Strong convergence theorem for family of minimization and monotone inclusion problems in Hadamard spaces

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Abstract:

In this paper, we introduce a modified Ishikawa-type proximal point algorithm for approximating a common solution of minimization problem, monotone inclusion problem and fixed point problem. We obtain a strong convergence of the proposed algorithm to a common solution of finite family of minimization problem, finite family of monotone inclusion problem and fixed point problem for asymptotically demicontractive mapping in Hadamard spaces. Numerical example is given to illustrate the applicability of our main result. Our results complement and extend some recent results in literature.

Keywords: CAT(0) spaces; Ishikawa-type proximal point algorithm; Monotone inclusion problem; Minimization problem; Fixed point problem; Asymptotically demicontractive mapping.

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1. Introduction

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a mapping c from a closed interval $[0, l] \subset \mathbf{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(k), c(k')) = |k - k'|$ for all $k, k' \in [0, l]$. In this case, c is called an isometry and $l = d(x, y)$. The image of c is called a geodesic segment joining x to y . When this image is unique, then it is denoted by $[x, y]$. The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and (X, d) is said to be uniquely geodesic space if every two points of X are joined by only one geodesic segment. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consist of three points x_1, x_2, x_3 in X (known as the vertices of Δ) and a geodesic segment between each pair of vertices (known as the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbf{R}^2$ such that $d(x_i, x_j) = d_{\mathbf{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$. A metric space (X, d) is called a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane $\mathbf{R}^2$. Furthermore, a geodesic space is a CAT(0) space if and only if it satisfies the following inequality, called the (CN) inequality of Bruhat and Tits [14] (see [12]): If x, y, z are points in X and y_0 is the midpoint of the segment $[y, z]$, then

$$(1.1) \quad d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z).$$

A complete CAT(0) space is called a Hadamard space. It is generally known that a CAT(0) space is a uniquely geodesic space. Examples of CAT(0) spaces includes: Euclidean spaces $\mathbf{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature, $\mathbf{R}$ -trees, Hilbert ball [22], Hyperbolic spaces [51]. See [4, 12, 47] for more equivalent definitions and properties of CAT(0) spaces.

Let $(1 - t)x \oplus ty$ denote the unique point z in the geodesic segment joining x to y for each x, y in a CAT(0) space such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$, where $t \in [0, 1]$. Let $[x, y] := \{(1 - t)x \oplus ty : t \in [0, 1]\}$, then a subset C of X is convex if $[x, y] \subseteq C$ for all $x, y \in C$. Let X be a metric space and C be a nonempty closed and convex subset of X . A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \rightarrow C$, if $Tx = x$. The set of fixed points of T is denoted by $F(T)$. Many mathematical models of real-world problems arising from biology, economics, engineering, computer science, among others are usually expressed in the

form of fixed point equation. Thus, the existence of solutions of these problems is equivalent to the existence of fixed points of some nonlinear mappings. The approximation of fixed points of nonlinear mappings is one of the most flourishing areas of research in nonlinear analysis and has continued to attract the interest of many researchers due to its extensive applications in diverse mathematical problems such as inverse problems, signal processing, image recovery, game theory, fuzzy theory, among others, see [17, 26, 27, 28, 49, 53, 56, 57, 58] and the references therein. In metric spaces, fixed point theory begins with the Banach contraction mapping principle which guarantees the existence and uniqueness of fixed points of contraction mappings when the space is complete and the sequence of Picard iterates converges to the unique fixed point. Since then, the principle has been constantly improved and extended in many directions. In particular, fixed point theorems in convex metric spaces have attracted the attention of many researchers (see, for example [6, 5, 39, 45]).

