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Convergence analysis for combination of equilibrium problems and k-nonspreading set-valued mappings

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Abstract:

We find a common solution of generalized equilibrium problems and the set of fixed points of a k-nonspreading setvalued mapping by using shrinking projection hybrid method. Finally, we compare the shrinking solution set after randomization by giving numerical example which justifies our main result.

Keywords: Shrinking projection hybrid method; Fixed point problem; k-nonspreading set-valued mappings.

MSC (2010): 90C33, 47H09, 49J30.

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1. Introduction

Throughout the paper, let E be a closed, convex subset of real Hilbert space H. $\langle .,. \rangle$ and $\|.\|$ are inner product and induced norm, respectively.

For $M : E \times E \to \mathbf{R}$, the *equilibrium problem* (for short, EP) is of finding $u \in E$ such that

$$M(u,v) \geq 0, \ \forall v \in E. \tag{1.1}$$

If $EP(M)$ is set of all solutions of above (1.1), then

$$EP(M) = \{u \in E : M(u,v) \geq 0, \ \forall v \in E\}. \tag{1.2}$$

Blum and Oettli [2] introduced Equilibrium problems which has helped to develop and solve many problems of game theory, economics and optimization problems. Recently many researchers studied and used its applications in many important areas, see for example [1, 2, 3, 4, 7, 8, 9, 10, 12, 15, 17, 18, 19, 20]. To solve various class of equilibrium problems different useful iterative results have been introduced; see for examples [1, 2, 3, 4, 7, 9, 10, 12, 15, 17, 18, 19, 20] and references therein.

Now we consider following generalized equilibrium problems: For each $i = 1, 2, ..., N$, let $M_i : E \times E \to \mathbf{R}$ be a bifunction and $a_i \in (0,1)$ with $\sum\limits_{i=1}^{N} a_i = 1$. Consider a mapping $\sum\limits_{i=1}^{N} a_i M_i : E \times E \to \mathbf{R}$. The *generalized equilibrium problems* (for short, GEP) is of finding $u \in E$ such that

$$\Big(\sum_{i=1}^{N} a_i M_i\Big)(u,v) \geq 0, \quad \forall v \in E. \tag{1.3}$$

It has been introduced by Suwannaut and Kangtunyakarn [17] and later on GEP(1.3) further generalized Bnouhachem [1] and Kazmi et al. [6].

Generalized equilibrium problem GEP(1.3) solutions set is denoted by $EP\Big(\sum\limits_{i=1}^{N} a_i M_i\Big)$, i.e.,

$$EP\Big(\sum_{i=1}^{N} a_i M_i\Big) = \Big\{u \in E : \Big(\sum_{i=1}^{N} a_i M_i\Big)(u,v) \geq 0, \ \forall v \in E\Big\}. \tag{1.4}$$

For each $i = 1, 2, ..., N$ we have $M_i = M$, above problem reduces to $EP(1.1)$.

Let the Hausdorff metric defined on the family of compact subsets $CO(E)$ is given by

$$H(U,V) = \max\Big\{\sup_{u\in U} d(u,V), \sup_{v\in V} d(v,U)\Big\}, \quad \text{for all } U,V \in CO(E),$$

where $d(u,V) = \inf_{b\in V} \|u-b\|$. An element $p \in E$ is called a *fixed point* of $T : E \to E$ (resp. $T : E \to CO(E)$) if $p = Tp$ (resp. $p \in Tp$). $F(T)$ is the set of all fixed points of T.

Mann [13], in 1953, introduced and studied the iterative sequence $\{u_n\}$ which is given by

$$u_{n+1} = \alpha_n u_n + (1-\alpha_n)Tu_n, \tag{1.5}$$

where $u_0 \in E$ is arbitrarily chosen and real sequence $\{\alpha_n\} \in [0,1]$. Mann's iterative method (1.5) is most extensively explored and successful method which is capable in constructing and handling nonexpansive mapping's fixed points. Recently, many authors extensively investigated and studied nonexpansive mappings by using various modified Mann's iterative methods. In 2003, Nakajo and Takahashi [14] studied a modified Mann's iterative method where the sequence $\{u_n\}$ is generated by

$$\begin{cases} u_1 \in E, \\ v_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ E_n = \{z \in E : \ \|v_n - z\| \le \|u_n - z\|\}, \\ K_n = \{z \in E : \ \langle u_1 - u_n, z - u_n\rangle \le 0\}, \\ u_{n+1} = P_{E_n \cap K_n} u_1, \ n \ge 1, \end{cases} \tag{1.6}$$

$\{\alpha_n\} \subset (0,1)$. Nakajo and Takahashi shown that above sequence converges strongly to $P_{F(T)}x_1$, where $P_{F(T)}$ is projection of metric on $F(T)$.

For finding a common solutions of EP(1.1)and the set of fixed points problems, in 2007, Tada and Takahashi [19], proposed the following scheme in H : Given $x_1 = x \in H$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by the scheme

$$\begin{cases} u_n \in E \text{ such that } M(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in E, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tu_n, \\ E_n = \{z \in H : \ \|y_n - z\| \le \|x_n - z\|\}, \\ K_n = \{z \in H : \ \langle x_1 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{E_n \cap K_n} x_1, \ n \ge 1, \end{cases} \tag{1.7}$$

$\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_\Omega x_1$, where $\Omega := EP(G) \cap F(T)$.

