Modules whose partial endomorphisms have a δ -small kernels

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Abstract:

Let R be a commutative ring and M a unital R -module. A submodule N is said to be δ -small, if whenever $N + L = M$ with M/L is singular, we have $L = M$. M is called δ -small monoform if any of its partial endomorphism has δ -small kernel. In this paper, we introduce the concept of δ -small monoform modules as a generalization of monoform modules and give some of their properties, examples and characterizations.

Keywords: Small submodules; δ -small submodules; Monoform modules; δ -small monoform modules; Artinian principal ideal ring.

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1. Introduction

Throughout all rings are associative, commutative with identity and all modules are unitary R -module. The concept monoform module is appeared in [12], where an R -module M is called monoform if for each nonzero submodule N of M and for each $f \in \text{Hom}(N, M)$, $f \neq 0$ implies $\text{Ker} f = \{0\}$. In [6], M. A. Inaam Hadi and al. introduced and studied the concept of small monoform, where a module M is called module small monoform if for each nonzero submodule N of M and for each nonzero $f \in \text{Hom}(N, M)$, $\text{Ker} f$ is small in N . A submodule L of a module M is called essential submodule of M , if $L \cap K \neq \{0\}$ for any nonzero submodule K of M . A submodule N of a module M is called e -small submodule of M if, for any essential submodule L of M , $N + L = M$ implies $L = M$. Recently in [5], A. D. Diallo and al introduced and studied the concept of e -small monoform, where a module M is called module e -small monoform if for each nonzero submodule N of M and for each nonzero $f \in \text{Hom}(N, M)$, $\text{Ker} f$ is e -small in N . In [13], Zhou. Y introduced and studied the concept of δ -small submodule, where a submodule N of M is called δ -small ($N \ll_{\delta} M$, for short) if whenever $N + L = M$ with M/L is singular, then $L = M$. An R -module M is called δ -small monoform if any of its partial endomorphism has δ -small kernel.

In this paper, we introduce and study the concept of δ -small monoform module as a new generalization of monoform module and give some of their properties, characterizations and examples. A module M is called compressible if it can be embedded in each of its nonzero submodules. Then, we have the obvious implications:

compressible $\Rightarrow$ monoform $\Rightarrow$ small monoform $\Rightarrow$ δ -small monoform $\Rightarrow$ e -small monoform. We will prove later that under certain conditions the properties compressible, monoform, small monoform and δ -small monoform are equivalent.

Moreover, we study the relations between δ -small monoform modules and other related modules as nonsingular module, quasi-injective module and fully retractable module.

Our paper is structured as follows:

In the first section, we give some known results which we will use or cite throughout this paper.

In the second section, we define the δ -small monoform modules. Our aim in this section is to work on the concept of δ -small monoform modules. We also show among others, the following results.

- (1) The following statements are equivalent for an R -module M :
- (a) M is a δ -small monoform module.
 - (b) For any $0 \neq N \leq M$ and for any nonzero $f \in \text{Hom}_R(N, M)$, if there exists $K \leq N$ such that $f(K) = f(N)$ and $(K + Z_2(N)) \leq_e N$, then $N = K$.
 - (c) For any $0 \neq N \leq M$ and for any nonzero $f \in \text{Hom}_R(N, M)$, $\text{Ker} f \subseteq \delta(N)$ and if there exists $K \leq N$ with $f(K) = f(N)$, then there exists a projective $L \leq^\oplus N$ such that $L \subseteq \text{Ker} f$ and $N = K \oplus L$.

Also we have a characterization of small monoform modules.

- (2) The following statements are equivalent for an R -module M :
- (a) M is a small monoform module.
 - (b) For any $0 \neq N \leq M$ and for any nonzero $f \in \text{Hom}_R(N, M)$, $\text{Ker} f \subseteq \text{Rad}(N)$ and if there exists $K \leq N$ with $f(K) = f(N)$, then there exists a projective $L \leq^\oplus N$ such that $L \subseteq \text{Ker} f$ and $N = K \oplus L$.

Finally, we prove the following theorem.

- (3) Let R be an artinian principal ideal ring and M be a singular prime R -module. Then the following statements are equivalent:
- (a) M is a δ -small monoform module.
 - (b) M is a monoform module.
 - (c) M is a small monoform module.
 - (d) M is a compressible module.
 - (e) M is a uniform module.
 - (f) M is a weakly co-Hopfian module.

2. Preliminaries

In this section, we give some known results which we will use or cite throughout this paper.

Definition 2.1. Let M be an R -module, put $Z(M) = \{m \in M : \text{ann}_R(m) \leq_e R\}$. M is called nonsingular if $Z(M) = \{0\}$, and singular if $Z(M) = M$. The Goldie torsion submodule $Z_2(M)$ of M is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. M is called Goldie torsion if $M = Z_2(M)$.

Definition 2.2. Let M be an R -module and $N \leq M$.

