Rainbow and strong rainbow connection number for some families of graphs

Yaqoub Ahmed Khan¹ orcid.org/0000-0001-9033-0916
Muhammad Naeem² orcid.org/0000-0002-0919-8207
Muhammad Kamran Siddiqui³ orcid.org/0000-0002-2607-4847
Mohammad Reza Farahani⁴ orcid.org/0000-0003-2185-5967

¹Government College University, Dept. of Mathematics, Lahore, Pakistan.

yaqoubahmedkhan@gmail.com

²Institute of Southern Punjab, Dept. of Mathematics and Statistics, Multan, Pakistan.

naeempkn@gmail.com

³COMSATS University Islamabad, Dept. of Mathematics, Lahore, Pakistan.

kamransiddiqui75@gmail.com

⁴Iran University of Science and Technology, Dept. of Applied Mathematics, Tehran, Iran.

mr_farahani@mathdep.iust.ac.ir

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Abstract:

Let G be a nontrivial connected graph. Then G is called a rainbow connect-graph if there exists a coloring $c: E(G) \rightarrow \{1, 2, \dots, k\}$, $k \in \mathbb{N}$, of the edges of G , such that there is a $u - v$ rainbow path between every two vertices of G , where a path P in G is a rainbow path if no two edges of P are colored the same. The minimum k for which there exists such a k -edge coloring is the rainbow connection number $rc(G)$ of G . If for every pair u, v of distinct vertices, G contains a rainbow $u - v$ geodesic, then G is called strong rainbow connected. The minimum k for which G is strong rainbow-connected is called the strong rainbow connection number $src(G)$ of G . The exact rc and src of the rotationally symmetric graphs are determined.

Keywords: Edge coloring; Rainbow coloring; Strong rainbow coloring.

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1. Introduction

G. Chartrand in [5] introduce the notion of a rainbow coloring of a graph. He gave some basic and characteristic theorems about the rc and src numbers of graphs. Since then the rainbow connection and strong rainbow connection numbers is studied by many mathematicians.

Let G be a nontrivial connected graph on which an *edge-coloring* $c : E(G) \rightarrow \{1, 2, \dots, n\}$, $n \in \mathbf{N}$, is defined, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of this are colored the same. An edge color graph G is called rainbow connected if there is a rainbow path between every two vertices of G . An edge-coloring under which G is rainbow connected is called a rainbow coloring. The rainbow connection number of a connected graph G , denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. A rainbow coloring using $rc(G)$ colors is called a minimum rainbow coloring.

For any two vertices u and v of G , a rainbow $u - v$ geodesic in G is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v in G . A graph G is strong rainbow connected if there exists a rainbow $u - v$ geodesic for any two vertices u and v in G . In this case, the coloring c is called a strong rainbow coloring of G . The strong rainbow connection number of a connected graph G , denoted by $src(G)$, is the smallest number of colors that are needed to make G a strong rainbow connected graph. A strong rainbow coloring of G using $src(G)$ colors is called a minimum strong rainbow coloring of G . Note that this number is also called the rainbow diameter number in [4].

If we assign distinct colors to each the edge of a connected graph then both coloring exists namely a rainbow coloring and a strong rainbow coloring, therefore every connected graph is rainbow-connected and strongly rainbow-connected with respect to some coloring of the edges of G . So, naturally the rainbow connection numbers $rc(G)$ and $src(G)$ are defined for every connected graph G and moreover $rc(G) \leq src(G)$. Also if G is a nontrivial connected graph of size m whose diameter (the largest distance between two vertices of G) is $diam(G)$, then

$$(1.1) \quad diam(G) \leq rc(G) \leq src(G) \leq m.$$

In this paper we have determined the exact value for rc and src number of the graphs of möbius ladder M_{2n} , antiprism A_n , convex polytope R_n and convex polytope Q_n .

For further reading in this regards the readers are referred to [15] and [13]. Moreover, Dewi Estetikasari and Syafrizal Sy in [6] determined the exact values of rainbow connection for some corona graph. For more details on different kind of labeling we refer to the articles [1,5,7,8,9,10,11,12,14,16,17,18].

2. Möbius ladders

The Möbius ladders M_{2n} is a cubic circulant graph with an even number of vertices, formed from an n -cycle by adding vedges (called rungs) connecting opposite pairs of vertices in the cycle, we denote the outer vertices as u_i , $1 \leq i \leq n$ and inner vertices as v_i , $1 \leq i \leq n$. It is so-named because (with the exception of $M_6 = K_{3,3}$) M_{2n} has exactly n number of 4-cycles which are linked together by their shared edges to form a topological Möbius strip. Möbius ladder can also be viewed as a prism with one twisted edge. Two different views of Möbius ladders M_{16} have been shown in Fig 2.1 Möbius ladders have many applications in chemistry, chemical stereography, electronics and computer science.

