\sum_{i=k}^{n} \frac{(-1)^i s_k s_{k+3} --- s_{i-3}}{r_k r_{k+3} --- r_i} a_i| S^{\frac{-1}{p_k}} < \infty\}$,

(v) $T_5(p) = \bigcap_{S>1}\{a = (a_k) \in w : sup_{K\in\wp} \sum_n |\sum_{k\in K} d_n^{(k)}(\hat{r}, \hat{s}) a_n| S^{\frac{-1}{p_k}} < \infty\}$,

(vi) $T_6(p) = \bigcap_{S>1}\{a = (a_k) \in w : \sum_k |\sum_{i=k}^{n} \frac{(-1)^i s_k s_{k+3} --- s_{i-3}}{r_k r_{k+3} --- r_i} a_i| S^{\frac{-1}{p_k}} < \infty\}$,

(vii) $T_7(p) = \{a = (a_k) \in w : \sum_n |\sum_k d_n^{(k)}(\hat{r}, \hat{s}) a_n| < \infty\}$.

Then we have, $[c_0(\hat{D}, p)]^\alpha = T_1(p), [c_0(\hat{D}, p)]^\beta = T_2(p) \cap T_3(p), [c_0(\hat{D}, p)]^\gamma = T_4(p)$,

$[c(\hat{D}, p)]^\alpha = T_1(p) \cap T_7(p), [c(\hat{D}, p)]^\beta = T_2(p) \cap T_3(p) \cap cs, [c(\hat{D}, p)]^\gamma = T_4(p) \cap bs$,

$[\ell_\infty(\hat{D}, p)]^\alpha = T_5(p), \quad [\ell_\infty(\hat{D}, p)]^\beta = T_6(p) \cap cs, \quad [\ell_\alpha(\hat{D}, p)]^\gamma = T_6(p)$.

Proof : To avoid repetition and similar arguments we give the proof only for the sequence space $c_0(\hat{D}, p)$.

Let $f = (f_n) \in w$ and define the matrix $A = (a_{nk})$ via the sequence $f = (f_n)$ by

$${}_{nk} = \begin{cases} {}^{k}_{n}(\hat{r}, \hat{s}) f_n, & (1 \leq k \leq n) \\ 0, & (k > n) \end{cases}$$

Using the relation (4.2), one can derive by straight forward calculation that

$f_n x_n = f_n\{\hat{D}^{-1}(\hat{r}, 0, 0, \hat{s})y\}_n = \sum_{k=1}^{n} d_n^{(k)}(\hat{r}, \hat{s}) y_k = (Ay)_n$ for any

$$n \in \mathbf{N}. \tag{4.3}$$

From the relation (4.3), we observe that $fx = (f_n x_n) \in \ell_1$ whenever $x = (x_k) \in c_0(\hat{D}, p)$ if and only if $Ay \in \ell_1$ whenever $y = (y_k) \in c_0(p)$. Hence, $f = (f_n) \in [c_0(\hat{D}, p)]^\alpha$ if and only if $A \in (c_0(p) : \ell_1)$. Then, in equation (5.14), by considering $q_n = 1$ for any $n \in \mathbf{N}$ we can write $f \in [c_0(\hat{D}, p)]^\alpha$ if and only if $sup_{K \in \wp} \sum_n |\sum_{k \in K} a_{nk}| S^{\frac{-1}{p_k}} < \infty$ and consequently, $[c_0(\hat{D}, p)]^\alpha = T_1(p)$.

Consider the equality

$$\sum_{k=1}^{n} f_k x_k = \sum_{k=1}^{n} (\sum_{j=1}^{k} d_k^{(j)}(\hat{r}, \hat{s}) y_j) f_k = (My)_n, \tag{4.4}$$

where $M = (m_{nk})$ is defined by

$${}_{nk} = \begin{cases} {}^{k}_{j=1} d_k^{(j)}(\hat{r}, \hat{s}) f_j, & 1 \leq k \leq n \\ 0, & k > n \end{cases} \tag{4.5}$$

for any $n, k \in \mathbf{N}$.