Kohsaka and Takahashi [11], in 2008, introduced a mapping $T : E \to E$ known as *nonspreading* if

$$2\|Tu - Tv\|^2 \leq \|Tu - v\|^2 + \|Tv - u\|^2, \quad \text{for all } u, v \in E.$$

Inspired by Iemoto et al. [5] and Liu [12], recently, Suantai et al. [15] proposed generalized k-nonspreading set-valued mappings by using Hausdorff metric. $T : E \to CO(E)$ is a *k-nonspreading set-valued mapping* if for $k > 0$

$$H(Tu, Tv)^2 \leq k\Big(d(Tu, v)^2 + d(u, Tv)^2\Big), \quad \text{for all } u, v \in E. \tag{1.8}$$

We can easily observe that for all $u \in E, k \in (0,1)$ and $p \in F(T)$, (1.8) implies

$$H(Tu, Tp) \leq \sqrt{\frac{k}{1-k}}\|u - p\|. \tag{1.9}$$

In particular, if T is a $\frac{1}{2}$-nonspreading and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. To find a common solution of the split equilibrium problem and the fixed point problem for a $\frac{1}{2}$-nonspreading set-valued mapping in Hilbert spaces, Suantai et al. [15] established a weak convergence result.

It is well known that strong convergence behaviour of iteration is more desirable than weak convergence, therefore in this paper, we propose shrinking projection hybrid method for finding a common solution of the set of $GEP(1.3)$, a combination of CQ-method and shrinking projection method and the set of fixed points of a k-nonspreading set-valued mapping with $k \in (0, \frac{1}{2}]$, which is more general than $\frac{1}{2}$-nonspreading set-valued mapping and we prove a strong convergence result. Finally by giving an numerical example we have verified that our proposed iterative method is more faster and effective than the results given in [6, 14, 15, 16, 17, 19].

2. Preliminaries

Now, we give some basic definitions and results before proving our main result.

Lemma 2.1 *For $u, v \in H$ and $\alpha \in [0,1]$, we have:*

(i) $\|\alpha u + (1-\alpha)v\|^2 = \alpha\|u\|^2 + (1-\alpha)\|v\|^2 - \alpha(1-\alpha)\|u-v\|^2$;

(ii) $\|u+v\|^2 \le \|u\|^2 + 2\langle v, u+v\rangle$;

(iii) *For a sequence* $\{u_n\}$ *which converges weakly to* $z \in H$*, then*

$$\limsup_{n\to\infty} \|u_n - v\|^2 = \limsup_{n\to\infty} \|u_n - z\|^2 + \|z-v\|^2.$$

Lemma 2.2 *Let* H *be a Hilbert space and* E *be a closed, convex subset of* H *and let* P_E *be the metric projection of* H *onto* E*. Let* $u \in H$ *and* $z \in E$*. Then* $z = P_E u$ *if and only if*

$$\langle u - z, v - z\rangle \le 0, \quad \forall v \in E.$$

Condition (A) A set-valued mapping $T : E \to CO(E)$ is said to satisfy *Condition (A)*, if $\|x - p\| = d(x, Tp)$, for all $x \in H$ and $p \in F(T)$.

Lemma 2.3 *Let* H *be a Hilbert space and* E *be a closed, convex subset of* H*. Let* $T : E \to CO(E)$ *be a* k*-nonspreading set-valued mapping with* $k \in (0, \frac{1}{2}]$ *and* $F(T) \neq \emptyset$*, then* $F(T)$ *is closed. Also* $F(T)$ *is convex, if* T *satisfies Condition (A).*

Proof. Let $u_n \to u$ as $n \to \infty$, then it follows that

$$\begin{aligned} d(u, Tu) &\le \|u - u_n\| + d(u_n, Tu) \le \|u - u_n\| + H(Tu_n, Tu) \\ &\le \|u - u_n\| + \sqrt{\frac{k}{1-k}}\Big\|u_n - u\Big\|, \end{aligned}$$

which implies $d(u, Tu) = 0$ as $n \to \infty$. Therefore $u \in F(T)$.

Now, let $x = sx_1 + (1-s)x_2$, where $x_1, x_2 \in F(T)$ and $s \in (0,1)$. Let $w \in Tp$. It follows from (1.8) and Lemma 2.1 that

$$\begin{aligned} \|x - w\|^2 &= \|s(w - x_1) + (1-s)(w - x_2)\|^2 \\ &= s\|w - x_1\|^2 + (1-s)\|w - x_2\|^2 - s(1-s)\|x_1 - x_2\|^2 \\ &= sd(w, Tx_1)^2 + (1-s)d(w, Tx_2)^2 - s(1-s)\|x_1 - x_2\|^2 \\ &\le sH(Tx, Tx_1)^2 + (1-s)H(Tx, Tx_2)^2 - s(1-s)\|x_1 - x_2\|^2 \end{aligned}$$

$$\leq \theta[s\|x - x_1\|^2 + (1-s)\|x - x_2\|^2] - s(1-s)\|x_1 - x_2\|^2$$
$$= s(1-s)^2\|x_1 - x_2\|^2 + (1-s)s^2\|x_1 - x_2\|^2 - s(1-s)\|x_1 - x_2\|^2$$
$$= 0,$$

where $\theta = \sqrt{\frac{k}{1-k}} < 1$ and hence $x = w$. Therefore, $x \in F(T)$. This completes the proof.