Let $T : C \rightarrow C$ be a nonlinear mapping. T is said to be

- (i) *L-Lipschitzian* if there exists $L > 0$ such that

$$d(Tx, Ty) \leq Ld(x, y) \quad \forall x, y \in C,$$

if $L \in [0, 1)$, then T is called *contraction*, while T is called *nonexpansive* if $L = 1$;

- (ii) *uniformly L-Lipschitzian* if there exists $L > 0$ such that

$$d(T^n x, T^n y) \leq Ld(x, y) \quad \forall x, y \in C, n \geq 1;$$

- (iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \quad \forall n \geq 1 \text{ and } x, y \in C;$$

- (iv) *asymptotically demicontractive* if $F(T) = \emptyset$ and there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, $\lim_{n \rightarrow \infty} k_n = 1$ such that

$$d^2(T^n x, y) \leq k_n d^2(x, y) + k d^2(x, T^n x)$$

for some $k \in [0, 1)$ and for all $n \geq 1$, $x \in C, y \in F(T)$;

- (v) $(\{\mu_n\}, \{v_n\}, \phi)$ -total asymptotically demicontractive if $F(T) \neq \emptyset$ and there exist a constant $k \in [0, 1)$ and sequences $\{\mu_n\}, \{v_n\} \subset [0, \infty)$ with $\mu_n \rightarrow 0$, $v_n \rightarrow 0$, and a strictly increasing continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$d^2(T^n x, y) \leq d^2(x, y) + \mu_n \phi(d(x, y)) + k d^2(x, T^n x) + v_n$$

for all $n \geq 1$, $x \in C$, $y \in F(T)$.

Let $\{x_n\}$ be a bounded sequence in X and $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ be a continuous functional defined by $r(x, \{x_n\}) = n \rightarrow \infty \limsup d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\}$ while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. It is generally known that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ -convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}n \rightarrow \infty \lim x_n = x$ (see [21]). The concept of Δ -convergence in metric spaces was first introduced and studied by Lim [42]. In 2008, Kirk and Panyanak [39] introduced and studied this concept in CAT(0) spaces and proved that it is very similar to the weak convergence in Banach space setting. Kirk [38] proved that every nonexpansive mapping defined on a nonempty closed convex and bounded subset of a CAT(0) space has a fixed point. In 2013, Kim [37] studied the Ishikawa-type iterative scheme for approximating fixed point of a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive mapping in a complete CAT(0) space. Under some compactness condition, Kim [37] obtained a strong convergence result. Several other authors have also studied fixed point theory for nonlinear mappings in CAT(0) spaces (see, for example [4, 6, 47]).

Let X be a CAT(0) space. A function $f : X \rightarrow (-\infty, +\infty]$ is called convex if

$$f(\lambda x \oplus (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in X, \lambda \in (0, 1).$$

The function f is called proper, if $D(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$, where $D(f)$ denotes the domain of f . A function $f : D(f) \rightarrow (-\infty, \infty]$ is said to be lower semi-continuous at a point $x \in D(f)$ if

$$(1.2) \quad f(x) \leq \liminf_{n \rightarrow \infty} f(x_n),$$

for each sequence $\{x_n\}$ which converges strongly to x . f is said to be lower semicontinuous on $D(f)$ if it is lower semi-continuous at any point in $D(f)$. For any $\mu > 0$, the resolvent of f in X is defined in [7] as

$$(1.3) \quad J_\mu^f(x) = \arg \min_{y \in X} \left[f(y) + \frac{1}{2\mu} d^2(y, x) \right].$$

It was established in [32] that J_μ^f is well defined and that it is a non-expansive mapping for all $\mu > 0$. For simplicity, we shall write J_μ for the resolvent of a proper convex and lower semi-continuous function f . The approximation of solutions of Minimization Problem (MP) has been of great interest in optimization theory, nonlinear analysis and geometry, see [1, 24]. The MP is defined as follows: Find $x \in X$ such that

$$(1.4) \quad f(x) = \min_{y \in X} f(y).$$

We denote the solution set of problem (1.4) by $\operatorname{argmin}_{y \in X} f(y)$. In [7], it was shown that $F(J_\mu)$ coincides with $\operatorname{argmin}_{y \in X} f(y)$.

