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Stability of solutions to fractional differential equations with time-delays

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Abstract

This paper deals with a fractional boundary value problem involving variable delays. Sufficient conditions for the existence of a unique solution are investigated. Moreover the stability of the unique solution is discussed. A numerical example that emphasizes the importance of the results obtained in this article is also included.

Keywords: *Fractional derivative, Existence of solution, Stability of solution, Boundary value problem.*

AMS classification: *34A08, 26A33.*

1. Introduction

Differential equations of fractional order have become very useful in recent years due to their many applications in applied sciences, fluid flows, optics, geology, viscoelastic materials, biosciences, ...Moreover, fractional differential equations are integro-differential equations and their numerical solution requires large computer memory and long runs of numerical simulations, this makes it very difficult to investigate the general properties of fractional dynamical systems. As a consequence, accurate approximation and a suitable numerical technique play an important role in identifying the solution behavior of such fractional equations and in exploring their applications (see, e.g.see [1,3–6,10-14, 16-20, 23-28] and the references therein. Recently, many works focus on the existence of solutions for fractional differential equations with delay, see [1,4-5,16,17,24,26-28]. In [16], the authors proved the existence results for a class of delay fractional differential equations of the form:

$$\begin{aligned} D_{0^+}^{\alpha} u\,(t) &= u(t) + f\,(t, u(t), u(t-\tau)))\,, 0<\alpha<1, 0<t\leq 1, \\ u\,(t) &= \varphi(t), t \in [-\tau, 0] \end{aligned}$$

with the boundary condition $\begin{cases} u(0) = \lim_{t\to 0^+} t^{1-\alpha}u(t) = c = u(1) \\ D_{0^+}^{1-\alpha}u(t)_{t=0} = c\Gamma(\alpha), \end{cases}$
where $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative and f is a continuous function.

In [28], the authors discussed the stability of the solutions for nonlinear fractional differential equations with constant delays and integral boundary conditions:

$$\begin{aligned} D_{0^+}^{\alpha} u\,(t) &= \sum_{j=1}^{n} a_j(t) f\,(t, u(t), u(t-\tau_j)) = 0, 0<\alpha<1, t>0, \\ u\,(t) &= \varphi(t), t<0, \\ I_{0^+}^{\alpha-1}u(t)_{t=0} &= 0, \lim \varphi(t)_{t\longrightarrow 0^-} = 0, \end{aligned}$$

here $f : \mathbf{R}^+ \times \mathbf{R}^2 \longrightarrow \mathbf{R}$ is a continuous function, a_j and φ are given continuous functions, $\tau_j \geq 0$, $j = 1, 2, ..., n$ are constants.

For more results on the stability of solution for fractional boundary value problem we refer to [11,20,27].

In this work, we discuss the existence, uniqueness and stability of solutions for a nonlinear fractional boundary value problem with variable delays that we denote by (P):

$$
(\mathrm{P})\begin{cases} D_{0^+}^{\alpha} u\left(t\right) = q(t) f\left(t, u(t-\theta_1(t)), u(t-\theta_2(t))\right), 2 \leq \alpha < 3, t > 0, \\ u\left(t\right) = \varphi(t), t \in \left[-\tau, 0\right], \\ u''(0) = 0, \lim_{t\to\infty} D_{0^+}^{\alpha-1} u(t) = \Gamma(\alpha) u(0), \end{cases}
$$

where $D_{0^+}^{\alpha}$ denotes the standard Riemann-Liouville fractional derivative of order α, the functions $\theta_i : [0, \infty) \to (0, \infty)$ are continuous functions, such $\lim_{t\to\infty}(t - \theta_i(t)) = +\infty$, $\tau = -\min_{0\leq i\leq 2} \min_{t\geq 0}(t - \theta_i(t))$. We assume that $q : [0, \infty) \to [0, \infty)$, the function f is continuous on $[0, \infty) \times \mathbf{R}^2$ and φ is a continuous function on the interval $[-\tau, 0]$.

Delay fractional differential equations arise in models representing biological phenomena when the time delays occurring in these phenomena are considered such as population dynamics, epidemiology, immunology, physiology, and neural networks. The memory or time-delays in these models are related to the duration of certain hidden processes, such as the stages of a life cycle, the time between the infection period and the immune one...

Mathematical models involving integer order differential equations have proven useful in understanding the dynamics of biological systems, however, most biological, physical, and engineering systems have long-range temporal memory [2], and long-range space interactions [21] .

Moreover, for a physical process, the fractional order derivative is related to the whole space, while the integer order derivative describes the local properties of a certain position, consequently and due to the properties of fractional derivatives and integrals such as their ability to describe hereditary and memory properties in different processes that exist in most biological systems, models of fractional order differential equations seem more consistent with real phenomena than those of integer order, we refer to [26-28] for some applications of fractional order systems in modeling and control. Furthermore, It has been successfully applied to system biology [7], physics [9,29], hydrology [22], medicine [15], and finance [8].