1. N is called small submodule of M ($N \ll M$, for short) if, for any submodule L of M , $N + L = M$ implies $L = M$
2. A submodule N is called δ -small ($N \ll_{\delta} M$, for short) if whenever $N + L = M$ and M/L is singular then $L = M$.

Remark 2.3. Each small submodule, or nonsingular semi-simple of a module M is δ -small submodule in M . The δ -small submodules of a singular module are small submodules.

Lemma 2.4. ([13], Lemma 1.2)

A submodule N of a module M is δ -small if and only if for all $X \leq M$: if $X + N = M$, then $M = X \oplus Y$ for a projective semi-simple submodule Y with $Y \leq N$.

Lemma 2.5. ([13], Lemma 1.3)

1. Let N , K and L be submodules of an R -module M such that $K \subseteq N$, then $N \ll_{\delta} M$ if and only if $K \ll_{\delta} M$ and $N/K \ll_{\delta} M/K$.
2. If $K \ll_{\delta} M$ and $f : M \rightarrow M'$ is a homomorphism, then $f(K) \ll_{\delta} M'$. In particular, if $K \ll_{\delta} M \subseteq M'$, then $K \ll_{\delta} M'$.
3. Assume that $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \ll_{\delta} M_1 \oplus M_2$ if and only if $K_1 \ll_{\delta} M_1$ and $K_2 \ll_{\delta} M_2$.

Definition 2.6. Let $\wp$ be the classe of all singular simples modules. For a module M , let $\delta(M) = \text{Rej}_M(\wp) = \cap \{N \subseteq M \mid M/N \in \wp\}$ be the rejet in M of $\wp$.

3. Main results

In this section, we introduce the notion of δ -small monoform module and obtain some of its basic properties. It is shown that every submodule of a δ -small monoform module inherits the property (Proposition 2.5). Also we characterize the δ -small monoform modules (Proposition 2.9 and Theorem 2.17). Moreover a characterization of small monoform modules is obtained (Proposition 2.22). Also we prove that under certain conditions, the properties compressible, monoform, small monoform, δ -small monoform and weakly co-Hopfian are equivalent (Theorem 2.32).

Definition 3.1. An R -module M is called a δ -small monoform if for each nonzero submodule N of M and for each $f \in \text{Hom}(N, M)$, $f \neq 0$ implies $\text{Ker } f \ll_{\delta} N$.

Example 3.2. Every nonsingular semi-simple module is a δ -small monoform. But the converse is not true in general.

In fact, the $\mathbf{Z}$ -module $\mathbf{Z}/4\mathbf{Z}$ is a δ -small monoform but it is neither semi-simple nor nonsingular.

Example 3.3. Let M be a Noetherian R -module. Suppose for any $0 \neq N \leq M$, every proper submodule of N is contained in $\delta(N)$. Then M is a δ -small monoform module.

Remark 3.4. (1) It is clear that every monoform module is a δ -small monoform. But the converse is not true in general, for example $\mathbf{Z}/4\mathbf{Z}$ as $\mathbf{Z}$ -module is a δ -small monoform but it is not monoform. (2) The homomorphic image of δ -small monoform module is not necessarily δ -small monoform; for example $\mathbf{Z}$ as $\mathbf{Z}$ -module is δ -small monoform.

Let $\pi : \mathbf{Z} \longrightarrow \mathbf{Z}/12\mathbf{Z}$, where π is the natural projection. However $\mathbf{Z}/12\mathbf{Z}$ as $\mathbf{Z}$ -module is not δ -small monoform because $\bar{0} \neq f = 4\bar{x} \in \text{End}_{\mathbf{Z}}(\mathbf{Z}/12\mathbf{Z})$ and

$\text{Ker } f = \langle \bar{3} \rangle \not\ll_{\delta} \mathbf{Z}/12\mathbf{Z}$. From this, it follows easily that quotients of δ -small monoform modules need not to be δ -small monoform in general.

Proposition 3.5. Every nonzero submodule of δ -small monoform module is a δ -small monoform.

Proof. Let M be a δ -small monoform R -module. For any $K \leq N$, $K \neq \{0\}$, let $f : K \longrightarrow N$ such that $f \neq 0$. Then $i \circ f \neq 0$, where i is the inclusion mapping. Thus, $\text{Ker}(i \circ f) \ll_e K$ because M is a δ -small monoform module. But $\text{Ker}(i \circ f) = \text{Ker } f$, hence $\text{Ker } f \ll_{\delta} K$ and so N is a δ -small monoform module.

Remark 3.6. The direct sum of δ -small monoform modules is not necessarily

δ -small monoform module. Consider the following examples:

1. Let $M = \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ as $\mathbf{Z}$ -module. Each of $\mathbf{Z}/4\mathbf{Z}$ and $\mathbf{Z}/3\mathbf{Z}$ is a δ -small monoform and $M = \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z} \cong \mathbf{Z}/12\mathbf{Z}$. However $\mathbf{Z}/12\mathbf{Z}$ as $\mathbf{Z}$ -module is not δ -small monoform.
2. The $\mathbf{Z}$ -module $\mathbf{Z} \oplus \mathbf{Z}$ is not δ -small monoform while $\mathbf{Z}$ is a δ -small monoform $\mathbf{Z}$ -module.