The Proximal Point Algorithm (PPA) is known to be one of the most effective and successful method for solving the MP (1.4). The PPA was introduced by Martinet [43] in 1970 and was further developed by Rockafellar [52] in Hilbert spaces as follows. Let f be a proper convex and lowersemi-continuous function defined on a Hilbert space H . The PPA is defined for arbitrary $x_1 \in H$ by

$$(1.5) \quad x_{n+1} = \arg \min_{y \in H} \left(f(y) + \frac{1}{2\mu_n} \|y - x_n\|^2 \right),$$

for each $n \geq 1$, where $\mu_n > 0$ for all $n \geq 1$. It was proved in [52] that if f has a minimizer in H and $\sum_{n=1}^{\infty} \mu_n = \infty$, then the sequence $\{x_n\}$ converges weakly to a minimizer of f (see also [13, 46]). Güler [23] gave an example to show that the PPA does not necessarily converges strongly in general. To overcome this challenge, Kamimura and Takahashi [34] combined the PPA with the Halpern's algorithm, resulting to the following Halpern-type PPA, for which they obtained a strong convergence result for approximating solution of (1.4).

$$(1.6) \quad \begin{cases} u, x_0 \in H, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\mu_n} x_n. \end{cases}$$

The PPA was later introduced and studied in CAT(0) spaces by Bačák [9] for approximating a solution of MP in CAT(0) spaces. He proposed the following algorithm: For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ by

$$(1.7) \quad x_{n+1} = \arg \min_{y \in X} \left(f(y) + \frac{1}{2\mu_n} d^2(y, x_n) \right),$$

for each $n \geq 1$, where $\mu_n > 0$ for all $n \geq 1$. Under the conditions that f has a minimizer in X and $\sum_{n=1}^{\infty} \mu_n = \infty$, he proved that $\{x_n\}$ Δ -converges to a minimizer of f . In 2014, Bačák [8] studied a split version of the PPA for minimizing sum of convex functions in complete CAT(0) spaces. Very recently, Suparatulatorn *et al.* [54] propose the following Halpern-type PPA for approximating a common solution of a MP and fixed point problem for nonexpansive mapping in CAT(0) spaces:

$$(1.8) \quad \begin{cases} u, x_1 \in X, \\ y_n = \arg \min_{y \in X} \left[f(y) + \frac{1}{2\mu_n} d^2(y, x_n) \right], \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\mu_n \geq \mu > 0$. They obtained a strong convergence result

under some mild conditions. For more results on PPA for MP, see [7, 10, 24, 59] and the references therein.

In 2008, Breg and Nikolaev [11] introduced the concept of quasilinearization mapping in CAT(0) spaces. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ which they called a vector and defined a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow \mathbf{R}$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} \left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \right), \quad (a, b, c, d \in X)$$

called the quasilinearization mapping. It is easy to verify that

$$\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b), \quad \langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle, \quad \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$$

and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a, b, c, d, e \in X$. It has been established that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality (see [11]). Recall that the space X is said to satisfy the Cauchy-Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \forall a, b, c, d \in X$.

In 2010, Ahmadi Kakavandi and Amini [3] introduced the concept of dual space in a complete CAT(0) space X by first introducing the concept of pseudometric space which they defined as the space $\mathbf{R} \times X \times X$ together with a pseudometric D defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (t, s \in \mathbf{R}, a, b, c, d \in X),$$

where $\Theta : \mathbf{R} \times X \times X \rightarrow C(X, \mathbf{R})$ is defined as $\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle$, ($t \in \mathbf{R}, a, b, x \in X$) and $L(\varphi) = \sup \left\{ \frac{\varphi(x) - \varphi(y)}{d(x, y)} : x, y \in X, x \neq y \right\}$ for any function $\varphi : X \rightarrow \mathbf{R}$. Furthermore, D defines an equivalence relation on $\mathbf{R} \times X \times X$, where the equivalence class of (t, a, b) is

$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0\}$. Then, $X^* = \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbf{R} \times X \times X\}$ is a metric space with D as the metric. The pair (X^*, D) is called the dual space of (X, d) . The dual space X^* acts on $X \times X$ by $\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle$, $x^* = [t\overrightarrow{ab}] \in X^*$, $x, y \in X$ (see [36]). It was shown in [3] that the dual of a closed and convex subset of a Hilbert space H with nonempty interior is H and $t(b - a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbf{R}$, $a, b \in H$.