2. Preliminaries

In this section, we introduce some necessary definitions and lemmas that will be used later and can be found in [18,23,25].

The Riemann fractional integral of order $\alpha > 0$ of a function f is given by

$$
I_{0^+}^{\alpha} f\left(t\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_0^t \left(t-s\right)^{\alpha-1} f\left(s\right) ds
$$

provided that the right side is pointwise defined on $(0, +\infty)$.

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function f is given by

$$D_{0^+}^{\alpha} f(t) = \frac{d^n}{dt^n}\left(I_{0^+}^{n-\alpha} f(t)\right),$$

provided that the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 1. *Let $\alpha > 0$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, c_i \in \mathbf{R}, i = 1, 2, \dots, n$$

as solution.

Lemma 2. *The solution of the following linear fractional boundary value problem*

$$\begin{aligned} D_{0^+}^{\alpha} u(t) &= e(t), 2 \leq \alpha < 3, t > 0, \\ u(t) &= \varphi(t), t \in [-\tau, 0], \\ u''(0) &= 0, \lim_{t \to \infty} D_{0^+}^{\alpha-1} u(t) = \Gamma(\alpha) u(0), \end{aligned}$$

is given by

$$u(t) = \begin{cases} \varphi(0) t^{\alpha-1} + \int\limits_0^{\infty} G(t,s) e(s) ds, t > 0 \\ u(t) = \varphi(t), t \in [-\tau, 0] \end{cases},$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, 0 \leq s \leq t < \infty \\ t^{\alpha-1}, 0 \leq t \leq s < \infty \end{cases}.$$

Proof. By Lemma 1, we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - I_{0^+}^{\alpha} e(t).$$

Since $u(0) = \varphi(0)$ and $u''(0) = 0$, we deduce that $c_3 = c_2 = 0$. Now, from $\lim_{t \to \infty} D_{0^+}^{\alpha-1} u(t) = \Gamma(\alpha) u(0)$, we get

$$c_1 = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int\limits_0^{\infty} e(s) ds,$$

then the solution is

$$u(t) = \varphi(0)t^{\alpha-1} + \int_0^\infty G(t,s)e(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1}, 0 \le s \le t < \infty \\ t^{\alpha-1}, 0 \le t \le s < \infty. \end{cases}$$

$\square$

Lemma 3. *The function G is continuous nonnegative and for all $s, t \geq 0$ satisfies:*

$$\frac{G(t,s)}{1+t^{\alpha-1}} \leq \frac{1}{\Gamma(\alpha)}.$$

Proof. The proof is easy, so we omit it. $\square$

Denote by $(X, \|.\|)$ the Banach space

$$X = \left\{ u \in C[-\tau, \infty) : \sup_{t \in [0,\infty)} \frac{|u(t)|}{1+t^{\alpha-1}} < \infty \right\}$$

according to the norm

$$\|u\|_X = \|u\|_0 + \|u\|_\infty^0,$$

where

$$\|u\|_0 = \max_{t \in [-\tau,0]} |u(t)|, \quad \|u\|_\infty^0 = \sup_{t \in [0,\infty)} \frac{|u(t)|}{1+t^{\alpha-1}}.$$

Define the operator $T : X \to X$ as

$$\text{Tu(t)=} \begin{cases} \varphi(0)t^{\alpha-1} + \int\limits_0^\infty G(t,s)q(s)f\left(s, u(s-\theta_1(s)), u(s-\theta_2(s))\right) ds, t > 0 \\ \varphi(t), t \in [-\tau, 0]. \end{cases}$$

Then the problem (P) has a solution if and only if the operator T has a fixed point in X.

3. Existence and uniqueness of a solution

Theorem 1. *Assume that: (H1) there exist two nonnegative functions L_1, $L_2 \in L^1(0,\infty)$ such that*

$$\left|f(t,(1+t^{\alpha-1})x_1,(1+t^{\alpha-1})y_1)-f(t,(1+t^{\alpha-1})x_2,(1+t^{\alpha-1})y_2)\right|$$

$$\leq L_1(t)\left|x_1-x_2\right|+L_2(t)\left|y_1-y_2\right|, \tag{3.1}$$

for all $x_1,y_1,x_2,y_2\in\mathbf{R}$, $t>0$ and

$$C=\max\left\{\int_0^\infty q(s)L_1(s)ds,\int_0^\infty q(s)L_2(s)ds\right\}<\frac{\Gamma(\alpha)}{2}. \tag{3.2}$$

(H2) There exist $t_i>0$, such that $t-\theta_i(t)<0$, if $0\leq t\leq t_i$, $t-\theta_i(t)\geq 0$, if $t>t_i$, $i=1,2$. Then the nonlinear fractional boundary value problem (P) has a unique solution in X.