Definition 3.7. An R -module M is called fully stable if for each $N \leq M$, N is stable, that is for each homomorphism $f : N \longrightarrow M$, $f(N) \subseteq N$.

Theorem 3.8. *Let M be a fully stable and R -module such that $M = M_1 \oplus M_2$, where M_1 and M_2 are two submodules of M and for each R -homomorphism $f : N_1 \oplus N_2 \longrightarrow M$, $f \neq 0$ implies $f(N_1) \neq \{0\}$ and $f(N_2) \neq \{0\}$ (i.e. $f|_{N_1} \neq 0, f|_{N_2} \neq 0$). Then M_1 and M_2 are δ -small monoforms if and only if M is a δ -small monoform.*

Proof.

$\Rightarrow$) Let $0 \neq N \leq M$, $f : N \longrightarrow M$ such that $f \neq 0$. Since M is fully stable, every submodule of M is stable. So N is stable. Thus, $N = (N \cap M_1) \oplus (N \cap M_2)$ by [1, Proposition 4.5, P.29].

Consider:

$$(N \cap M_1) \xrightarrow{i_1} N \xrightarrow{f} M \xrightarrow{p_1} M_1$$

$$(N \cap M_2) \xrightarrow{i_2} N \xrightarrow{f} M \xrightarrow{p_2} M_2$$

where i_1, i_2 are inclusion mappings and p_1, p_2 are projection mappings.

Then $p_1 \circ f \circ i_1 : (N \cap M_1) \longrightarrow M_1$ and $p_2 \circ f \circ i_2 : (N \cap M_2) \longrightarrow M_2$. Let $N_1 = N \cap M_1$ and $N_2 = N \cap M_2$. By hypothesis, $f|_{N_1} \neq 0$, so there exists $n_1 \in N \cap M_1$,

$n_1 \neq 0$ and $f(n_1) \neq 0$. Also, $f|_{N_2} \neq 0$, so there exists $n_2 \in N \cap M_2, n_2 \neq 0$ and $f(n_2) \neq 0$. On the other hand, $f \circ i_1 : (N \cap M_1) \longrightarrow M$ implies $f \circ i_1(n_1) = f(n_1) \neq 0$ and $f \circ i_2 : (N \cap M_2) \longrightarrow M$ implies $f \circ i_2(n_2) = f(n_2) \neq 0$. Thus,

$f \circ i_1(N \cap M_1) \subseteq N \cap M_1$ because N_1 and N_2 are stable, $f(N \cap M_1) \subseteq N \cap M_1$. Similarly $f(N \cap M_2) \subseteq N \cap M_2$. Since $f(n_1) \in N \cap M_1$ and $f(n_1) \neq 0$, so $p_1 \circ f \circ i_1(n_1) = f(n_1) \neq 0$. Similarly, $p_2 \circ f \circ i_2(n_2) = f(n_2) \neq 0$. Thus $p_1 \circ f \circ i_1 \neq 0$ and $p_2 \circ f \circ i_2 \neq 0$. Since M_1 and M_2 are δ -small monoforms, $\text{Ker}(p_1 \circ f \circ i_1) \ll_\delta (N \cap M_1)$ and $\text{Ker}(p_2 \circ f \circ i_2) \ll_\delta (N \cap M_2)$. Then by Lemma 1.5,

$$\text{Ker}(p_1 \circ f \circ i_1) \oplus \text{Ker}(p_2 \circ f \circ i_2) \ll_\delta (N \cap M_1) \oplus (N \cap M_2) = N.$$

Let $x = x_1 + x_2 \in \text{Ker} f$, where $x_1 \in N \cap M_1$ and $x_2 \in N \cap M_2$. So $f(x_1) + f(x_2) = 0$. Hence, $f(x_1) = -f(x_2) \in (N \cap M_1) \cap (N \cap M_2) = \{0\}$, so $f(x_1) = 0$ and $f(x_2) = 0$. So, $p_1 \circ f \circ i_1(x_1) = p_1 \circ f(x_1) = f(x_1) = 0$ and $p_2 \circ f \circ i_2(x_2) = p_2 \circ f(x_2) = f(x_2) = 0$. Hence, $x = x_1 + x_2 \in \text{Ker}(p_1 \circ f \circ i_1) \oplus \text{Ker}(p_2 \circ f \circ i_2) \ll_\delta N$. It follows that $\text{Ker} f \subseteq \text{Ker}(p_1 \circ f \circ i_1) \oplus \text{Ker}(p_2 \circ f \circ i_2) \ll_\delta N$. Thus by Lemma 1.5, $\text{Ker} f \ll_\delta N$ and so M is a δ -small monoform.