Now, from the relation (4.4), we observe that, $fx = (f_n x_n) \in cs$ whenever $x = (x_k) \in c_0(\hat{D}, p)$ if and only if $My \in cs$ whenever $y = (y_k) \in c_0(p)$. Hence, $f = (f_n) \in [c_0(\hat{D}, p)]^\beta$ if and only if $M \in (c_0(p) : c)$.

Again, we derive from the equation (5.10) and (5.11), by taking $q_n = 1$ for any $n \in \mathbf{N}$ and some $S \in N_2$ we have $\sum_{k=0}^{n} |m_{nk}| S^{\frac{-1}{p_k}} < \infty$ and there exists scalar $\beta_k \in \mathbf{C}$ for any $k \in \mathbf{N}$ such that $sup_{n \in \mathbf{N}} |\sum_{k=1}^{n} m_{mk} - \beta_k| S^{\frac{-1}{p_k}} < \infty$, respectively, which implies that $[c_0(\hat{D}, p)]^\beta = T_2(p) \cap T_3(p)$. Now, we deduce from equation (5.9) that $fx = (f_n x_n) \in bs$ whenever $x = (x_k) \in c_0(\hat{D}, p)$ if and only if $My \in \ell_\infty$ whenever $y = (y_k) \in c_0(p)$. Hence, $f = (f_n) \in [c_0(\hat{D}, p)]^\gamma$ if and only if $M \in (c_0(p) : \ell_\infty)$. Therefore, by equation (5.9) with $q_n = 1$ for any $n \in \mathbf{N}$, we attain $[c_0(\hat{D}, p)]^\gamma = T_4(p)$.

5. Matrix Transformations

In summability theory, different classes of matrices have been investigated. Characterization of matrix classes is found in Rath and Tripathy [20], Tripathy and Sen [23] and many others.

Let λ denote any of the sequence spaces c_0, c or ℓ_∞ and μ be any given sequence space. In this section, we characterize the classes $(\lambda(\hat{D}, p) : \mu)$ and $(\mu : \lambda(\hat{D}, p))$ of infinite matrices. Throughout we consider, $\hat{b}_{nk} = \sum_{i=k}^{\infty} \frac{(-1)^i s_i s_{i+3} - - - s_{n-3}}{r_i r_{i+3} - - - r_n} b_{ni}$, for any $k, n \in \mathbf{N}$.

Theorem 5.1: *Suppose that the elements of the infinite matrices $U = (u_{nk})$ and $V = (v_{nk})$ be connected with the relation*

$$u_{nk} = r_k v_{nk} + s_{k+3} v_{n,k+3} or v_{nk} = \sum_{i=k}^{\infty} \frac{(-1)^i s_i s_{i+3} - - - s_{n-3}}{r_i r_{i+3} - - - r_n} u_{ni} for any k, n \in \mathbf{N}$$

(5.1)

and λ be any of the spaces c_0, c or ℓ_∞ and μ be any given sequence space.

Then, $U \in (\lambda(\hat{D}, p) : \mu)$ if and only if

$$V \in (\lambda(p) : \mu) and$$

$\mathrm{V}^n \in (\lambda(p) : c)$(5.2)
for any fixed $n \in \mathbf{N}$, where $V^n = (v_{mk}^{(n)})$ with

$$v_{mk}^{(n)} = \begin{cases} \sum_{i=k}^{m} \frac{(-1)^i s_i s_{i+3} - - - s_{n-3}}{r_i r_{i+3} - - - r_n} u_{ni}, & 0 \le k \le m \\ 0, \quad k > m \end{cases}$$

for any $m, k \in \mathbf{N}$.

Proof: Suppose that the infinite matrices $U = (u_{nk})$ and $V = (v_{nk})$ be connected with the relation (5.1) and let μ be any given sequence space. One can easily prove that the spaces (λ, p) and $\lambda(p)$ are paranorm isomorphic.