2.1. Two views of Möbius ladders M_{2n}

Möbius ladder has $2n$ vertices and $3n$ edges for all $n \geq 2$ with diameter of M_{2n} is equal to $\lceil n/2 \rceil$ then its easy to see from (1.1) that

$$(2.1) \quad src(M_{2n}) \geq rc(M_{2n}) \geq \lceil n/2 \rceil$$

Theorem 1. The rainbow connection number and the strong rainbow connection number of Möbius ladder M_{2n} is $\lceil n/2 \rceil$ for $n \geq 2$.

Proof. It is easy to see that M_{2n} contain a $2n$ -cycle, $C_{2n} : u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, u_1$. We denote the edges of M_{2n} as: $e_{1i} = u_i u_{i+1}$, $e_{2i} = v_i v_{i+1}$ for $1 \leq i \leq n-1$, $e_{3i} = u_i v_i$ for $1 \leq i \leq n$, $e_{4n} = u_n v_1$ and $e_{5n} = v_n u_1$.

Let us take $\lceil n/2 \rceil = k$ for our convenience. To show that $src(M_{2n}) \leq k$ we define the edge coloring $c : E(M_{2n}) \rightarrow \{1, 2, 3, \dots, k\}$ as follows:

$$c(e_{ji}) = \begin{cases} i, & \text{for } j = 1, 2 \text{ and } 1 \leq i \leq k; \\ i - k, & \text{for } j = 1, 2 \text{ and } k < i \leq n - 1; \\ i, & \text{for } j = 3 \text{ and } 1 \leq i \leq k; \\ i - k, & \text{for } j = 1, 2 \text{ and } k < i \leq n; \\ \lceil n/2 \rceil, & \text{for } j = 3, 4 \text{ and } i = n. \end{cases}$$

From c we get $src(M_{2n}) \leq k = \lceil n/2 \rceil$. Thus from (1.1) and (2.1) we obtain $src(M_{2n}) = rc(M_{2n}) = \lceil n/2 \rceil$. To understand the above coloring see the Fig 2.2 for M_{16} . $\square$

2.2. Rainbow and strong rainbow coloring of M_{16}

3. The graph of an antiprism

The antiprism A_n , $n \geq 3$, is a 4-regular graph and, for $n = 3$, it is the octahedron. This graph has $2n$ vertices and $4n$ edges for $n \geq 3$, A_n consists

of an outer n -cycle $v_1v_2 \dots v_n$, an inner n -cycle $u_1u_2 \dots u_n$, and a set of n spokes u_iv_i and $u_{i+1}v_i$ for $i = 1, 2, \dots, n$ with indices taken modulo n see (Fig. 3.1).

3.1. The graph of antiprism A_n

The diameter of A_n for $n \geq 3$ is $\lceil n/2 \rceil$. So by (1.1) we obtain

$$(3.1) \quad src(A_n) \geq rc(A_n) \geq \lceil n/2 \rceil$$

Theorem 2. The rainbow connection number and the strong rainbow connection number of Antiprism A_n , for $n \geq 3$ is $\lceil n/2 \rceil$.

Proof. We denote the edges of A_n as: $e_{1i} = v_iv_{i+1}$, $e_{2i} = u_iu_{i+1}$ for $1 \leq i \leq n$, $e_{3i} = u_iv_i$ for $1 \leq i \leq n$, $e_{4n} = u_{i+1}v_i$ for $1 \leq i \leq n$. Let us take $\lceil n/2 \rceil = k$ for our convenience. From (3.1) we already know that $src(A_n) \geq k$. To show that $src(A_n) \leq k$ we define the edge coloring $c : E(A_n) \rightarrow \{1, 2, 3, \dots, k\}$ as follows:

$$c(e_{ji}) = \begin{cases} i, & \text{for } 1 \leq j \leq 4 \text{ and } 1 \leq i \leq k; \\ i - k, & \text{for } 1 \leq j \leq 4 \text{ and } k < i \leq n. \end{cases}$$

This coloring c on $E(A_n)$ give us a rainbow $u - v$ geodesic for any two vertices u and v in A_n . From c we get $src(A_n) \leq k = \lceil n/2 \rceil$. Thus from (1.1) and (3.1) we obtain $src(A_n) = rc(A_n) = \lceil n/2 \rceil$. To understand the

above coloring see the Fig 3.2 for A_8 . $\square$

4. The graph of convex polytope R_n

For $n \geq 3$, by R_n , we denote the graph of convex polytope defined in [2,3,10,11,12,14] which is obtained as a combination of the graph of a prism and the graph of an antiprism (Fig. 4.1). The graph R_n has $3n$ vertices and $6n$ edges for $n \geq 3$. The $\text{diam}(R_n) = \lceil n/2 \rceil + 1$. By using (1.1)

$$(4.1) \quad \text{src}(R_n) \geq \text{rc}(R_n) \geq \lceil n/2 \rceil + 1$$

4.1. The graph of convex polytop R_n

Theorem 3. For $n \geq 3$, the rainbow connection number and strong connection number of convex polytop R_n is $\lceil n/2 \rceil + 1$.