($\Leftarrow$) It is clear by Proposition 2.5.

Proposition 3.9. *Let M be a nonzero Noetherian R -module. Then M is a δ -small monoform if and only if every nonzero 3-generated submodule of M is a δ -small monoform.*

Proof. $\Rightarrow$) It follows from Proposition 2.5.

($\Leftarrow$) Suppose that every nonzero 3-generated submodules is a δ -small monoform. Let $0 \neq N \leq M$ and let $f : N \rightarrow M$ such that $f \neq 0$. If $\text{Ker } f = \{0\}$, then $\text{Ker } f \ll_{\delta} N$. If $\text{Ker } f \neq \{0\}$, let $x \neq 0$ and $x \in \text{Ker } f$. Let $y \in N$ and let $f(y) = z$. Put $L = Rx + Ry + Rz$, L is 3-generated submodule of M .

By hypothesis, L is a δ -small monoform. Let $H = Rx + Ry$, and $g = f|_H : H \rightarrow L$. Hence $\text{Ker } g \ll_{\delta} H \leq N$ because L is a δ -small monoform. But $x \in \text{Ker } g$, so

$\langle x \rangle \subseteq \text{Ker } g \ll_{\delta} N$. Since M is Noetherian, $\text{Ker } f$ is finitely generated. Hence $\text{Ker } f = \sum_{i=1}^n Rx_i$, for some $x_1, x_2, \dots, x_n \in M$. We have $\langle x_i \rangle \ll_{\delta} N$ for each $i = 1, \dots, n$. Thus by Lemma 1.5, $\text{Ker } f = \sum_{i=1}^n Rx_i \ll_{\delta} N$. So M is a δ -small monoform R -module.

Recall that a module is called weakly co-Hopfian if, for any injective endomorphism f of M , $f(M) \leq_e M$.

Corollary 3.10. *Let R be an artinian principal ideal ring. Then the following statements are equivalent for a nonzero weakly co-Hopfian R -module M :*

- (1) M is a δ -small monoform module.
- (2) Every nonzero 3-generated submodule of M is a δ -small monoform.

Proof.

Suppose M is a weakly co-Hopfian R -module. Then, according to Theorem 3.8 in [3], M is a finitely generated module. Hence M is a Noetherian module because R is an artinian ring. Therefore, in view of Proposition 2.9, the result is obtained.

Definition 3.11. *An R -module M is called small monoform if for each nonzero submodule N of M and for each $f \in \text{Hom}(N, M)$, $f \neq 0$ implies $\text{Ker } f \ll N$.*

Remark 3.12. *It is clear that every small monoform module is a δ -small monoform. But the converse is not true in general; for example let R be*

an artinian semi-simple ring and M be a nonzero R -module which is not of finite length. Clearly, M is a δ -small monoform module. But it is not small monoform.

In fact, M is a nonsingular semi-simple module and so it is a δ -small monoform R -module. Now, let $f : M \rightarrow M$ be a nonzero endomorphism. Then, $\text{Ker } f \neq \{0\}$. Since M is a semi-simple module, $\text{Ker } f \not\leq M$. Therefore, M is not small monoform.

Definition 3.13. (1) A submodule N of a module M is called e -small ($N \ll_e M$, for short) if, for any essential submodule L of M , $N + L = M$ implies $L = M$.

(2) An R -module M is called e -small monoform if for each nonzero submodule N of M and for each $f \in \text{Hom}(N, M)$, $f \neq 0$ implies $\text{Ker } f \ll_e N$.

Remark 3.14. It is clear that every δ -small monoform module is an e -small monoform. But the converse is not true in general; for example $\mathbf{Z}/6\mathbf{Z}$ is e -small monoform as $\mathbf{Z}$ -module. But it is not δ -small monoform. So, according to Remark, it is easily seen that e -small monoform modules need not be δ -small monoform or small monoform in general.

Proposition 3.15. Every nonsingular e -small monoform R -module is a δ -small monoform module.

Proof.

Suppose M is a nonsingular e -small monoform module. Let $0 \neq N \leq M$ and $f : N \rightarrow M$ be a nonzero homomorphism. Suppose $\text{Ker } f + K = N$ with N/K singular for some $K \leq N$. Since M is a nonsingular module, N is also a nonsingular module. Thus, $K \leq_e N$. Since M is assumed e -small monoform, $K = N$. Therefore, M is a δ -small monoform module.

Lemma 3.16. ([5], Proposition 2.24)

Let M be a uniform R -module. Then M is small monoform if and only if M is an e -small monoform.

Theorem 3.17. The following statements are equivalent for an R -module M :

- (1) M is a δ -small monoform module.
- (2) For any $0 \neq N \leq M$ and for any nonzero $f \in \text{Hom}_R(N, M)$, if there exists $K \leq N$ such that $f(K) = f(N)$ and $(K + Z_2(N)) \leq_e N$, then $N = K$.