Let X be a complete CAT(0) space and X^* be its dual space. A multivalued operator $A : X \rightarrow 2^{X^*}$ with domain $\mathbf{D}(A) := \{x \in X : Ax \neq \emptyset\}$ is monotone if and only if for all $x, y \in \mathbf{D}(A)$, $x^* \in Ax$, $y^* \in Ay$, we have

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0 \quad (\text{see [36]}).$$

The resolvent of the operator A of order $\lambda > 0$ is the multivalued mapping $J_\lambda^A : X \rightarrow 2^X$ defined in [36] as

$$J_\lambda^A(x) := \{z \in X \mid [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}.$$

The operator A satisfies the range condition if for every $\lambda > 0$, $\mathbf{D}(J_\lambda^A) = X$ (see [36]). For simplicity, we shall write J_λ for the resolvent of a monotone operator A .

Monotone operator theory is an area of research in nonlinear and convex analysis that has continued to attract the interest of many researchers due to the role it plays in diverse mathematical problems such as optimization problems, equilibrium problems, variational inequality problems, saddle point problems, among others (see [2, 29, 30, 31, 47, 55] and the references therein). Many of these problems can be modelled as a Monotone Inclusion Problem (MIP) which is considered the most important problems in monotone operator theory (see [36, 50] and the references therein). Let $A : X \rightarrow 2^{X^*}$ be a monotone operator, the MIP is

to find

$$(1.9) \quad x \in \mathbf{D}(A) \text{ such that } 0 \in Ax.$$

The solution set of problem (1.9) is denoted by $A^{-1}(0)$, which is known to be closed and convex (see [50], Remark 3.1).

The PPA is one of the most popular method for finding solution of (1.9). In 1976, Rockafellar [52] studied the following PPA for approximating a solution of MIP (1.9) in Hilbert space H , for $x_0 \in H$

$$(1.10) \quad x_{n-1} - x_n \in \lambda_n A(x_n),$$

where $\{\lambda_n\}$ is a sequence of positive real numbers. Rockafellar [52] proved that the sequence $\{x_n\}$ generated by Algorithm (1.10) is weakly convergent to a solution of (1.9), provided $\lambda_n \geq \lambda > 0$ for each $n \geq 1$. Since then, numerous authors have developed and studied different methods for finding solutions of (1.9) in Hilbert spaces and Banach spaces (see [15, 34, 35, 48] and the references therein). Recently, researchers are beginning to study MIP in CAT(0) spaces. In 2016, Khatibzadeh and Ranjbar [36] introduced and studied the following PPA in CAT(0) spaces for approximating a solution of MIP, for which they obtained both strong and Δ -convergence result, using the algorithm

$$(1.11) \quad \begin{cases} x_0 \in X, \\ x_n = J_{\lambda_n} x_{n-1}, \end{cases}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Very recently, Ranjbar and Khatibzadeh [50] proposed the following Mann-type and Halpern-type PPA in complete CAT(0) spaces for finding a solution of MIP:

$$(1.12) \quad \begin{cases} x_0 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J_{\lambda_n} x_n \end{cases}$$

and

$$(1.13) \quad \begin{cases} u, x_0 \in X, \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J_{\lambda_n} x_n, \end{cases}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$. They obtained a Δ -convergence result using the Mann-type PPA while they obtained a strong convergence result using the Halpern-type PPA.

In this paper, we propose the Ishikawa-type PPA for approximating a common solution of finite family of MP, finite family of MIP and fixed point problem for asymptotically demicontractive mapping in Hadamard spaces. Our results extend and complement the results of Kim [37], Bačák [9], Suparatulatorn *et al.* [54], Khatibzadeh and Ranjbar [36] and Ranjbar and Khatibzadeh [50].

2. Preliminary

We state some known and useful results which will be needed in the proof of our main results.