Proof. Let $u,v\in X$, we have

$$\left\|Tu-Tv\right\|_0=\max_{t\in[-\tau,0]}\left|Tu(t)-Tv(t)\right|=0. \tag{3.3}$$

On the other hand, for $t>0$ we get by computations

$$\begin{aligned}\left|\frac{Tu(t)-Tv(t)}{1+t^{\alpha-1}}\right| &\leq \frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)\left|f\left(s,u(s-\theta_1(s)),u(s-\theta_2(s))\right)\right.\\ &\quad\left.-f\left(s,v(s-\theta_1(s)),v(s-\theta_2(s))\right)\right|ds\end{aligned}$$

$$\begin{aligned}\leq\ &\frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)\left|f\left(s,\frac{(1+s^{\alpha-1})\,u(s-\theta_1(s))}{1+s^{\alpha-1}},\frac{(1+s^{\alpha-1})\,u(s-\theta_2(s))}{1+s^{\alpha-1}}\right)\right.\\ &\left.-f\left(s,\frac{(1+s^{\alpha-1})\,v(s-\theta_1(s))}{1+s^{\alpha-1}},\frac{(1+s^{\alpha-1})\,,v(s-\theta_2(s))}{1+s^{\alpha-1}}\right)\right|ds\end{aligned}$$

$$\begin{aligned}\leq\ &\frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)L_1(s)\left|\frac{u(s-\theta_1(s))-v(s-\theta_1(s))}{1+s^{\alpha-1}}\right|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^\infty q(s)L_2(s)\left|\frac{u\left(s-\theta_2(s)\right)-v\left(s-\theta_2(s)\right)}{1+s^{\alpha-1}}\right|ds\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} q(s)L_1(s) \left| \frac{u(s-\theta_1(s)) - v(s-\theta_1(s))}{1+s^{\alpha-1}} \right| ds$$
$$+\frac{1}{\Gamma(\alpha)} \int_{t_1}^{\infty} q(s)L_1(s) \left| \frac{u(s-\theta_1(s)) - v(s-\theta_1(s))}{1+s^{\alpha-1}} \right| ds$$
$$+\frac{1}{\Gamma(\alpha)} \int_0^{t_2} q(s)L_2(s) \left| \frac{u(s-\theta_2(s)) - v(s-\theta_2(s))}{1+s^{\alpha-1}} \right| ds$$
$$+\frac{1}{\Gamma(\alpha)} \int_{t_2}^{\infty} q(s)L_2(s) \left| \frac{u(s-\theta_2(s)) - v(s-\theta_2(s))}{1+s^{\alpha-1}} \right| ds$$

$$\leq \frac{2C}{\Gamma(\alpha)} \left(\|u-v\|_0 + \|u-v\|_0^{\infty} \right), \tag{3.4}$$

hence

$$\|Tu - Tv\|_{\infty}^{0} \leq \frac{2C}{\Gamma(\alpha)} \|u-v\|_X .$$

Thanks to (3.3) and (3.4), it yields

$$\|Tu - Tv\|_X \leq \frac{2C}{\Gamma(\alpha)} \|u-v\|_X .$$

Taking (3.2) into account, we deduce that T is a contraction and then T has a unique fixed point in X that is the unique solution for problem (P). □

4. Stability of solution

In this section, we study the stability of the solution for the nonlinear fractional boundary value problem (P). Let $\tilde{u}$ be a solution of the following fractional boundary value problem

$$(\widetilde{P}) \begin{cases} D_{0^+}^{\alpha} \tilde{u}(t) - q(t) f\left(t, \tilde{u}(t-\theta_1(t)), \check{u}(t-\theta_2(t))\right) = 0, 2 \leq \alpha < 3, t > 0 \\ \tilde{u}(t) = \tilde{\varphi}(t), t \in [-\tau, 0] \\ \tilde{u}''(0) = 0, \lim_{t\to\infty} D_{0^+}^{\alpha-1} \tilde{u}(t) = \Gamma(\alpha)\tilde{u}(0). \end{cases}$$

Definition 1. *The solution of the fractional boundary value problem (P) is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any two solutions*

u and $\tilde{u}$ of problems (P) and ($\widetilde{P}$) respectively, one has $\|\varphi - \tilde{\varphi}\|_0 \leq \delta$, then $\|u - \tilde{u}\|_X < \varepsilon$.

