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Edge irregularity strength of certain families of comb graph

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Abstract:

Edge irregular mapping or vertex mapping $h : V(U) \longrightarrow \{1, 2, 3, 4, ..., s\}$ is a mapping of vertices in such a way that all edges have distinct weights. We evaluate weight of any edge by using equation $wt_h(cd) = h(c)+h(d)$, $\forall c, d \in V(U)$ and $\forall cd \in E(U)$. Edge irregularity strength denoted by $es(U)$ is a minimum positive integer use to label vertices to form edge irregular labeling. In this paper, we find exact value of edge irregularity strength of different families of comb graph.

Keywords: Irregular assignment; Irregularity strength; Edge irregularity strength; Comb graphs.

MSC (2020): 05C78.

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1. Introduction:

In this paper, we consider finite, simple and undirected graphs. The procedure of assignment of numbers (either positive or negative) to the elements of a graph G is termed as labeling. Vertex set $V(G)$ and edge set $E(G)$ are the elements of a graph G. If we label vertices or edges then this labeling is categorized as vertex labeling or edge labeling respectively. If we label both vertices and edges, then this labeling is termed as total labeling.
Chartrand et al.[13] had introduced edge labeling for a graph G. We call this labeling as irregular assignments because all vertices have distinct weights. Irregularity strength $s(G)$ is a minimum positive integer which is used to form irregular labeling. Results regarding irregularity strength can be seen in [8, 12, 14, 18]
Vertex irregular mapping or edge mapping $h : E(G) \longrightarrow \{1, 2, 3, 4, ..., s\}$ is a mapping of edges in such a way that all vertices have distinct weights. We evaluate weight of any vertex by using equation $wt_h(c) = \Sigma h(cd)$, $\forall c, d \in V(G)$ and $cd \in E(G)$. Motivated by Chartrand's work, Bača at [10] introduced new labeling named as vertex irregular total labeling. Vertex irregular total labeling for a graph G is a mapping $h : E(G) \cup V(G) \longrightarrow \{1, 2, 3, 4, ..., s\}$ in such a way that the total vertex weight is different for all vertices. We can evaluate total vertex weight by using the relation $wt_h(c) = h(c) + \Sigma h(cd)$, $\forall c, d \in V(G)$ and $\forall cd \in E(G)$. Total vertex irregularity strength denoted by $tvs(G)$ is a minimum positive integer use to label vertices to form vertex irregular total labeling. For more results on labeling and trees we refer to see [1,2,4,5,9,11,15,16,17,19,20,21,22,24].
Both edge irregular and vertex irregular was a new labeling categorized as totally irregular total labeling developed by Marzuki in [10] by the motivation of previous improvements. Total irregularity strength for a graph U is denoted as $ts(U)$. Results related to irregular total labeling were developed in [10].
Because of provocation of preceding results Ahmed et. al in [2] developed a new concept of edge irregularity strength denoted by $es(U)$ which is a minimum positive integer use to label vertices to form edge irregular labeling. Inspired by this, more results were developed in [5,6,23].

Theorem 1.1. *[2] Let G be a simple graph with maximum degree $\Delta = \Delta(G)$. Then $es(G) \geq \max\{\lceil(|E(G)| + 1)/2\rceil, \Delta(G)\}$.*

2. Comb Graph:

Let us consider a path graph P_n, having $n \geq 1$ vertices and $(n-1)$ edges. The comb graph Cb_n is defined by $P_n \bigodot K_1$. It has $2n$ vertices and $(2n-1)$ edges.

3. Comb Graph Ca_n:

Comb graph Ca_n is formed by vertex set $V(Ca_n) = \{a_i^j; 1 \leq i \leq j+1, 1 \leq j \leq n\}$ and edge set $E(Ca_n) = \{a_1^j a_1^{j+1}; 1 \leq j \leq n-1\} \bigcup \{a_i^j a_{i+1}^j; 1 \leq j \leq n, 1 \leq i \leq j\}$. It has $(\frac{n^2+3n}{2})$ vertices and $(\frac{n^2+3n-2}{2})$ edges.