Lemma 2.4 **[15]** *Let Let H be a Hilbert space and E be a closed, convex subset of H. Let $T : E \to CO(E)$ be a k-nonspreading set-valued mapping such that $k \in (0, \frac{1}{2}]$. If $u, v \in E$ and $a \in Tu$, then there exists $b \in Tv$ such that*

$$\|a - b\|^2 \leq H(Tu, Tv)^2 \leq \frac{k}{1-k}\Big(\|u - v\|^2 + 2\langle u - a, v - b\rangle\Big).$$

Lemma 2.5 **[15]** *Let H be a Hilbert space and E be a closed, convex subset of H. Let $T : E \to CO(E)$ be a k-nonspreading set-valued mapping such that $k \in (0, \frac{1}{2}]$. Let $\{x_n\}$ be a sequence in E which converges weakly to p and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

Theorem 2.1 **[2]** *Let $M : E \times E \to \mathbf{R}$ is a mapping with $M(u, u) = 0$, $\forall u \in E$. If M is monotone, upper hemicontinuous and for each $u \in E$ fixed, the function $v \to M(u, v)$ is convex and lower semicontinuous, then for fixed $r > 0$ and $z \in E$, there exists a nonempty compact convex subset K of H and $u \in E \cap K$ such that*

$$M(v, u) + \frac{1}{r}\langle v - u, u - z\rangle < 0, \ \forall v \in E\backslash K.$$

Lemma 2.6 **[4]** *Let for each $u \in H$ and $r > 0$, a mapping $T_r : H \to E$ is given by*

$$T_r(u) = \Big\{z \in E : \ M(z, v) + \frac{1}{r}\langle v - z, v - u\rangle \geq 0, \ \forall v \in E\Big\}, \ \ u \in H \qquad .$$

If $M : E \times E \to \mathbf{R}$ satisfies Theorem 2.1., then the following hold:

(i) T_r is nonempty and firmly nonexpansive, i.e., for any $u, v \in H$,

$$\|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v\rangle;$$

(ii) $F(T_r) = EP(M)$ *and* $EP(M)$ *is convex and closed.*

Lemma 2.7 [17] *Let* H *be a Hilbert space and* E *be a closed, convex subset of* H*. For each* $i \in N$*, let a mapping* $M_i : E \times E \to \mathbf{R}$ *follows Theorem 2.1 with* $\bigcap\limits_{i=1}^{N} EP(M_i) \neq \emptyset$*.*
Then

$$EP\Big(\sum_{i=1}^{N} a_i M_i\Big) = \bigcap_{i=1}^{N} EP(M_i),$$

where $a_i \in (0,1)$ *for* $i = 1, 2, ..., N$ *and* $\sum\limits_{i=1}^{N} a_i = 1$*.*

Example 2.1 Let

$$M_1(u,v) = \frac{1}{2}(u-1)(5v^2 - u^2 - 4uv),$$
$$M_2(u,v) = (u-1)(v^2 - u^2),$$
$$M_3(u,v) = (u-1)(uv + v^2 - 2u^2), \quad \forall u, v \in \mathbf{R},$$

where $M_i : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ for $i = 1, 2, 3$. It can be seen that $M_i(u,v)$ satisfies Theorem 2.1 for each i and $\bigcap\limits_{i=1}^{3} EP(M_i) = \{0, 1\}$.
If we take $a_1 = \frac{1}{4}, a_2 = \frac{1}{12}$ and $a_3 = \frac{2}{3}$, then

$$\sum_{i=1}^{3} a_i M_i(u,v) = \frac{1}{24}(u-1)(4uv + 33v^2 - 37u^2),$$

which yields $EP\Big(\sum\limits_{i=1}^{N} a_i M_i\Big) = \{0, 1\}$.

Remark 2.1 [17] From Lemma 2.6, we obtain

$$F(T_r^{\sum}) = EP\Big(\sum_{i=1}^{N} a_i M_i\Big) = \bigcap_{i=1}^{N} EP(M_i),$$

where

$$T_r^{\sum}(u) = \Big\{z \in E : \ \Big(\sum_{i=1}^{N} a_i M_i\Big)(z,v) + \frac{1}{r}\langle v-z, z-u\rangle \geq 0, \ \forall v \in E\Big\},$$

and $a_i \in (0,1)$, for each i and $\sum\limits_{i=1}^{N} a_i = 1$.

3. Main Result

Now we give our main Theorem of finding strong convergence result for common solutions of $GEP(1.3)$ and the fixed points of a k-nonspreading set-valued mapping.

Theorem 3.1 *Let H be a Hilbert space and E be a closed, convex subset of H. For each $i \in N$, suppose $M_i : E \times E \to \mathbf{R}$ be a bi mapping satisfying Theorem 2.1. Let $T : E \to CO(E)$ be a k-nonspreading set-valued mapping with $k \in (0, \frac{1}{2}]$. Assume $\Omega = \bigcap_{i=1}^{N} EP(M_i) \cap F(T) \neq \emptyset$. For given initial point $x_1 \in H$ with $K_1 = E$, let $\{u_n\}, \{y_n\}$ and $\{x_n\}$ are given by*

$$\begin{cases} u_n \in E \text{ such that } \sum_{i=1}^{N} a_i M_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\ y_n \in \alpha_n u_n + (1-\alpha_n) T u_n, \\ E_n = \{z \in E : \ \|y_n - z\| \leq \|x_n - z\|\}, \\ K_n = \{z \in K_{n-1} : \ \langle x_1 - x_n, z - x_n\rangle \leq 0\}, n \geq 2, \\ x_{n+1} = P_{E_n \cap K_n} x_1, \ n \geq 1, \end{cases} \tag{3.1}$$

If T satisfies Condition (A) and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$ with $\{\alpha_n\} \subset (0,1)$, then $\{x_n\}$ converges strongly to $P_\Omega x_1$.