- (3) For any $0 \neq N \leq M$ and for any nonzero $f \in \text{Hom}_R(N, M)$, if there exists $K \leq N$ such that $((K + Z_2(N))/Z_2(N)) \leq_e N/Z_2(N)$ and $f(K) = f(N)$, then $K = N$.
- (4) For any $0 \neq N \leq M$ and for any nonzero $f \in \text{Hom}_R(N, M)$, $\text{Ker} f \subseteq \delta(N)$ and if there exists $K \leq N$ with $f(K) = f(N)$, then there exists a projective $L \leq^\oplus N$ such that $L \subseteq \text{Ker} f$ and $N = K \oplus L$.

Proof.

(1) $\Rightarrow$ (2) Suppose that M is a δ -small monoform module. Let $0 \neq N \leq M$ and $0 \neq f \in \text{Hom}_R(N, M)$, $f \neq 0$. Let $K \leq N$ such that $f(K) = f(N)$. Thus, $\text{Ker} f + K = N$. Since M is a δ -small monoform module, $\text{Ker} f \ll_\delta N$. By Lemma 1.4, $N = K \oplus L$ for some projective semi-simple submodule L with $L \subseteq \text{Ker} f$. This implies that $N/K \cong L$. Since $(K + Z_2(N)) \leq_e N$, N/K is Goldie torsion by ([2], Proposition 2.2). Thus, $N/K \cong L$ is Goldie torsion. Therefore, $L = \{0\}$, i.e., $N = K$.

(2) $\Rightarrow$ (1) Let $0 \neq N \leq M$ and $f : N \rightarrow M$ be a nonzero homomorphism. Suppose $\text{Ker} f + K = N$ with N/K singular for some $K \leq N$. It is clear that N/K is Goldie torsion. So, in view of Proposition 2.2 in [2], we have $(K + Z_2(N)) \leq_e N$. Therefore, $N = K$.

(2) $\Leftrightarrow$ (3) Follows from Proposition 2.2 in [2].

(1) $\Leftrightarrow$ (3) According to the argument used in the proof of (1) $\Rightarrow$ (2) and in view of Proposition 2.2 in [2], this equivalence holds true.

(1) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) Let $0 \neq N \leq M$ and $f : N \rightarrow M$ be a nonzero homomorphism. Suppose $\text{Ker} f + K = N$ with N/K singular for some $K \leq N$. This implies that

$f(N) = f(K)$. Then by hypothesis, there is a projective $L \leq^\oplus N$ with $L \subseteq \text{Ker} f \subseteq \delta(N)$ such that $N = K \oplus L$. Hence $\delta(L) \oplus \delta(K) = \delta(N) = L \oplus \delta(N)$. Thus $\delta(L) = L$. Since L is projective, L is semi-simple by Proposition 2.13 in [9]. Thus $N/K \cong L = \{0\}$, i.e., $N = K$. Therefore, M is a δ -small monoform module.

Corollary 3.18. *Every δ -small module which does not have any nonzero projective semi-simple is a small monoform module.*

Proof.

Let M be a δ -small module, N be a nonzero submodule of M and $f : N \rightarrow M$ be a nonzero homomorphism. Assume $\text{Ker} f + K = N$ for some $K \leq N$. Since M is assumed δ -small monoform, $\text{Ker} f \ll_\delta N$. Then by

Theorem 2.17, $N = K \oplus L$ for some projective semi-simple submodule L with $L \subseteq \text{Ker} f$. By hypothesis, $L = \{0\}$. This implies that $N = K$ and so $\text{Ker} f \ll N$. Hence, M is a small monoform module.

Corollary 3.19. *The following statements are equivalent for a uniform R -module M :*

- (1) M is a δ -small monoform module.
- (2) M is an e -small monoform module.
- (3) M is a small monoform module.

Proposition 3.20. *Let M be a free $\mathbf{Z}$ -module. Then the following conditions are equivalent:*

- (1) M is an e -small monoform.
- (2) M is a δ -small monoform.

Proof. (1) $\Rightarrow$ (2) Suppose M is an e -small monoform module. Let $0 \neq N \leq M$ and $f : N \rightarrow M$ be a nonzero $\mathbf{Z}$ -homomorphism. Suppose $\text{Ker} f + K = N$ with N/K singular for some $N \leq K$. For any complement L to K in N , we have $K \oplus L \leq_e N$. It is clear that $K + L + \text{Ker} f = N$. So $K \oplus L = N$ because $\text{Ker} f \ll_e N$. By Theorem 2.17, it is easy to see that K is a semi-simple submodule. Since M is a free $\mathbf{Z}$ -module, N is also a free $\mathbf{Z}$ -module. Hence N is a projective $\mathbf{Z}$ -module. It follows that K is projective. Now, K is a projective semi-simple module. Thus, according to Theorem 2.17, $N = K$. Therefore, M is a δ -small monoform module.