Theorem 3.1. *Let Ca_n be a comb graph. Then $es(Ca_n) = \lceil \frac{n(n+3)}{4} \rceil$*

Proof. Let Ca_n be a comb graph. We have to show that $es(Ca_n) = \lceil \frac{n(n+3)}{4} \rceil$. From Theorem 1.1 we get lower bound $es(Ca_n) \geq \lceil \frac{n(n+3)}{4} \rceil$. For converse, we have to prove that $es(Ca_n) \leq \lceil \frac{n(n+3)}{4} \rceil$. For this define a labeling on vertex set such that

If j is odd,

$$f(a_i^j) = \begin{cases} \lceil \frac{i}{2} \rceil, & \text{if } j = 1, 1 \leq i \leq j+1 \\ \lceil \frac{j^2+3j-2i+2}{4} \rceil, & \text{if } j \equiv 3(mod4), 1 \leq i \leq j \\ \lceil \frac{j^2+3j-2i+4}{4} \rceil - 2, & \text{if } j \equiv 3(mod4), i = j+1 \\ \lceil \frac{j^2+3j-2i+2}{4} \rceil, & \text{if } j \equiv 1(mod4), 1 \leq i \leq j+1 \end{cases}$$

If j is even,

$$f(a_i^j) = \begin{cases} \lceil \frac{i+2}{2} \rceil, & \text{if } j = 2, 1 \leq i \leq j+1 \\ \lceil \frac{j^2+j+2i-2}{4} \rceil, & \text{if } j > 2, 1 \leq i \leq j+1 \end{cases}$$

Now we evaluate weights for all edges as

$$w_t(a_1^j a_1^{j+1}) = \begin{cases} \frac{j^2+3j+2}{2}, & \text{if j is odd} \\ \frac{j^2+3j+4}{2}, & \text{if } j \equiv 2(mod4) \\ \frac{j^2+3j+2}{2}, & \text{if } j \equiv 4(mod4) \end{cases}$$

If j is odd,

$$w_t(a_i^j a_{i+1}^j) = \begin{cases} 2, & \text{if } i=1, j=1 \\ \frac{j^2+3j-2i+2}{2}, & \text{if } j \equiv 3(mod4), 1 \leq i \leq j-1 \\ \frac{j^2+3j-2i}{2}, & \text{if } j \equiv 3(mod4), i=j \\ \frac{j^2+3j-2i+2}{2}, & \text{if } j \equiv 1(mod4), 1 \leq i \leq j \end{cases}$$

If j is even,

$$w_t(a_i^j a_{i+1}^j) = \frac{j^2+j+2i}{2}, \text{if } 1 \leq i \leq j$$

On the basis of above calculations we see that all edges have distinct weights.
Hence, $es(Ca_n) = \lceil \frac{n(n+3)}{4} \rceil$ □

4. Comb Graph Cd_n:

Comb graph Cd_n, is formed by vertex set $V(Cd_n) = \{a_i^j; 1 \leq j \leq n, 1 \leq i \leq \lfloor \frac{j+3}{2} \rfloor\}$ and edge set $E(Cd_n) = \{a_i^j a_{i+1}^j; 1 \leq j \leq n\ , 1 \leq i \leq \lfloor \frac{j+1}{2} \rfloor\} \bigcup \{a_1^j a_1^{j+1}; 1 \leq j \leq n-1\}$. It has $(\frac{n^2+6n}{4})$ vertices and $(\frac{n^2+6n-4}{4})$ edges.

Theorem 4.1. *Let Cd_n be a comb graph. Then $es(Cd_n) = \frac{n^2+6n}{8}$.*

Proof. Let Cd_n be a comb graph. We have to show that $es(Cd_n) = \frac{n^2+6n}{8}$. From Theorem 1.1 we get lower bound $es(Cd_n) \geq \frac{n^2+6n}{8}$. For converse, we have to prove that $es(Ca_n) \leq \frac{n^2+6n}{8}$.. For this define a labeling on vertex set such that

$$f(a_i^j) = \begin{cases} \lceil \frac{i+2j-2}{2} \rceil, & j=1,2, 1 \leq i \leq \lfloor \frac{j+3}{2} \rfloor \\ \lceil \frac{j^2+4j+4i-5}{8} \rceil, & j>1, odd, 1 \leq i \leq \lfloor \frac{j+3}{2} \rfloor \\ \lceil \frac{j^2+6j-4i+4}{8} \rceil, & j>2, even, 1 \leq i \leq \lfloor \frac{j+3}{2} \rfloor \end{cases}$$

Now we evaluate weights for all edges as follows:

$$w_t(a_1^j a_1^{j+1}) = \lceil \frac{j^2+6j+4}{4} \rceil,\ 1 \leq j \leq n-1$$

$$w_t(a_i^j a_{i+1}^j) = \begin{cases} 2, & i = 1, j = 1 \\ \frac{j^2+4j+4i-1}{4}, & j > 1, odd; 1 \leq i \leq \lfloor \frac{j+1}{2} \rfloor \\ \frac{j^2+6j-4i+4}{4}, & j \geq 2, even; 1 \leq i \leq \lfloor \frac{j+1}{2} \rfloor \end{cases}$$