Proof. The proof can be divided into the following steps.

Step 1. We claim that $\{x_n\}$ is well defined.
It can easily seen from the definition that E_n is closed and K_n is closed and convex for every $n \in \mathbf{N}$. We claim that E_n is convex. Since $E_n = \{z \in E : \ \|y_n - z\| \leq \|x_n - z\|\}$ which can be given as

$$E_n = \{z \in E : \ \|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0\},$$

which implies that E_n is convex. Therefore, $E_n \cap K_n$ is closed and convex subset of H for each $n \in \mathbf{N}$. Hence $P_{E_n \cap K_n} x_1$ is well defined and as a result $\{x_n\}$ is well defined.
From Lemma 2.6 and Remark 2.1 it can be concluded that $EP(\sum_{i=1}^{N} a_i M_i)$ is closed and convex. Further from Lemma 2.3, it can be concluded that $F(T)$ is closed and convex. Consequently, Ω is closed and convex and therefore $P_\Omega x_1$ is well defined.

Step 2. We claim that $\Omega \subset E_n \cap K_n$.

Let $p \in \bigcap_{i=1}^{N} EP(M_i) \cap F(T)$, then by using Lemma 2.6, we have $u_n = T_{r_n}^{\sum} x_n$ and

$$\|u_n - p\| = \|T_{r_n}^{\sum} x_n - T_{r_n}^{\sum} p\| \leq \|x_n - p\| \tag{3.2}$$

for all $n \in \mathbf{N}$. Now we have

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n u_n + (1-\alpha_n) z_n - p\| \\
&\leq \alpha_n \|u_n - p\| + (1-\alpha_n)\|z_n - p\| \\
&= \alpha_n \|u_n - p\| + (1-\alpha_n) d(z_n, Tp) \\
&\leq \alpha_n \|u_n - p\| + (1-\alpha_n) H(Tu_n, Tp) \\
&\leq \alpha_n \|u_n - p\| + (1-\alpha_n)\theta \|u_n - p\| \\
&\leq \|u_n - p\|,
\end{aligned} \tag{3.3}$$

for all $z_n \in Tu_n$, where $\theta = \sqrt{\frac{k}{1-k}} < 1$. So, we have $p \in E_n$ and hence

$$\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset E_n, \quad \text{for all } n \in \mathbf{N}. \tag{3.4}$$

Further, we claim that

$$\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset E_n \cap K_n, \quad \text{for all } n \in \mathbf{N}. \tag{3.5}$$

It can be proved by using induction. For $n = 1$, we have $\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset E_1$ and $K_1 = H$, we get $\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset E_1 \cap K_1$. Let $\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset E_n \cap K_n$ for some n. Since $x_{n+1} = P_{E_n \cap K_n} x_1$, then $x_{n+1} \in E_n \cap K_n$ and

$$\langle x_1 - x_{n+1}, z - x_{n+1} \rangle \leq 0, \quad \text{for all } z \in E_n \cap K_n.$$

Since $\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset E_n \cap K_n$, for all $z \in \bigcap_{i=1}^{N} EP(M_i) \cap F(T)$

$$\langle x_1 - x_{n+1}, z - x_{n+1} \rangle \leq 0,$$

and hence $z \in K_{n+1}$. So, we get

$$\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset K_{n+1} \quad \text{for all } n \in \mathbf{N}.$$

By using (3.4) we have

$$\bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset E_{n+1} \cap K_{n+1}, \quad \text{for all } n \in \mathbf{N}.$$

Hence $\Omega \subset E_n \cap K_n$, for all $n \in \mathbf{N}$.

Step 3. We claim that $\lim_{n\to\infty} \|x_n - x_1\|$ exists.
Since nonempty set Ω is closed and convex therefore there exists a unique $v \in \Omega$ in H such that $v = P_\Omega x_1$. From $x_{n+1} = P_{E_n \cap K_n} x_1$, it follows that

$$\|x_{n+1} - x_1\| \leq \|z - x_1\|, \quad \text{for all } z \in E_n \cap K_n \text{ and all } n \in \mathbf{N}.$$

Since $v \in \Omega \subset E_n \cap K_n$, we have

$$\|x_{n+1} - x_1\| \leq \|v - x_1\|, \quad \text{for all } n \in \mathbf{N}. \tag{3.6}$$

Therefore, $\{x_n\}$ is bounded. Again (3.2) and (3.3) $\{u_n\}$ and $\{y_n\}$ are bounded.
Since $x_n = P_{K_n} x_1$ and $x_{n+1} \in K_n$, for all n, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \text{for all } n \in \mathbf{N}. \qquad .$$

As $\{x_n\}$ is bounded, therefore $\{\|x_n - x_1\|\}$ is nondecreasing and bounded. Therefore $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Step 4. We claim that $\lim_{n\to\infty} x_n = w \in E$.
Since $m > n$, therefore from K_n we have $K_m \subset K_n$. Since $x_m = P_{K_m} x_1 \subset K_n$ and $x_n = P_{K_n} x_1$, it follows from (2.1) that

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2.$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, above inequality gives

$$\lim_{n\to\infty} \|x_m - x_n\| = 0, \tag{3.7}$$

therefore $\{x_n\}$ is a Cauchy sequence in E and there exists $w \in E$ such that $\lim_{n\to\infty} x_n = w$. Particularly if $m = n+1$, then (3.7) gives