(2) $\Leftarrow$ (1) It is easy to see.

Corollary 3.21. *Let M be a projective $\mathbf{Z}$ -module. Then the following conditions are equivalent:*

- (1) M is an e -small monoform.
- (2) M is a δ -small monoform.

Proof.

It is clear.

In the following, we give a characterization of small monoform modules.

Proposition 3.22. *The following statements are equivalent for an R -module M :*

- (1) M is a small monoform module.
 (2) For any $0 \neq N \leq M$ and for any nonzero $f \in \text{Hom}_R(N, M)$, $\text{Ker} f \subseteq \text{Rad}(N)$ and if there exists $K \leq N$ with $f(K) = f(N)$, then there exists a projective $L \leq^\oplus N$ such that $L \subseteq \text{Ker} f$ and $N = K \oplus L$.

Proof.

(1) $\Rightarrow$ (2) It is clear.

(2) $\Rightarrow$ (1) Let $0 \neq N \leq M$ and $f : N \rightarrow M$ be a nonzero homomorphism. Suppose $\text{Ker} f + K = N$ for some $K \leq N$. By hypothesis, there exists a projective direct summand L of N with $L \subseteq \text{Ker} f \subseteq \text{Rad}(N)$ such that $N = K \oplus L$. So $\text{Rad}(L) \oplus \text{Rad}(K) = L \oplus \text{Rad}(K)$. This implies that $\text{Rad}(L) = L$. Thus, $L = \{0\}$ because L is projective. Hence $K = N$. Therefore, M is a small monoform module.

Definition 3.23. Let M be an R -module.

- (1) M is called prime if $\text{Ann}_R(M) = \text{Ann}_R(N)$ for each $0 \neq N \leq M$.
 (2) M is called compressible if it can be embedded in each of its nonzero submodules.

Remark 3.24. Every compressible R -module is a δ -small monoform see [10, Corollary 26.3.5]. The converse is not true in general; for example; the $\mathbf{Z}$ -module $\mathbf{Q}$ is a δ -small monoform. But it is not compressible since $\text{Hom}_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Z}) = \{0\}$.

Proposition 3.25. Let M be a δ -small monoform semi-simple R -module. Then every proper submodule of M is nonsingular.

Proof.

Let M be a δ -small monoform semi-simple module and N be any proper submodule of M . Then $N \not\leq_e M$ because M is semi-simple. If $N = \{0\}$, then N is nonsingular. Suppose $N \neq \{0\}$. So, there exists a relative complement K of N in M such that $N \oplus K \leq_e M$. Let $f : N \oplus K \rightarrow M$ define by $f(n + k) = k$ for all $n + k \in N \oplus K$. It is clear that f is well-defined and $f \neq 0$. Since M is δ -small monoform, $\text{Ker} f \ll_\delta N \oplus K$. But $\text{Ker} f = N$. Thus, $N \oplus \{0\} \ll_\delta N \oplus K$. So according to Lemma 2.3 in [11], N is projective semi-simple. Hence, N is nonsingular.

Corollary 3.26. Let M be a δ -small monoform semi-simple R -module. Then any surjection from a CS R -module to a proper submodule of M splits.

Proof. Suppose M_1 is a CS module and N is a proper submodule of M . By Proposition 2.25, N is nonsingular. Let $f : M_1 \longrightarrow N$ be a surjection. Since M_1 is a CS module, there exists a direct summand K of M_1 such that $\text{Ker} f \leq_e K$. Then $K/\text{Ker} f$ is a singular module and it injects into M_1 . Therefore, $K/\text{Ker} f = \{0\}$, i.e., $\text{Ker} f = K$. This implies that $\text{Ker} f$ is a direct summand of M_1 .

Corollary 3.27. *Let M be a δ -small monoform semi-simple R -module. If M is compressible, then M is nonsingular. In this case, M is isomorphic to a nonzero ideal of a R .*

Proof.

In view of Proposition 2.25, every nonzero proper submodule N of M is nonsingular. Since M is a compressible, there exists a monomorphism $f : M \longrightarrow N$. Thus, $f(Z(M)) \subseteq Z(N)$. Since N is nonsingular, $Z(N) = \{0\}$. Thus, $f(Z(M)) = \{0\} = f(\{0\})$. Since f is a monomorphism, $Z(M) = \{0\}$. Therefore, M is a nonsingular module. Thus, in view of Theorem 26.4.3 in [12], M is isomorphic to a nonzero ideal of a R .

Corollary 3.28. *Let M be a δ -small monoform semi-simple R -module and N be a proper submodule of M . If K and L are two injective submodules of N , so is $K + L$.*

Proof.

Let N a proper submodule of M . Consider the naturel surjection $f : K \oplus L \longrightarrow K + L$. By Proposition 2.25, N is nonsingular. So, $K + L \subseteq N$ is also nonsingular. Since $K \oplus L$ is injective, it is a CS module. According to the Corollary, f splits. Therefore, $K + L$ is isomorphic to a direct summand of $N \oplus L$, so it is injective.