On the basis of above calculations we see that all edges have distinct weights.
Hence, $es(Cd_n) = \frac{n^2+6n}{8}$ □

5. Comb Graph Ce_n:

Comb graph Ce_n is formed by vertex set $V(Ce_n) = \{a_i^j; 1 \leq j \leq n, 1 \leq i \leq 2j+1\}$ and edge set $E(Ce_n) = \{a_{i+1}^j a_{i+2}^{j+1}; 1 \leq j \leq n-1, i = j\} \bigcup \{a_i^j a_{i+1}^j; 1 \leq j \leq n, 1 \leq i \leq 2j\}$. It has n^2+2n vertices and n^2+2n-1 edges.

Theorem 5.1. *Let Ce_n be a comb graph. Then $es(Ce_n) = \lfloor \frac{(n+1)^2}{2} \rfloor$*

Proof. Let Ce_n be a comb graph. We have to show that $es(Ce_n) = \frac{(n+1)^2}{2}$. From Theorem 1.1 we get lower bound $es(Ce_n) \geq \lfloor \frac{(n+1)^2}{2} \rfloor$. For converse, we have to prove that $es(Ce_n) \leq \lfloor \frac{(n+1)^2}{2} \rfloor$. For this define a labeling on vertex set such that

$$f(a_i^j) = \lceil \frac{j^2+i-1}{2} \rceil, \text{if } 1 \leq i \leq 2j+1, 1 \leq j \leq n$$

Now we evaluate weights for all edges as follows:

$$w_t(a_{i+1}^j a_{i+2}^{j+1}) = (i+1)^2, \text{if } i = j, 1 \leq j \leq n-1$$

$$w_t(a_i^j a_{i+1}^j) = i + j^2, \text{if } 1 \leq i \leq 2j, 1 \leq j \leq n.$$

On the basis of above calculations we see that all edges have distinct weights.
Hence,$es(Ce_n) = \lfloor \frac{(n+1)^2}{2} \rfloor$ □

6. Comb Graph Cf_n:

Let Cf_n be a comb graph, which is formed by vertex set $V(Cf_n) = \{a_i^j; 1 \leq i \leq l, 1 \leq j \leq n\}$ and edge set $E(Cf_n) = \{a^j_{\frac{l+1}{2}} a^{j+1}_{\frac{l+1}{2}}; 1 \leq j \leq n-1\} \bigcup \{a_i^j a_{i+1}^j; 1 \leq j \leq n, 1 \leq i \leq l-1\}$. It has (nl) vertices and $(nl-1)$ edges and l is the length of vertices in each j

Theorem 6.1. *Let Cf_n be a comb graph. Then $es(Cf_n) = \lceil \frac{nl}{2} \rceil$*

Proof. Let Cf_n be a comb graph. We have to show that $es(Cf_n) = \lceil \frac{nl}{2} \rceil$. From Theorem 1.1 we get lower bound $es(Cf_n) \geq \lceil \frac{nl}{2} \rceil$. For converse, we have to prove that $es(Cf_n) \leq \lceil \frac{nl}{2} \rceil$. For this define a labeling on vertex set such that

$$f(a_i^j) = \lceil \frac{l(j-1)+i}{2} \rceil \; 1 \leq i \leq l, 1 \leq j \leq n$$

Now we evaluate weights for all edges as follows:

$$w_t(a^j_{\frac{l+1}{2}} a^{j+1}_{\frac{l+1}{2}}) = lj + 1, \; 1 \leq j \leq n-1$$

$$w_t(a_i^j a_{i+1}^j) = i + l(j-1) + 1, \; 1 \leq i \leq l-1, 1 \leq j \leq n$$

On the basis of above calculations we see that all edges have distinct weights.
Hence, $es(Cf_n) = \lceil \frac{nl}{2} \rceil$ □

7. Comb Graph Cg_n:

Comb graph Cg_n is formed by vertex set $V(Cg_n) = \{a_i^j; 1 \leq i \leq j+1, 1 \leq j \leq \lceil \frac{n}{2} \rceil\} \bigcup \{a_i^j; \lceil \frac{n}{2} \rceil < j \leq n, i = n-j+2\}$ and edge set $E(Cg_n) = \{a_1^j a_1^{j+1}; 1 \leq j \leq n-1\} \bigcup \{a_i^j a_{i+1}^j; 1 \leq j \leq \lceil \frac{n}{2} \rceil, 1 \leq i \leq j\} \bigcup \{a_i^j a_{i+1}^j; \lceil \frac{n}{2} \rceil < j \leq n, i = n-j+1\}$. It has $(\frac{n^2+6n+1}{4})$ vertices and $(\frac{n^2+6n-3}{4})$ edges.