Let $U \in (\lambda(\hat{D}, p) : \mu)$ and $y \in \lambda(p)$. Then, $V\hat{D}(\hat{r}, 0, 0, \hat{s})$ clearly exists and $(u_{nk})_{k\in\mathbf{N}} \in [\lambda(\hat{D}, p)]^{\beta}$, which implies that (5.2) is necessary and $(v_{nk})_{k\in\mathbf{N}} \in [\lambda(p)]^{\beta}$ for each $n \in \mathbf{N}$. Thus, Vy exists for all $y \in \lambda(p)$ and hence by letting $m \to \infty$ in the equality

$$\sum_{k=1}^{m} u_{nk}x_k = \sum_{k=1}^{m}[\sum_{i=k}^{m} \frac{(-1)^i s_i s_{i+3} - - - s_{n-3}}{r_i r_{i+3} - - - r_n} u_{ni}]y_k \tag{5.3}$$

Now, using (5.1) we have $Ux = Vy$, which concludes that $V \in (\lambda(p) : \mu)$.

Conversely, let $V \in (\lambda(p) : \mu)$ and suppose it holds the relation (5.2). Also let $y = (y_k) \in \lambda(p)$. Then, we have $(v_{nk})_{k\in\mathbf{N}} \in [\lambda(p)]^{\beta}$, which gives together with the relation (5.2) that $(u_{nk})_{k\in\mathbf{N}} \in [\lambda(\hat{D}, p)]^{\beta}$ for each $n \in \mathbf{N}$.

Thus, Ux exists and consequently, from the relation (5.3) by letting $m \to \infty$ we have $Vy = Ux$ and hence $U \in (\lambda(\hat{D}, p) : \mu)$.

Theorem 5.2 *Suppose that the elements of the infinite matrices $E = (e_{nk})$ and $F = (f_{nk})$ are connected with relation $f_{nk} = r_n e_{nk} + s_{n-3}e_{n-3,k}$ for any $k, n \in \mathbf{N}, \lambda$ is any of the spaces c_0, c or ℓ_∞ and μ be any given sequence space. Then, $E \in (\mu : \lambda(\hat{D}, p))$ if and only if $F \in (\mu : \lambda(p))$.*

Proof: Let $z = (z_k) \in \mu$ and consider the equality $\sum_{k=1}^{m} f_{nk}z_k = \sum_{k=1}^{m}(r_n e_{nk} + s_{n-3}e_{n-3,k})z_k$ for any $m, n \in \mathbf{N}$. Then by letting $m \to \infty$, we have $(Fz)_n = \{\hat{D}(\hat{r}, 0, 0, \hat{s})Ez\}_n$ for any $n \in \mathbf{N}$. Hence, we observe that $Ez \in \lambda(\hat{D}, p)$ whenever $z \in \mu$ if and only if $Fz \in \lambda(p)$ whenever $z \in \mu$. This completes the proof.

Let (q_n) be a non-decreasing bounded sequence of positive real numbers. Also, let S and T denote the natural numbers. Finally, the sets K_1 and K_2 are defined by
$K_1 = \{k \in \mathbf{N} : p_k \leq 1\}$ and $K_2 = \{k \in \mathbf{N} : p_k > 1\}$.

The following lemmas are consider from Grosse-Erdmann[9] which gives the characterization of matrix mappings between Maddox's sequence spaces $\ell(p), c_0(p), c(p)$ and $\ell_\infty(p)$.

Lemma 5.3 *Let $B = (b_{nk})$ be an infinite matrix. Then, the following statements hold*:

(i) $B \in (\ell_\infty(p) : \ell_\infty(q))$ *if and only if*

(5.4) $$\forall S > 1 \ni sup_{n\in \mathbf{N}}(\sum_{k\in \mathbf{N}} |b_{nk}|S^{\frac{1}{p_k}})^{q_n} < \infty,$$

(ii) $B \in (\ell_\infty(p) : c(q))$ *if and only if*

(5.5) $$\forall S > 1 \ni sup_{n\in \mathbf{N}} \sum_{k\in \mathbf{N}} |b_{nk}|S^{\frac{1}{p_k}} < \infty,$$