Proof. The graph R_n contains three n -cycles, $A_n : a_1, a_2, a_3, \dots, a_n, a_1$, $B_n : b_1, b_2, b_3, \dots, b_n, b_1$ and $C_n : c_1, c_2, c_3, \dots, c_n, c_1$, with edges denoted as $e_{1i} = a_i a_{i+1}$, $e_{2i} = b_i b_{i+1}$, $e_{3i} = c_i c_{i+1}$ respectively for $1 \leq i \leq n$. The cycle A_n is connected to B_n by n spoke edges denoted as $e_{4i} = a_i b_i$ and $e_{5i} = a_{i+1} b_i$ for $1 \leq i \leq n$. The cycle B_n is connected to C_n by the edges denoted as $e_{6i} = b_i c_i$ for $1 \leq i \leq n$.

Let $\lceil n/2 \rceil + 1 = k + 1$, then from (4.1) we have $src(R_n) \geq k + 1$. To show that $src(R_n) \leq k$ we define the edge coloring $c : E(R_n) \rightarrow \{1, 2, 3, \dots, k + 1\}$ as follows:

$$c(e_{ji}) \begin{cases} i, & \text{for } 1 \leq j \leq 5 \text{ and } 1 \leq i \leq k; \\ i - k, & \text{for } 1 \leq j \leq 5 \text{ and } k < i \leq n; \\ k + 1, & \text{for } j = 6 \text{ and } 1 \leq i \leq n. \end{cases}$$

The coloring function c gave us a coloring for R_n in which there exists a rainbow path as well as a rainbow geodesic path for each pair of vertices of R_n . This shows that $src(R_n) \leq k + 1$. Thus the result follows from (1.1) and the formulas $c(e_{ji})$ in above equation. $\square$

5. The graph of convex polytope Q_n

In this section, we shall investigate the $rc(Q_n)$ and $src(Q_n)$ of the graph of convex polytope Q_n for $n \geq 3$ defined in consisting of 3 - sided faces, 4 - sided faces, 5 - sided faces and n -sided faces (Fig. 5.1). The graph of Q_n have $4n$ vertices and $7n$ edges, for $n \geq 3$ with $\text{diam}(Q_n) = \lceil n/2 \rceil + 2$. So from (1.1) it follows

5.1 The graph of convex polytopon Q_n

$$(5.1) \quad src(Q_n) \geq rc(Q_n) \geq \lceil n/2 \rceil + 2$$

Theorem 4. For $n \geq 3$, the rainbow connection number of convex polytopon Q_n is $\lceil n/2 \rceil + 2$.

Proof. The graph of Q_n contains three n -cycles $A_n : a_1, a_2, a_3, \dots, a_n, a_{n+1} = a_1$, $B_n = b_1, b_2, b_3, \dots, b_n, b_{n+1} = b_1$ and $D_n = d_1, d_2, d_3, \dots, d_n, d_{n+1} = d_1$ having the edges denoted as $e_{1i} = a_i a_{i+1}$, $e_{2i} = b_i b_{i+1}$, and $e_{3i} = d_i d_{i+1}$ respectively for $1 \leq i \leq n$. The cycle A_n is connected to B_n by the edges denoted by $e_{4i} = a_i b_i$ for $1 \leq i \leq n$. We denote as $e_{5i} = b_i c_i$ and $e_{6i} = b_{i+1} c_i$, and the edges defined $c_i d_i$ are denoted as e_{7i} for $1 \leq i \leq n$.

For $n = 3$, consider 4-coloring $c : E(Q_n) \rightarrow \{1, 2, 3, 4\}$, which gives us a rainbow coloring of Q_3 , defined as $c(e_{ji}) = i$ for $j = 1, 2, 3, 4, 5, 7$, $1 \leq i \leq 3$ and $c(e_{6i}) = 4$ $1 \leq i \leq 3$. Since the $\text{diam}(Q_3) = 4$, so it results that $rc(Q_3) = 4$.

Now for $n \geq 4$, first we let $\lceil n/2 \rceil + 2 = k + 2$ then we define $(k + 2)$ -

rainbow coloring $c : E(Q_n) \rightarrow \{1, 2, 3, \dots, k+2\}$ as follows:

$$c(e_{ji}) = \begin{cases} i, & \text{for } 1 \leq j \leq 6, j \neq 4 \text{ and } 1 \leq i \leq k; \\ i - k, & \text{for } 1 \leq j \leq 6, j \neq 4 \text{ and } k < i \leq n; \\ k + 1, & \text{for } j = 4 \text{ and } 1 \leq i \leq k; \\ k + 1, & \text{for } j = 7 \text{ and } k < i \leq n; \\ k + 2, & \text{for } j = 7 \text{ and } 1 \leq i \leq k; \\ k + 2, & \text{for } j = 4 \text{ and } k < i \leq n. \end{cases}$$

Under this coloring there is a rainbow path between each pair of vertices of Q_n , for $n \geq 4$. So

$rc(Q_n) \leq \lceil n/2 \rceil + 2$ Finally by relating with equation (5.1), $rc(Q_n) = \lceil n/2 \rceil + 2$

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