Theorem 7.1. *Let Cg_n be a comb graph, then $es(Cg_n) = \lfloor \frac{n^2+6n+5}{8} \rfloor$*

Proof. Let Cg_n be a comb graph. We have to show that $es(Cg_n) = \lfloor \frac{n^2+6n+5}{8} \rfloor$. From Theorem 1.1 we get lower bound $es(Cg_n) \geq \lfloor \frac{n^2+6n+5}{8} \rfloor$. For converse, we have to prove that $es(Cg_n) \leq \lfloor \frac{n^2+6n+5}{8} \rfloor$. For this define a labeling on vertex set such that

$$f(a_i^j) = \begin{cases} \lceil \frac{i}{2} \rceil, & \text{if } i = 1,2; j = 1 \\ \lfloor \frac{j^2+2j+2i-1}{4} \rfloor, & \text{if } 1 \leq i \leq j+1, odd; 2 \leq j \leq \lceil \frac{n}{2} \rceil \\ \lfloor \frac{j^2+2i}{4} \rfloor, & \text{if } 2 \leq i \leq j+1, even; 2 \leq j \leq \lceil \frac{n}{2} \rceil \end{cases}$$

If $n \equiv 5(mod4)$,

$$f(a_i^j) = \begin{cases} \lfloor \frac{(\lceil \frac{n+2}{2} \rceil + 1)^2}{4} \rfloor + \lfloor \frac{i}{2} \rfloor, & \text{if } j = \lceil \frac{n+2}{2} \rceil; 1 \leq i \leq j-1, \\ odd & \\ \lfloor \frac{(\lceil \frac{n+2}{2} \rceil + 1)^2}{4} \rfloor + \sum_{j > \lceil \frac{n+2}{2} \rceil}^{n} \lceil \frac{n-j+2}{2} \rceil + \lfloor \frac{i}{2} \rfloor, & \text{if } \lceil \frac{n+2}{2} \rceil < j \leq n; \\ 1 \leq i \leq n-j+2, odd & \\ \lceil \frac{(\lceil \frac{n+2}{2} \rceil^2 + 3)}{4} \rceil + \frac{i-2}{2}, & \text{if } j = \lceil \frac{n+2}{2} \rceil; 2 \leq i \leq j-1, \\ even & \\ \lceil \frac{(\lceil \frac{n+2}{2} \rceil^2 + 3)}{4} \rceil + \sum_{j > \lceil \frac{n+2}{2} \rceil}^{n} \lceil \frac{n-j+3}{2} \rceil + \frac{i-2}{2}, & \text{if } \lceil \frac{n+2}{2} \rceil < j \leq n; \\ 2 \leq i \leq n-j+2, even & \end{cases}$$

If $n \equiv 3(mod4)$,

$$f(a_i^j) = \begin{cases} \lfloor \frac{(\lceil \frac{n+2}{2} \rceil + 1)^2}{4} \rfloor + \lfloor \frac{i}{2} \rfloor, & \text{if } j = \lceil \frac{n+2}{2} \rceil; 1 \leq i \leq j-1, \\ odd & \\ \lfloor \frac{(\lceil \frac{n+2}{2} \rceil + 1)^2}{4} \rfloor + \sum_{j > \lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+4}{2}) + \lfloor \frac{i}{2} \rfloor, & \text{if } \lceil \frac{n+2}{2} \rceil < j \leq n, odd; \\ 1 \leq i \leq n-j+2, odd & \\ \lceil \frac{(\lceil \frac{n+2}{2} \rceil^2 + 3)}{4} \rceil + \frac{i-2}{2}, & \text{if } j = \lceil \frac{n+2}{2} \rceil; 2 \leq i \leq j-1, \\ even & \\ \lceil \frac{(\lceil \frac{n+2}{2} \rceil^2 + 3)}{4} \rceil + \sum_{j > \lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+2}{2}) + \frac{i-2}{2}, & \text{if } \lceil \frac{n+2}{2} \rceil < j \leq n, odd; \\ 2 \leq i \leq n-j+2, even & \\ \lfloor \frac{(\lceil \frac{n+2}{2} \rceil + 1)^2}{4} \rfloor + \sum_{j > \lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+1}{2}) + \lfloor \frac{i}{2} \rfloor, & \text{if } \lceil \frac{n+2}{2} \rceil < j \leq n, even; \\ 1 \leq i \leq n-j+2, odd & \\ \lceil \frac{(\lceil \frac{n+2}{2} \rceil^2 + 3)}{4} \rceil + \sum_{j > \lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+5}{2}) + \frac{i-2}{2}, & \text{if } \lceil \frac{n+2}{2} \rceil < j \leq n, even; \\ 2 \leq i \leq n-j+2, even & \end{cases}$$