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.8}$$

Step 5. We claim that $w \in F(T)$.
As $x_{n+1} \in E_n$, therefore

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \le 2\|x_n - x_{n+1}\|.$$

Since $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$, therefore we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.9}$$

Since $p \in \bigcap_{i=1}^{N} EP(M_i) \cap F(T)$ and $T_r^{\sum}$ is firmly nonexpansive, we have

$$\begin{aligned}\|u_n - p\|^2 = \|T_{r_n}^{\sum} x_n - T_{r_n}^{\sum} p\|^2 &\le \langle T_{r_n}^{\sum} x_n - T_{r_n}^{\sum} p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\Big\}.\end{aligned}$$

Hence,

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.10}$$

For $z_n \in Tu_n$, it follows from (3.2) that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(z_n, Tp)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(Tu_n, Tp)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\theta^2\|u_n - p\|^2.\end{aligned}$$

Since $\alpha < 1$, it follows from (3.10) that

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - (1-\alpha_n)\theta^2\|x_n - u_n\|^2,$$

which can be written as

$$(1-\alpha_n)\theta^2\|x_n-u_n\|^2 \le \|x_n-p\|^2-\|y_n-p\|^2.$$
$$\le \|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|). \tag{3.11}$$

Since $(1-\alpha_n)\theta^2 > 0$, it follows from (3.9) and (3.11) that

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.12}$$

Since $y_n \in \alpha_n u_n + (1-\alpha_n)Tu_n$, then for any $z_n \in Tu_n$ we have

$$(1-\alpha_n)\|z_n-u_n\| = \|y_n-u_n\| \le \|y_n-x_n\|+\|x_n-u_n\|. \tag{3.13}$$

Since $(1-\alpha_n) > 0$, it follows from (3.9), (3.12) and (3.13) that

$$\lim_{n\to\infty}\|z_n-u_n\| = 0. \tag{3.14}$$

It follows from (3.12) and (3.14) that the sequences $\{x_n\}, \{u_n\}$ and $\{z_n\}$ all have the same asymptotic behaviour and hence $u_n \to w$ and $z_n \to w$ as $n \to \infty$. Hence, by Lemma 2.5, we have $w \in F(T)$.

Step 6. We claim that $w \in \bigcap_{i=1}^{N} EP(M_i)$.

Since $u_n = T_{r_n}^{\sum} x_n$, we have

$$\sum_{i=1}^{N} a_i M_i(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y \in E.$$

From monotonicity of Theorem 2.1, above can be written as

$$\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge \sum_{i=1}^{N} a_i M_i(y, u_n), \quad \forall y \in E. \tag{3.15}$$

Since $\liminf_{n\to\infty} r_n > 0$, there exists $r > 0$ such that $r_n > r$, $\forall n$. Hence , it follows that

$$\lim_{n\to\infty}\frac{\|u_n-x_n\|}{r_n} < \lim_{n\to\infty}\frac{\|u_n-x_n\|}{r} = 0. \tag{3.16}$$

It follows from (3.12), (3.15), (3.16) and Theorem 2.1 that

$$\sum_{i=1}^{N} a_i M_i(y, w) \le 0, \quad \forall y \in E.$$

For $s \in (0,1]$ and $y \in E$, assume $y_s := sy + (1-s)w$. For each $y \in E$, we have $y_s \in E$ and therefore $\sum_{i=1}^{N} a_i M_i(y_s, w) \leq 0$. Now we have

$$
\begin{aligned}
0 &= \sum_{i=1}^{N} a_i M_i(y_s, y_s) \\
&= \sum_{i=1}^{N} a_i M_i(y_s, sy + (1-s)w) \\
&\leq s \sum_{i=1}^{N} a_i M_i(y_s, y) + (1-s) \sum_{i=1}^{N} a_i M_i(y_s, w)) \\
&\leq s \sum_{i=1}^{N} a_i M_i(y_s, y).
\end{aligned}
$$

After dividing by s, it follows that

$$\sum_{i=1}^{N} a_i M_i(ty+(1-t)w, y) \geq 0 \ \ \forall y \in E.$$

From Theorem 2.1 and taking $t \downarrow 0$, we have

$$\sum_{i=1}^{N} a_i M_i(w, y) \geq 0 \ \ \forall y \in E.$$

Which implies, $w \in EP(\sum_{i=1}^{N} a_i M_i)$. By using Lemma 2.7,

$$EP(\sum_{i=1}^{N} a_i M_i) = \bigcap_{i=1}^{N} EP(M_i).$$

Therefore, we obtain $w \in \bigcap_{i=1}^{N} EP(M_i) \cap F(T)$.

Step 7. We claim that $w = P_{\Omega} x_1$.
Since $x_n = P_{K_n} x_1$ and $w \in \bigcap_{i=1}^{N} EP(M_i) \cap F(T) \subset K_n$, we have

$$\langle x_1 - x_n, x_n - p \rangle \geq 0, \quad \forall p \in K_n. \tag{3.17}$$

Applying $n \to \infty$ in (3.17), we have

$$\langle x_1 - w, w - p\rangle \geq 0, \quad \forall p \in K_n.$$

Since $\Omega \subset K_n$, we have

$$\langle x_1 - w, w - p\rangle \geq 0, \quad \forall p \in \Omega,$$

which gives $w = P_\Omega x_1$.