Theorem 2. *Under the assumptions of Theorem 4, the solution of the fractional boundary value problem (P) is stable.*

Proof. Let u and $\tilde{u}$ be solutions of problems (P) and ($\widetilde{P}$) respectively, we have

$$\|u - \tilde{u}\|_0 = \max_{t\in[-\tau,0]} |u(t) - \tilde{u}(t)| = \max_{t\in[-\tau,0]} |\varphi(t) - \tilde{\varphi}(t)| = \|\varphi - \tilde{\varphi}\|_0 . \tag{4.1}$$

On the other hand for $t \geq 0$, we have

$$\left|\frac{u(t) - \tilde{u}(t)}{1 + t^{\alpha-1}}\right| \leq \frac{t^{\alpha-1}}{1 + t^{\alpha-1}} |\varphi(0) - \tilde{\varphi}(0)|$$

$$+\frac{1}{\Gamma(\alpha)} \int_0^\infty q(s) \left| f\left(s, u(s - \theta_1(s)), u(s - \theta_2(s))\right) - f\left(s, \tilde{u}(s - \theta_1(s)), \tilde{u}(s - \theta_1(s))\right) \right| ds$$

$$\begin{aligned} \leq\ & \frac{t^{\alpha-1}}{1 + t^{\alpha-1}} |\varphi(0) - \tilde{\varphi}(0)| \\ &+\frac{1}{\Gamma(\alpha)} \int_0^\infty q(s) L_1(s) \left|\frac{u(s - \theta_1(s)) - \tilde{u}(s - \theta_1(s))}{1 + s^{\alpha-1}}\right| ds \\ &+\frac{1}{\Gamma(\alpha)} \int_0^\infty q(s) L_2(s) \left|\frac{u(s - \theta_2(s)) - \tilde{u}(s - \theta_1(s))}{1 + s^{\alpha-1}}\right| ds, \end{aligned}$$

reasoning as in the proof of Theorem 4, we get

$$\|u - \tilde{u}\|_\infty^0 \leq \left(1 + \frac{2C}{\Gamma(\alpha)}\right) \|\varphi - \tilde{\varphi}\|_0 + \frac{2C}{\Gamma(\alpha)} \|u - \tilde{u}\|_\infty^0 ,$$

hence

$$\|u - \tilde{u}\|_\infty^0 \leq \left(\frac{\Gamma(\alpha) + 2C}{\Gamma(\alpha) - 2C}\right) \|\varphi - \tilde{\varphi}\|_0 . \tag{4.2}$$

In view of (4.1) and (4.2), we obtain

$$\|u-\tilde{u}\|_X \le \left(\frac{2\Gamma(\alpha)}{\Gamma(\alpha)-2C}\right)\|\varphi-\tilde{\varphi}\|_0\,,$$

therefore, for $\epsilon>0$, we can find $\delta=\left(\frac{2\Gamma(\alpha)}{\Gamma(\alpha)-2C}\right)^{-1}\epsilon$ such that if $\|\varphi-\tilde{\varphi}\|_0<\delta$ then $\|u-\tilde{u}\|_X<\epsilon$, which proves that then unique solution is stable. □

Now we give a numerical example.

Example 3. *Consider the following fractional boundary value problem*

$$(P1)\begin{cases} D_{0^+}^{\alpha}u\left(t\right)=q(t)f\left(t,u(t-\theta_1(t)),u(t-\theta_2(t))\right),t>0,\\ u\left(t\right)=\varphi(t),t\in\left[-\tau,0\right],\\ u''(0)=0,\lim_{t\to\infty}D_{0^+}^{\alpha-1}u(t)=\Gamma(\alpha)u(0),\end{cases}$$

where $\alpha=\frac{12}{5}$, $f\left(t,x,y\right)=\frac{e^{-t}}{6}\left(x+ty-\frac{1}{1+x^2}\right)$, $\varphi\left(t\right)=t^2$, $q\left(t\right)=\frac{1}{1+t^{\alpha-1}}$, $\theta_1(t)=\frac{t}{2}+\frac{1}{2}$, $\theta_2(t)=\frac{2t}{3}+\frac{1}{3}$, *then* $\tau=\frac{1}{2}$. *Let us check the hypotheses of Theorem 4. -(H1) holds if we choose* $L_1(t)=\frac{e^{-t}}{6}\left(1+t^{\alpha-1}+\left(1+t^{\alpha-1}\right)^2\right)$, $L_2(t)=\frac{te^{-t}\left(1+t^{\alpha-1}\right)}{6}$ *and then* $C=0.540\,36$. *-There exist* $t_1=t_2=1$, *such that* $t-\theta_i(t)<0$, *for* $0\le t\le 1$, *and* $t-\theta_i(t)\ge 0$, *if* $t>1$, $i=1,2$. *Hence the Hypothesis (H2) is satisfied.*

By Theorems 4 and 5, we conclude that the problem (P1) has a unique solution that is stable in X.

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