$\exists\ (\beta_k) \in w$ *and*

(5.6) $$\forall S > 1 \ni lim_{n\to\infty}(\sum_{k} |b_{nk} - \beta_k|S^{\frac{1}{p_k}})^{q_n} = 0,$$

(iii) $B \in (\ell_\infty(p) : c_0(q))$ *if and only if*

$$\forall S > 1 \ni lim_{n\to\infty}(\sum_{k} |b_{nk}|S^{\frac{1}{p_k}})^{q_n} = 0,$$

(5.7)

(iv) $B \in (\ell_\infty(p) : \ell(q))$ *if and only if* $\sup_{K\in\wp} \sum_n |\sum_{k\in K} b_{nk}S^{\frac{1}{p_k}}|^{q_n} < \infty$

forany

$q_n \geq 1$ *and for any* $S > 1$. (5.8)

Lemma 5.4 *Let* $B = (b_{nk})$ *be an infinite matrix. Then, the following statements hold*:

(i) $B \in (c_0(p) : \ell_\infty(q))$ *if and only if*

(5.9) $$\forall S > 1 \ni sup_{n\in \mathbf{N}}(\sum_{k\in \mathbf{N}} |b_{nk}|S^{\frac{-1}{p_k}})^{q_n} < \infty,$$

(ii) $B \in (c_0(p) : c(q))$ *if and only if*

(5.10) $$\forall S > 1 \ni sup_{n\in \mathbf{N}} \sum_{k\in \mathbf{N}} |b_{nk}|S^{\frac{-1}{p_k}} < \infty,$$

$\forall T, \exists S > 1 \ni sup_{n\in \mathbf{N}} \sum_{k\in K_2} |b_{nk} - \beta_k| T^{\frac{1}{q_n}} S^{\frac{-1}{p_k}} < \infty, (5.11)$

$$\exists(\beta_k) \in w \ni lim_{n\to\infty}|b_{nk} - \beta_k|^{q_n} = 0, \tag{5.12}$$

(iii) $B \in (c_0(p) : c_0(q))$ *if and only if* (5.11)*holds with*

$$\beta_k = 0$$

for any $k \in \mathbf{N}$ *and* $lim_{n\to\infty}|b_{nk}|^{q_n} = 0$ *for each fixed* $k \in \mathbf{N}, (5.13)$
$(iv) B \in (c_0(p) : \ell(q))$ *if and if*

$$\exists S > 1 \ni sup_{K\in\wp} \sum_n |\sum_{k\in K} b_{nk} S^{\frac{-1}{p_k}}|^{q_n} < \infty$$

for any $q_n \geq 1$. (5.14)

Lemma 5.5 *Let* $B = (b_{nk})$ *be an infinite matrix. Then, the following statements hold*: $(i) B \in (c(p) : \ell(q))$ *if and only if* (5.9) *holds and*

$$sup_{n\in\mathbf{N}}|\sum_k b_{nk}|^{q_n} < \infty, \tag{5.15}$$

(ii) $B \in (c(p) : c(q))$ *if and only if* (5.10), (5.11) *and* (5.12) *hold and*

$$\exists \beta \in \mathbf{C} \ni lim_{n\to\infty}|\sum_k b_{nk} - \beta|^{q_n} = 0, \tag{5.16}$$

(iii) $B \in (c(p) : c_0(q))$ *if and only if* (5.13),*holds and*

$$\forall T, \exists S > 1 \ni sup_{n\in\mathbf{N}} \sum_{k\in K_2} |b_{nk}| T^{\frac{1}{q_n}} S^{\frac{-1}{p_k}} < \infty, \tag{5.17}$$

$$\lim_{n\to\infty} |\sum_k b_{nk}|^{q_n} = 0, \tag{5.18}$$

(iv) $B \in (c(p) : \ell(q))$ *if and only if* (5.14) *holds and*

$$\sum_n |\sum_k b_{nk}|^{q_n} < \infty$$

for any $q_n \geq 1$(5.19)