Now we evaluate weights for all edges as follows:

$$w_t(a_i^j a_{i+1}^j) = \frac{j^2+j+2i}{2}, \text{if } 1 \le j \le \lceil \frac{n}{2} \rceil, 1 \le i \le j$$

$$w_t(a_1^j a_1^{j+1}) = \frac{j^2+3j+2}{2}, \text{if } 1 \le j \le \lceil \frac{n}{2} \rceil$$

If $n \equiv 5(mod4)$,

$$w_t(a_i^j a_{i+1}^j) = \begin{cases} \alpha+\beta, & \text{if } j = \lceil \frac{n+2}{2} \rceil; 1 \le i \le n-j+1 \\ \gamma+\delta, & \text{if } \lceil \frac{n+2}{2} \rceil < j \le n; 1 \le i \le n-j+1 \end{cases}$$

where

$\alpha = \lfloor \frac{(\lceil \frac{n+2}{2} \rceil+1)^2}{4} \rfloor + \lfloor \frac{i}{2} \rfloor$
$\beta = \lceil \frac{((\lceil \frac{n+2}{2} \rceil)^2+3)}{4} \rceil + \lfloor \frac{i-1}{2} \rfloor$
$\gamma = \lfloor \frac{(\lceil \frac{n+2}{2} \rceil+1)^2}{4} \rfloor + \sum_{j>\lceil \frac{n+2}{2} \rceil}^{n} \lceil \frac{n-j+2}{2} \rceil + \lfloor \frac{i}{2} \rfloor$
$\delta = \lceil \frac{((\lceil \frac{n+2}{2} \rceil)^2+3)}{4} \rceil + \sum_{j>\lceil \frac{n+2}{2} \rceil}^{n} \lceil \frac{n-j+3}{2} \rceil + \lfloor \frac{i-1}{2} \rfloor$.

If $n \equiv 3(mod4)$,

$$w_t(a_i^j a_{i+1}^j) = \begin{cases} \alpha+\beta, & \text{if } j = \lceil \frac{n+2}{2} \rceil; 1 \le i \le n-j+1 \\ \gamma+\delta, & \text{if } \lceil \frac{n+2}{2} \rceil < j \le n, odd; 1 \le i \le n-j+1 \\ \lambda+\mu, & \text{if } \lceil \frac{n+2}{2} \rceil < j \le n, even; 1 \le i \le n-j+1 \end{cases}$$

where
$\alpha = \lfloor \frac{(\lceil \frac{n+2}{2} \rceil+1)^2}{4} \rfloor + \lfloor \frac{i}{2} \rfloor$
$\beta = \lceil \frac{((\lceil \frac{n+2}{2} \rceil)^2+3)}{4} \rceil + \lfloor \frac{i-1}{2} \rfloor$
$\gamma = \lfloor \frac{(\lceil \frac{n+2}{2} \rceil+1)^2}{4} \rfloor + \sum_{j>\lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+4}{2}) + \lfloor \frac{i}{2} \rfloor$
$\delta = \lceil \frac{((\lceil \frac{n+2}{2} \rceil)^2+3)}{4} \rceil + \sum_{j>\lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+2}{2}) + \lfloor \frac{i-1}{2} \rfloor$
$\lambda = \lfloor \frac{(\lceil \frac{n+2}{2} \rceil+1)^2}{4} \rfloor + \sum_{j>\lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+1}{2}) + \lfloor \frac{i}{2} \rfloor$
$\mu = \lceil \frac{((\lceil \frac{n+2}{2} \rceil)^2+3)}{4} \rceil + \sum_{j>\lceil \frac{n+2}{2} \rceil}^{n} (\frac{n-j+5}{2}) + \lfloor \frac{i-1}{2} \rfloor$

On the basis of above calculations we see that all edges have distinct weights.
Hence we can say that $es(Cg_n) = \lfloor \frac{n^2+6n+5}{8} \rfloor$ □

Conclusion:

In this paper, we obtained exact values of edge irregularity strength of different families of comb graph.

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