Based on Theorem 3.1, we have following consequences.

Corollary 3.1 *Let H be a Hilbert space and E be a closed, convex subset of H. Let $M : E \times E \to \mathbf{R}$ satisfying Theorem 2.1 such that $EP(M) \neq \emptyset$. For a given initial point $x_1 \in H$ with $K_1 = E$, let $\{u_n\}$ and $\{x_n\}$ is given by*

$$\begin{cases} u_n \in E \text{ such that } M(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\ E_n = \{z \in E : \ \|y_n - z\| \leq \|x_n - z\|\}, \\ K_n = \{z \in K_{n-1} : \ \langle x_1 - x_n, z - x_n\rangle \leq 0\}, n \geq 2, \\ x_{n+1} = P_{E_n \cap K_n} x_1, \ n \geq 1, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{EP(M)}x_1$.

Proof. By taking $M_i = M$, for each i and $T = I$ with $\alpha_n = 1$, the Theorem 3.1, reduces to Corollary 3.1.

Corollary 3.2 *Let H be a Hilbert space and E be a closed, convex subset of H. Let $T : E \to CO(E)$ be a k-nonspreading set-valued mapping with $k \in (0, \frac{1}{2}]$ such that $F(T) \neq \emptyset$. For a given initial point $x_1 \in H$ with $K_1 = E$, let the sequences $\{u_n\}, \{y_n\}$ and $\{x_n\}$ are given by*

$$\begin{cases} u_n \in E \text{ such that } \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in E, \\ y_n \in \alpha_n u_n + (1 - \alpha_n)Tu_n, \\ E_n = \{z \in E : \ \|y_n - z\| \leq \|x_n - z\|\}, \\ K_n = \{z \in K_{n-1} : \ \langle x_1 - x_n, z - x_n\rangle \leq 0\}, n \geq 2, \\ x_{n+1} = P_{E_n \cap K_n} x_1, \ n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$. If T satisfies condition (A), then $\{x_n\}$ converges strongly to $P_{F(T)}x_1$.

Proof. By taking $M_i = 0$, for each i with $r_n = 1$ the Theorem 3.1 reduces to Corollary 3.2.

4. Numerical Illustrations

Now, an example to understand and verify the convergence nature of main result 3.1 is given as follows:

Example 4.1 Let $M_i : [1,4] \times [1,4] \to \mathbf{R}$ be defined by

$$M_i(x,y) = i(y^2 - 2x^2 + xy + 3x - 3y), \quad \text{for all } x, y \in [1,4],$$

for each $i \in N$ and $\mathbf{R} = H, E = [1,4]$. Further, let $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$ such that $\sum_{i=1}^{N} a_i = 1$, where $i \in N$. Now

$$\sum_{i=1}^{N} a_i M_i(x,y) = \sum_{i=1}^{N} \left(\frac{4}{5^i} + \frac{1}{N5^N}\right) i(y^2 - 2x^2 + xy + 3x - 3y) = \Psi(y^2 - 2x^2 + xy + 3x - 3y),$$

where $\Psi = \sum_{i=1}^{N} \left(\frac{4}{5^i} + \frac{1}{N5^N}\right) i$. It can be easily seen that $\sum_{i=1}^{N} a_i M_i$ satisfies Theorem 3.1 and

$$EP\Big(\sum_{i=1}^{N} a_i M_i\Big) = \bigcap_{i=1}^{N} EP(M_i) = \{1\}.$$

Let be a mapping $T : E \to CO(E)$ by

$$Tx = \begin{cases} \{1\}, & x \in [1,3] \\ [\frac{x}{x+1}, 1], & x \in (3,4]. \end{cases}$$

Now, we show that T is $\frac{1}{2}$-nonspreading set-valued mapping. In fact, we have the following cases:
Case 1: if $x, y \in [1,3]$, then $H(Tx, Ty) = 0$.
Case 2: if $x \in [1,3]$ and $y \in (3,4]$, then

$$2H(Tx,Ty)^2 = 2\Big(1 - \frac{y}{y+1}\Big)^2 < 2 < d(Tx,y)^2 + d(x,Ty)^2.$$

Case 3: if $x, y \in (3,4]$, then

$$2H(Tx,Ty)^2 = 2\Big(\frac{x}{x+1} - \frac{y}{y+1}\Big)^2 < 2 < d(Tx,y)^2 + d(x,Ty)^2,$$

which shows that T is $\frac{1}{2}$-nonspreading set-valued mapping.
It is easy to see that $\bigcap_{i=1}^{N} EP(M_i) \cap F(T) = \{1\}$.

Step 1. Find $\{u_n\}$ in $E = [1, 4]$.
For $r_n > 0$, we have $\{x_n\}$ and $\{u_n\}$ in E such that

$$\sum_{i=1}^{N} a_i M_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in E,$$

it can be written as

$$S(y) := \Psi r_n y^2 + (\Psi u_n r_n + u_n - x_n - 3\Psi r_n)y + 3\Psi r_n u_n - u_n^2 - 2\Psi r_n u_n^2 + u_n x_n \geq 0, \ \forall y \in E.$$

As $S(y) = ay^2 + by + c \geq 0$, for all $y \in E$ then $b^2 - 4ac = (u_n - 3\Psi r_n + 3\Psi r_n u_n - x_n)^2 \leq 0$. Therefore, $(u_n - 3\Psi r_n + 3\Psi r_n u_n - x_n)^2 = 0$ which implies that

$$u_n = \frac{x_n + 3\Psi r_n}{1 + 3\Psi r_n}.$$

Therefore, $u_n = T_{r_n}^{\sum} x_n = \frac{x_n + 3\Psi r_n}{1 + 3\Psi r_n}$ for each $r_n > 0$.