Let X be a CAT(0) space. Then, for all $x, y, z \in X$ and $t, s \in [0, 1]$, the following hold:

- (i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$, (see [47, 59]),
- (ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$, (see [47, 59]),
- (iii) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s|d(x, y)$, (see [16]),
- (iv) $d(tx \oplus (1-t)y, tx \oplus (1-t)z) \leq (1-t)d(y, z)$, (see [38]).

[19] Let X be a CAT(0) space and $a, b, c \in X$. Then for each $t \in [0, 1]$,

$$d^2(ta \oplus (1-t)b, c) \leq t^2d^2(a, c) + (1-t)^2d^2(b, c) + 2t(1-t)\langle \overrightarrow{ac}, \overrightarrow{bc} \rangle.$$

[59, 41] Every bounded sequence in a complete CAT(0) space has a Δ -convergence subsequence.

[33] Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ -converges to x if and only if $n \rightarrow \infty \limsup \langle \overrightarrow{x_n x}, \overrightarrow{y x} \rangle \leq 0 \forall y \in X$.

[20] Let X be a complete CAT(0) space and $T : X \rightarrow X$ be a non-expansive mapping, then the conditions that $\{x_n\}$ Δ -converges to x and $\{d(x_n, Tx_n)\}$ converges strongly to 0, implies that $x = Tx$.

[25] Let C be a closed and convex subset of a complete CAT(0) space X and $T : C \rightarrow C$ be a uniformly L -Lipschitzian and $(\{\mu\}, \{v_n\}, \phi)$ -total asymptotically demicontractive mapping with $F(T) = \emptyset$. If $\{x_n\}$ is a bounded sequence in C such that $\Delta\text{-}n \rightarrow \infty \lim x_n = p$ and $n \rightarrow \infty \lim d(x_n, Tx_n) = 0$. Then $TP = p$.

Remark 2.1. If $\phi(\lambda) = \lambda^2$ and $v_n = 0$ for each $n \geq 1$, then in Lemma 2.6, T is an asymptotically demicontractive mapping.

[62]. Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

- (i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,
- (ii) $\limsup_{n \rightarrow \infty} \delta_n \leq 0$,
- (iii) $\gamma_n \geq 0$ ($n \geq 0$), $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

[44]. Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1} \forall j \in \mathbf{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbf{N}$ such that $m_k \rightarrow \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbf{N}$:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

[40] Let (X, d) be a complete CAT(0) space and $f : X \rightarrow (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in X$ and $\mu > 0$, we have

$$(2.1) \quad \frac{1}{2\mu} d^2(J_\mu x, y) - \frac{1}{2\mu} d^2(x, y) + \frac{1}{2\mu} d^2(x, J_\mu x) + f(J_\mu x) \leq f(y).$$

Definition 2.2. Let X be a complete CAT(0) space and C be a nonempty closed and convex subset of X . A mapping $T : C \rightarrow C$ is said to be a firmly nonexpansive mapping (see [36]) if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \quad \forall x, y \in C.$$

From Cauchy-Schwartz inequality, it is clear that the class of nonexpansive mappings is more general than the class of firmly nonexpansive mappings (see [36]).

[36] Let X be a CAT(0) space and J_λ be the resolvent of a multivalued operator $A : X \rightarrow 2^{X^*}$ of order λ . Then, we have the following;

- (i) for any $\lambda > 0$, $F(J_\lambda) = A^{-1}(0)$,
- (ii) if A is monotone, then J_λ is a single-valued and a nonexpansive mapping.

3. Main Results

We make the following remark which is a consequence of Theorem 2.12.

Remark 3.1. *If X is a $CAT(0)$ space and J_λ is the resolvent of a monotone operator $A : X \rightarrow 2^{X^*}$ of order $\lambda > 0$, then*

$$d^2(u, J_\lambda x) + d^2(J_\lambda x, x) \leq d^2(u, x),$$

for all $u \in A^{-1}(0)$ and $x \in \mathbf{D}(J_\lambda)$.