Corollary 3.29. *Let R be a self-injective ring and M be a δ -small monoform semi-simple R -module with finite uniform dimension. Then for any proper submodule N of M , $\text{End}_R(N)$ is a CS ring.*

Proof.

Let N be a proper submodule of M . Then, according to Proposition 2.25, N is a nonsingular module and has finite uniform dimension. Since R is assumed self-injective, N is a CS module by ([8], Exerc.). Since M is semi-simlpe, N is also semi-simple. Therefore, according to Theorem 3.2 in [7], $\text{End}_R(N)$ is a CS ring.

Proposition 3.30. *The following statements are equivalent for a singular semi-simple R -module M :*

- (1) M is a δ -small monoform module.
- (2) M is a monoform module.
- (3) M is a simple module.
- (4) M is a compressible module.

Proof.

(2) $\Leftrightarrow$ (3) Follows from Remark 1.3 in [7].

(1) $\Rightarrow$ (2) Suppose M is a δ -small monoform module. Since M is assumed semi-simple, every proper submodule of M is nonsingular by Proposition 2.25. Since M is singular, every proper submodule of M is also singular. From this, every proper submodule of M must be zero. Thus, M is simple module. Therefore, M is a monoform module.

(2) $\Rightarrow$ (1) It is clear.

(1) $\Leftrightarrow$ (4) By argument used in the proof of (1) $\Rightarrow$ (2) and in view of Remark, this equivalence holds true.

Lemma 3.31. *Any δ -small monoform singular R -module is a uniform module.*

Proof.

Let M be a δ -small monoform singular module. Suppose there exists a nonzero submodule $N \leq M$ such that $N \not\leq_e M$. So, there exists a relative complement K of N in M such that $N \oplus K \leq_e M$. Let $f : N \oplus K \rightarrow M$ define by $f(n+k) = n$ for all $n+k \in N \oplus K$. It is clear that f is well-defined and $f \neq 0$. Since M is δ -small monoform, $\text{Ker } f = \{0\} \oplus K \ll_\delta N \oplus K$. In view of Proposition 2.25, K is nonsingular. Since M is singular, K is also singular. From this, K must be zero. This implies that $N \leq_e M$. This is a contradiction. Therefore any δ -small monoform singular module is a uniform module.

Theorem 3.32. *Let R be an artinian principal ideal ring and M be a singular prime R -module. Then the following statements are equivalent:*

- (1) M is a δ -small monoform module.
- (2) M is a monoform module.
- (3) M is a small monoform module.
- (4) M is a compressible module.
- (5) M is a uniform module.
- (6) M is a weakly co-Hopfian module.

Proof.

(1) $\Rightarrow$ (2) Suppose M is a δ -small monoform module. Let $0 \neq N \leq M$ and $f : N \longrightarrow M$ be a homomorphism. Since M is assumed singular, it is a uniform module by Lemma 2.31. Thus, M is a weakly co-Hopfian module. Since R is an artinian principal ideal ring, M is a finitely generated module by Theorem 3.8 in [3]. So, there exists an epimorphism $g : R \longrightarrow M$ such that $R/\text{ann}_R(M) \cong M$. Since M is a prime module, $\text{ann}_R(M)$ is a prime ideal of R . Hence, $\text{ann}_R(M)$ is a maximal ideal of R because R is artinian. This implies that M is a simple module. Hence, M is a monoform module.

(2) $\Rightarrow$ (1) It is clear.

(1) $\Leftrightarrow$ (3) It is easy to see.

(4) $\Rightarrow$ (1) Results from Remark 2.24.

(1) $\Rightarrow$ (4) According to the argument used in the proof of (1) $\Rightarrow$ (2), we get that M is a uniform prime finitely generated module. Therefore, in view of Lemma 26.2.9 in [10], M is a compressible module.

(2) $\Rightarrow$ (5) It is clear.

(5) $\Rightarrow$ (2) Suppose M is a uniform module. According to the proof of

(1) $\Rightarrow$ (2), M is a simple module. Therefore, M is a monoform module.

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (5) Suppose M is a weakly co-Hopfian module. Then, M is a simple module. Therefore, M is a uniform module.

Remark 3.33. *It is clear that a simple module is a δ -small monoform prime module. But the converse is not true in general. For example $\mathbf{Z}$ as $\mathbf{Z}$ -module is a delta-small monoform prime module. However, it is not simple.*

Corollary 3.34. *Let R be an artinian principal ideal ring. Then the following statements are equivalent for a singular R -module M :*

1. M is a δ -monoform prime module.
2. M is a simple module.

Proof. F(1) $\Rightarrow$ (2). Suppose M is a δ -monoform prime module. Then $\text{ann}_R(M)$ is a prime ideal of R . According to Theorem 2.32, M is a simple module.