Step 2. Find $y_n \in \alpha_n u_n + (1 - \alpha_n)T u_n$.
By choosing $\alpha_n = r_n = \frac{n}{100n+1}$, we have $y_n \in \frac{n}{100n+1} u_n + (1 - \frac{n}{100n+1}) z_n$, where

$$z_n \in \begin{cases} \{1\}, & u_n \in [1, 3] \\ [\frac{u_n}{u_{n+1}}, 1], & u_n \in (3, 4]. \end{cases}$$

n	u_n	y_n	$E_n \cap K_n$	X_n
1	3.8926	0.9296	[1, 2.4648]	4.0000
2	2.4121	1.0141	[1, 1.7394]	2.4648
3	1.7128	1.0071	[1, 1.3773]	1.7394
4	1.3598	1.0036	[1, 1.1884]	1.3733
5	1.1816	1.0018	[1, 1.0951]	1.1884
6	1.0917	1.0009	[1, 1.0480]	1.0951
7	1.0463	1.0005	[1, 1.0242]	1.0480
8	1.0234	1.0002	[1, 1.0122]	1.0242
9	1.0118	1.0001	[1, 1.0062]	1.0122
10	1.0060	1.0001	[1, 1.0031]	1.0062
11	1.0030	1.0000	[1, 1.0016]	1.0031
12	1.0015	1.0000	[1, 1.0008]	1.0016
13	1.0008	1.0000	[1, 1.0004]	1.0008
14	1.0004	1.0000	[1, 1.0002]	1.0004
15	1.0002	1.0000	[1, 1.0001]	1.0002
16	1.0001	1.0000	[1, 1.0001]	1.0001
17	1.0000	1.0000	[1, 1.0000]	1.0001
18	1.0000	1.0000	[1, 1.0000]	1.0000

Table 1: The values of $\{u_n\}$ and $\{x_n\}$ with $n = N = 18$

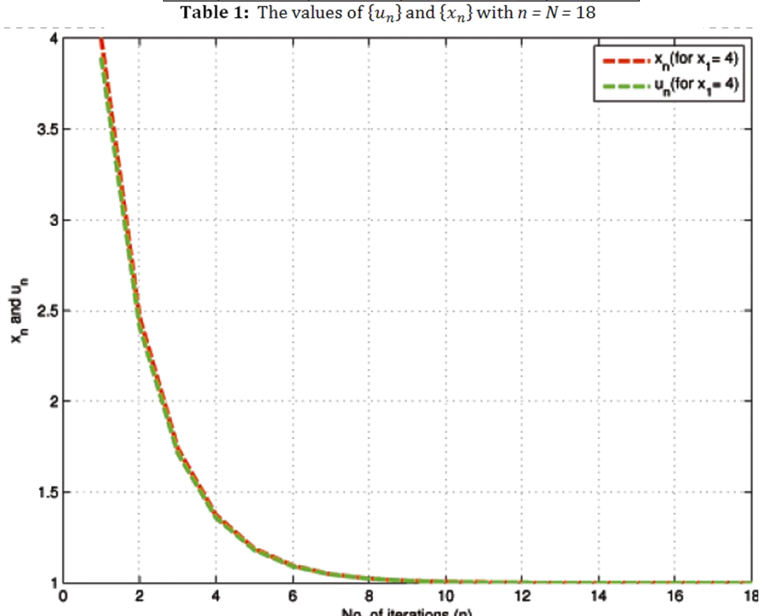

Figure 1: Convergence of x_n and u_n

Table 1 and Figure 1: z_n: being randomized in the first time

Step 3. Find $E_n = \{z \in E : \ \|y_n - z\| \leq \|x_n - z\|\}$.
Since $(2z - (x_n + y_n))(x_n - y_n) \leq 0$, therefore we have:
Case 1: If $x_n - y_n = 0$, then $E_n = E, \ \forall n \geq 1$.
Case 2: If $x_n - y_n > 0$, then $E_n = [1, \frac{x_n+y_n}{2}], \ \forall n \geq 1$.
Case 3: If $x_n - y_n < 0$, then $E_n = [\frac{x_n+y_n}{2}, 4], \ \forall n \geq 1$.

Step 4. Find $K_n = \{z \in K_{n-1} : \langle x_1 - x_n, z - x_n \rangle \leq 0\}$.
Since $(x_1 - x_n)(z - x_n) \leq 0$, therefore we have:
Case 1: If $x_1 - x_n = 0$, then $K_n = E$, $\forall n \geq 2$.
Case 2: If $x_1 - x_n > 0$, then $K_n = K_{n-1} \cap [1, x_n]$, $\forall n \geq 2$.
Case 2: If $x_1 - x_n < 0$, then $K_n = K_{n-1} \cap [x_n, 4]$, $\forall n \geq 2$.