Proof. Indeed for any $u \in A^{-1}(0)$, $x \in \mathbf{D}(J_\lambda)$ and $\lambda > 0$, we obtain from Theorem 2.12 and Definition 2.11 that

$$\begin{aligned} d^2(J_\lambda x, u) &\leq \langle \overrightarrow{J_\lambda x u}, \overrightarrow{xu} \rangle \\ &= \frac{1}{2} \left(d^2(J_\lambda x, u) + d^2(u, x) - d^2(J_\lambda x, x) \right), \end{aligned}$$

which implies

$$d^2(u, J_\lambda x) + d^2(J_\lambda x, x) \leq d^2(u, x).$$

□

Let X be a Hadamard space and X^* be its dual space. For each $i = 1, 2, \dots, N$, let $A_i : X \rightarrow 2^{X^*}$ be multivalued monotone mappings that satisfy the range condition and let J_λ^i be the resolvent of A_i for any $\lambda > 0$. For each $j = 1, 2, \dots, m$, let $f_j : X \rightarrow (-\infty, \infty]$ be proper convex and lower semi continuous functions and J_μ^j be the resolvent of f_j for any $\mu > 0$. Then,

$$F(\Phi_\lambda^N \circ \Psi_\mu^m) = \left(\cap_{i=1}^N A_i^{-1}(0) \right) \cap \left(\cap_{j=1}^m \arg \min_{y \in X} f_j(y) \right),$$

where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \dots \circ J_\lambda^2 \circ J_\lambda^1$, $\Phi_\lambda^0 = I$ and $\Psi_\mu^m = J_\mu^N \circ J_\mu^{m-1} \circ \dots \circ J_\mu^2 \circ J_\mu^1$, $\Psi_\mu^0 = I$.

Proof. Observe that $\left(\cap_{i=1}^N A_i^{-1}(0) \right) \cap \left(\cap_{j=1}^m \arg \min_{y \in X} f_j(y) \right) = F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \dots \cap F(J_\lambda^2) \cap F(J_\lambda^1) \cap F(J_\mu^m) \cap F(J_\mu^{m-1}) \cap \dots \cap F(J_\mu^2) \cap F(J_\mu^1)$. Clearly, $\left(\cap_{i=1}^N A_i^{-1}(0) \right) \cap \left(\cap_{j=1}^m \arg \min_{y \in X} f_j(y) \right) \subseteq F(\Phi_\lambda^N \circ \Psi_\mu^m)$.

We now show that $F(\Phi_\lambda^N \circ \Psi_\mu^m) \subseteq \left(\cap_{i=1}^N A_i^{-1}(0) \right) \cap \left(\cap_{j=1}^m \arg \min_{y \in X} f_j(y) \right)$.

Let $x \in F(\Phi_\lambda^N \circ \Psi_\mu^m)$ and $y \in \left(\left(\cap_{i=1}^N A_i^{-1}(0) \right) \cap \left(\cap_{j=1}^m \arg \min_{y \in X} f_j(y) \right) \right)$. Then,

$$\begin{aligned}
 d^2(x, y) &= d^2(\Phi_\lambda^N \Psi_\mu^m x, \Phi_\lambda^N \Psi_\mu^m y) \\
 &\leq d^2(\Phi_\lambda^{N-1} \Psi_\mu^m x, \Phi_\lambda^{N-1} \Psi_\mu^m y) \\
 (3.1) \qquad &= d^2(\Phi_\lambda^{N-1} \Psi_\mu^m x, y).
 \end{aligned}$$

From Remark 3.1, we obtain that

$$\begin{aligned}
 d^2(J_\lambda^N(\Phi_\lambda^{N-1} \Psi_\mu^m x), \Phi_\lambda^{N-1} \Psi_\mu^m x) &\leq d^2(\Phi_\lambda^{N-1} \Psi_\mu^m x, y) - d^2(J_\lambda^N(\Phi_\lambda^{N-1} \Psi_\mu^m x), y) \\
 &\vdots \\
 &\leq d^2(x, y) - d^2(\Phi_\lambda^N \Psi_\mu^m x, y) \\
 &= d^2(x, y) - d^2(x, y),
 \end{aligned}$$