Lemma 5.6 *Let $B = (b_{nk})$ be an infinite matrix. Then, the following statements hold*:

(i) $B \in (\ell(p) : c(q))$ *if and only if* (5.12) *holds and*

$$sup_{n\in\mathbf{N}} sup_{k\in K_1} |b_{nk}|^{p_k} < \infty, \tag{5.20}$$

$$\exists S > 1 \ni sup_{n\in\mathbf{N}} \sum_k |b_{nk}S^{-1}|^{p_k} < \infty, \tag{5.21}$$

$\exists(\beta_k) \in w$ *and* $\forall T > 1 \ni sup_{n\in\mathbf{N}} sup_{k\in K_1} (|b_{nk} - \beta_k| T^{\frac{1}{q_n}})^{p_k} < \infty, (5.22)$

$\exists(\beta_k) \in w$ *and* $\forall T, \exists S > 1 \ni sup_{n\in\mathbf{N}} sup_{k\in K_2} (|b_{nk} - \beta_k| T^{\frac{1}{q_n}} S^{-1})^{p_k} < \infty, (5.23)$

(ii) $B \in (\ell(p) : c_0(q))$ *if and only if* (5.13) *holds and*

$\sup_{n\in\mathbf{N}} sup_{k\in K_1} |a_{nk}T^{\frac{1}{q_n}}|^{p_k} < \infty, \forall T > 1, (5.24)$

$\exists S > 1 \ni sup_{n\in\mathbf{N}} \sum |a_{nk}T^{\frac{1}{q_n}} S^{-1}|^{p_k} < \infty$ *for any* $T > 1, (5.25)$

(iii) $B \in (\ell(p) : \ell\infty(q))$ *if and only if*

$\exists T > 1 \ni sup_{n\in\mathbf{N}} sup_{k\in K_2} |b_{nk}T^{\frac{-1}{q_n}}|^{p_k} < \infty,$

$\exists T > 1 \ni sup_{n\in\mathbf{N}} sup_{k\in K_2} |a_{nk}T^{\frac{-1}{q_n}}|^{p_k} < \infty. (5.26)$

Now, the following we may quote our theorems without proof on the characterization of some matrix classes concerning with the sequence spaces $c_0(\hat{D}, p), c(\hat{D}, p)$ *and* $\ell_\infty(\hat{D}, p)$.

Theorem 5.7 *Let $B = (b_{nk})$ be an infinite matrix. Then, the following statements hold*:

(i) $B \in (c_0(\hat{D}, p) : \ell(q))$ *if and only if* (5.14) *also holds with* $\hat{b}_{nk}$ *instead of* b_{nk} *and* (5.2) *also holds with* $\lambda = c_0$.

(ii) $B \in (c_0(\hat{D}, p) : c(q))$ *if and only if* (5.10), (5.11) *and* (5.12) *hold with* $\hat{b}_{nk}$ *instead of* b_{nk} *and* (5.2) *also holds with* $\lambda = c_0$.
(iii) $B \in (c_0(\hat{D}, p) : \ell_\infty(q))$ *if and only if* (5.9) *also holds with* $\hat{b}_{nk}$ *instead of* b_{nk} *and* (5.2) *also holds with* $\lambda = c_0$.

Theorem 5.8 *Let* $B = (b_{nk})$ *be an infinite matrix. Then, the following statements hold*:

(i) $B \in (c(\hat{D}, p) : \ell(q))$ *if and only if* (5.14) *and* (5.19) *hold with* $\hat{b}_{nk}$ *instead of* b_{nk} *and* (5.2) *also holds with* $\lambda = c$.

(ii) $B \in (c(\hat{D}, p) : c(q))$ *if and only if* (5.10), (5.11), (5.12) *and* (5.16) *hold with* $\hat{b}_{nk}$ *instead of* b_{nk} *and* (5.2) *also holds with* $\lambda = c$.

(iii) $B \in (c(\hat{D}, p) : \ell_\infty(q))$ *if and only if* (5.9) *and* (5.15) *also hold with* $\hat{b}_{nk}$ *instead of* b_{nk} *and* (5.2) *also holds with* $\lambda = c$.