n	u_n	y_n	$E_n \cap K_n$	X_n
1	3.8926	0.8603	[1, 2.4302]	4.0000
2	2.3787	1.0137	[1, 1.7219]	2.4302
3	1.6959	1.0069	[1, 1.3644]	1.7219
4	1.3513	1.0035	[1, 1.1840]	1.3644
5	1.1773	1.0018	[1, 1.0929]	1.1840
6	1.0895	1.0009	[1, 1.0469]	1.0929
7	1.0452	1.0005	[1, 1.0237]	1.0469
8	1.0228	1.0002	[1, 1.0119]	1.0237
9	1.0115	1.0001	[1, 1.0060]	1.0119
10	1.0058	1.0001	[1, 1.0030]	1.0060
11	1.0029	1.0000	[1, 1.0015]	1.0030
12	1.0015	1.0000	[1, 1.0008]	1.0015
13	1.0007	1.0000	[1, 1.0004]	1.0008
14	1.0004	1.0000	[1, 1.0002]	1.0004
15	1.0002	1.0000	[1, 1.0001]	1.0002
16	1.0001	1.0000	[1, 1.0001]	1.0001
17	1.0000	1.0000	[1, 1.0000]	1.0001
18	1.0000	1.0000	[1, 1.0000]	1.0000

Table 2: The values of $\{u_n\}$ and $\{x_n\}$ with $n = N = 18$

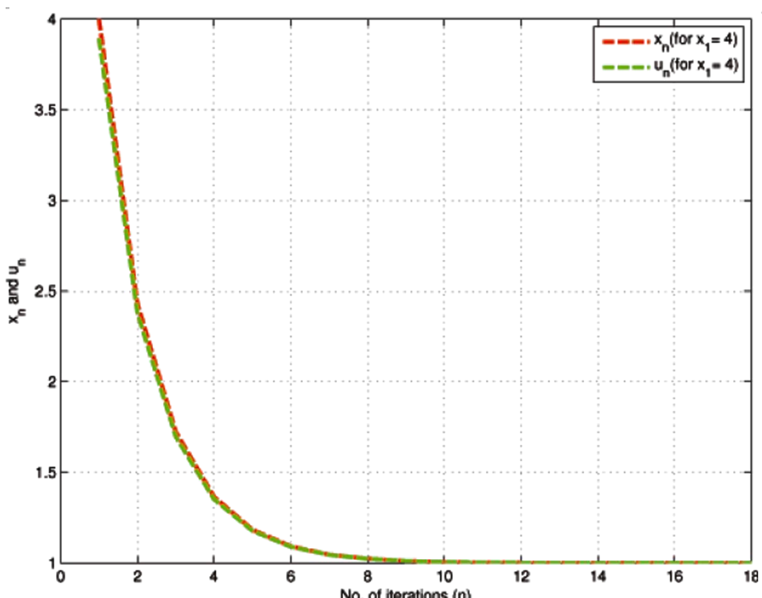

Figure 2: Convergence of x_n and u_n

Table 2 and Figure 2: Chosen z_n randomly

Step 5. Solve $x_{n+1} = P_{E_n \cap K_n} x_1$.
By taking $n = N = 18$ and choosing $x_1 = 4$, the scheme (3.1) converges to 1 (see the table 1, table 2 and figure 1, figure 2).

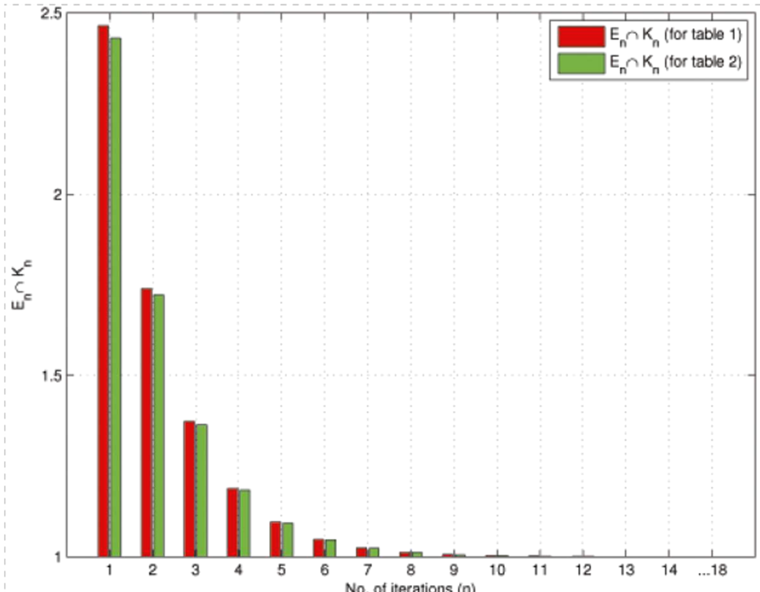

Figure 3: Shrinking behaviors of the set $E_n \cap K_n$

Figure 3: Shrinking behavior of $E_n \cap K_n$

Figure 3, shows the trend of $E_n \cap K_n$, that is $E_n \cap K_n \subset E_{n-1} \cap K_{n-1} ... \subset E_2 \cap K_2 \subset E_1 \cap K_1 \subset E$. It can be concluded that the iteration of $E_n \cap K_n$ will be shrinked till we obtain the approximate result.

Conclusion

In this work, we studied a new method which is known as shrinking projection hybrid iteration technique for finding simultaneous solution of a generalized equilibrium problems GEP (1.3) and a fixed point problem for a k-nonspreading set-valued mapping with $k \in (0, \frac{1}{2}]$ in Hilbert space. Next, we establish a main result for the converging sequences given by the shrinking projection hybrid method and given some of consequences. Finally, we show by an example that our method is better than existing methods. By making use of iteration method presented in this work, we can find the main result for the problems considered in [17].

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