(2) $\Rightarrow$ (1) Follows from Remark 2.33.

Corollary 3.35. *Let R be an artinian principal ideal ring. Then the following assertions are verified for a singular δ -small monoform R -module:*

1. $\text{End}_R(M)$ is a local ring.
2. M is a direct sum of a semi-simple module N and a module K such that $\delta(K) \leq_e K$.

Proof.

1. Since M is a finitely generated module over an artinian ring, M is of finite length. Thus, M is an indecomposable module of finite because M is a uniform module. Therefore, $\text{End}_R(M)$ is a local ring.
2. For $\delta(M)$, there exists $N \leq M$ such that $N \cap \delta(M) = \{0\}$ and $N \oplus \delta(M) \leq_e M$. According to 1), M is a finitely generated module over an artinian ring. Thus, in view of Example 2.25 in [11], there exists $K \leq M$ such that $N + K = M$ and $N \cap K \ll_\delta K$. Since $N \cap K = N \cap (N \cap K) \leq N \cap \delta(K) \leq N \cap \delta(M) = \{0\}$, $M = N \oplus K$. This implies that N is a semi-simple module. The last part is clear because $N \oplus \delta(K) \leq_e N \oplus K$.

For an R -module the dual module $\text{Hom}_R(M, R)$ is denoted by M^* . If the natural map $M \longrightarrow (M^*)^*$ is injective, then M will be called torsionless. M is called reflexive if the natural map $M \longrightarrow (M^*)^*$ is bijective.

Proposition 3.36. *Let R be a quasi-Frobenius principal ideal ring and M be a δ -small monoform singular R -module. Then the following assertions are verified:*

1. M is reflexive.
2. M^* and $E(M^*)$ are finitely generated.

Proof.

1. In view of Theorem 2.32, M is a finitely generated R -module. Therefore, according to Theorem 15.11 in [8], M is reflexive.
2. Since M^* is finitely generated and A is artinian, $E(M^*)$ is finitely generated.

Proposition 3.37. *Let M be a δ -small monoform R -module and f a surjective endomorphism of M . If $N \leq M$, then $f(N) \ll_\delta M$ if and only if $N \ll_\delta M$.*

Proof. $\Rightarrow$) Let $N + B = M$ with M/B singular for some $B \leq M$. Then $f(N) + f(B) = M$. Then $f(B) = M$ because $f(N) \ll_{\delta} M$ and $M/f(B)$ singular. This implies that $\text{Ker} f + B = M$. Since M is a δ -small monoform R -module, $\text{Ker} f \ll_{\delta} M$. Hence $B = M$. Therefore $N \ll_{\delta} M$.
 $(\Leftarrow)$ holds true from Lemma 1.5.

Definition 3.38. An R -module is called fully retractable if for every nonzero submodule N of M and every nonzero element $g \in \text{Hom}_R(N, M)$, we have $\text{Hom}_R(M, N)g \neq \{0\}$.

Proposition 3.39. Let M be a fully retractable R -module such that for each $0 \neq N \leq M$, the kernel of any nonzero endomorphism of N is δ -small. Then M is a δ -small monoform module.

Proof. Let $0 \neq N \leq M$ and $f : N \rightarrow M$ such that $f \neq 0$. Since M is fully retractable, there exists $g : M \rightarrow N, g \neq 0$. Consider $N \xrightarrow{f} M \xrightarrow{g} N$. We have $g \circ f \neq 0$ because M is fully retractable. By hypothesis, $\text{Ker}(g \circ f) \ll_{\delta} N$. Since $\text{Ker} f \subseteq \text{Ker}(g \circ f)$, thus by Lemma 1.5, $\text{Ker} f \ll_{\delta} N$. So M is an δ -small monoform module.

Definition 3.40. A submodule N of a module M is called δ -coclosed if N/K is singular and $N/K \ll_{\delta} M/K$ for some $K \leq N$ implies $K = N$.

Proposition 3.41. Let M be a quasi-injective R -module such that every submodule of M is δ -coclosed. If the kernel of any nonzero endomorphism of M is δ -small, then M is a δ -small monoform.

Proof. $\Rightarrow$) Let $0 \neq N \leq M$ and $f \in \text{Hom}_R(N, M), f \neq 0$. Since M is quasi-injective, there exists $g \in \text{End}_R(M)$ such that $g \circ i = f$ where i is the inclusion map. Hence, $g(n) = f(n)$ for each $n \in N$ and so $\text{Ker} f \leq \text{Ker} g$. By hypothesis, $\text{Ker} g \ll_{\delta} M$. So $\text{Ker} f \ll_{\delta} M$. On the other hand, $\text{Ker} f \leq N$ and by [4, Corollary 2.6], $\text{Ker} f \ll_{\delta} M$ implies $\text{Ker} f \ll_{\delta} N$. Therefore, M is a δ -small monoform module.
 $(\Leftarrow)$ It is clear.

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