Theorem 5.9 *Let* $B = (b_{nk})$ *be an infinite matrix. Then, the following statements hold*:
(i) $B \in (\ell_\infty(\hat{D}, p) : \ell_\infty(q))$ *if and only if* (5.8) *also holds with* b_{nk} *replaced by* $\hat{b}_{nk}$ *and* (5.2) *also holds with* $\lambda = \ell_\infty$.
(ii) $B \in (\ell_\infty(\hat{D}, p) : c_0(q))$ *if and only if* (5.7) *also holds with* b_{nk} *replaced by* $\hat{b}_{nk}$ *and* (5.2) *also holds with* $\lambda = \ell_\infty$.
(iii) $B \in (\ell_\infty(\hat{D}, p) : c(q))$ *if and only if* (5.5) *and* (5.6) *hold with* b_{nk} *replaced by* $\hat{b}_{nk}$ *and* (5.2) *also holds with* $\lambda = \ell_\infty$.

(iv) $B \in (\ell_\infty(\hat{D}, p) : \ell_\infty(q))$ *if and only if* (5.4) *also holds with* b_{nk} *replaced by* $\hat{b}_{nk}$ *and* (5.2) *also holds with* $\lambda = \ell_\infty$.

Theorem 5.10 *Let* $B = (b_{nk})$ *be an infinite matrix. Then, the following statements hold*:

(i) $B \in (c_0(p) : c(\hat{D}, q))$ *if and only if* (5.10), (5.11) *and* (5.12) *holds with* b_{nk} *replaced by* v_{nk}.
(ii) $B \in (c_0(p) : \ell_\infty(\hat{D}, q))$ *if and only if* (5.9) *also holds with* b_{nk} *replaced by* v_{nk}.

Theorem 5.11 *Let $B = (b_{nk})$ be an infinite matrix. Then, the following statements hold*:

(i) $B \in (c(p) : c(\hat{D}, q))$ *if and only if* (5.10), (5.11), (5.12) *and* (5.16) *holds with b_{nk} replaced by v_{nk}.*
(ii) $B \in (c(p) : \ell_\infty(\hat{D}, q))$ *if and only if* (5.9) *and* (5.15) *holds with b_{nk} replaced by v_{nk}.*

Theorem 5.12 *Let $B = (b_{nk})$ be an infinite matrix. Then, the following statements hold*:

(i) $B \in (\ell_\infty(p) : c_0(\hat{D}, q))$ *if and only if* (5.7) *also holds with b_{nk} replaced by v_{nk}.*
(ii)) $B \in (\ell_\infty(p) : c(\hat{D}, q))$ *if and only if* (5.5) *and* (5.6) *holds with b_{nk} replaced by v_{nk} .*
(iii) $B \in (\ell_\infty(p) : \ell_\infty(\hat{D}, q))$ *if and only if* (5.4) *also holds with b_{nk} replaced by v_{nk}.*

Theorem 5.13 *Let $B = (b_{nk})$ be an infinite matrix. Then, the following statements hold*:
(i) $B \in (\ell(p) : c_0(\hat{D}, q))$ *if and only if* (5.13) *and* (5.24)-(5.25) *holds with b_{nk} replaced by v_{nk}*
(ii) $B \in (\ell(p) : c(\hat{D}, q))$ *if and only if* (5.12) *and* (5.20)-(5.23) *holds with b_{nk} replaced by v_{nk}.*
(iii) $B \in (\ell(p) : \ell_\infty(\hat{D}, q))$ *if and only if* (5.26) *and* (5.27) *holds with b_{nk} replaced by v_{nk}.*

Conclusion:

The spectrum of the matrix class $D(r, 0, 0, s)$ has been investigated by Tripthy and Paul [22] is a special case of $\hat{D}(\hat{r}, 0, 0, \hat{s})$ if we consider $\hat{r} = re$ and $\hat{s} = se$. The results investigated in this paper are more general.

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