which implies

$$(3.2) \qquad \Phi_\lambda^N \Psi_\mu^m x = \Phi_\lambda^{N-1} \Psi_\mu^m x.$$

From Remark 3.1 and (3.1), we obtain

$$\begin{aligned}
 d^2(J_\lambda^{N-1}(\Phi_\lambda^{N-2} \Psi_\mu^m x), \Phi_\lambda^{N-2} \Psi_\mu^m x) &\leq d^2(\Phi_\lambda^{N-2} \Psi_\mu^m x, y) - d^2(J_\lambda^{N-1}(\Phi_\lambda^{N-2} \Psi_\mu^m x), y) \\
 &\vdots \\
 &\leq d^2(x, y) - d^2(\Phi_\lambda^{N-1} \Psi_\mu^m x, y) \\
 &\leq d^2(\Phi_\lambda^{N-1} \Psi_\mu^m x, y) - d^2(\Phi_\lambda^{N-1} \Psi_\mu^m x, y),
 \end{aligned}$$

which implies

$$(3.3) \qquad \Phi_\lambda^{N-1} \Psi_\mu^m x = \Phi_\lambda^{N-2} \Psi_\mu^m x,$$

where the last inequality follows from (3.1).

Also, from Remark 3.1 and (3.1), we have

$$\begin{aligned}
d^2(\Phi_\lambda^{N-2}\Psi_\mu^m x, \Phi_\lambda^{N-3}\Psi_\mu^m x) &\leq d^2(\Phi_\lambda^{N-3}\Psi_\mu^m x, y) - d^2(\Phi_\lambda^{N-2}\Psi_\mu^m x, y) \\
&\vdots \\
&\leq d^2(x, y) - d^2(\Phi^{N-2}\Psi_\mu^m x, y) \\
&\leq d^2(\Phi_\lambda^{N-1}\Psi_\mu^m x, y) - d^2(\Phi_\lambda^{N-1}\Psi_\mu^m x, y),
\end{aligned}$$

which implies

$$(3.4) \quad \Phi_\lambda^{N-2}\Psi_\mu^m x = \Phi_\lambda^{N-3}\Psi_\mu^m x.$$

Continuing in this manner, we obtain that

$$\Phi_\lambda^{N-3}\Psi_\mu^m x = \Phi_\lambda^{N-4}\Psi_\mu^m x = \dots = \Phi_\lambda^2\Psi_\mu^m x = \Phi_\lambda^1\Psi_\mu^m x = \Phi_\lambda^0\Psi_\mu^m x = \Psi_\mu^m x.$$

(3.5)

Furthermore, we obtain from Lemma 2.10 and (3.1) that

$$\frac{1}{2\mu}d^2(\Psi_\mu^m x, y) - \frac{1}{2\mu}d^2(\Psi_\mu^{m-1}x, y) + \frac{1}{2\mu}d^2(\Psi_\mu^{m-1}x, \Psi_\mu^m x) + f(\Psi_\mu^m x) \leq f(y).$$

Since $f(y) \leq f(\Psi_\mu^m x)$, we have that

$$\begin{aligned}
d^2(\Psi_\mu^m x, \Psi_\mu^{m-1}x) &\leq d^2(\Psi_\mu^{m-1}x, y) - d^2(\Psi_\mu^m x, y) \\
&\vdots \\
&\leq d^2(x, y) - d^2(\Phi_\lambda^{N-1}\Psi_\mu^m x, y) \\
&\leq d^2(\Phi_\lambda^{N-1}\Psi_\mu^m x, y) - d^2(\Phi_\lambda^{N-1}\Psi_\mu^m x, y),
\end{aligned}$$

which implies

$$(3.6) \quad \Psi_\mu^m x = \Psi_\mu^{m-1}x.$$

Following similar argument as above, we obtain that

$$(3.7) \quad \Psi_\mu^{m-1}x = \Psi_\mu^{m-2}x = \Psi_\mu^{m-3}x = \dots = \Psi_\mu^2x = \Psi_\mu^1x = \Psi_\mu^0x = x.$$

Hence, from (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7), we have

$$(3.8) \quad \begin{aligned} \Phi_\lambda^N \Psi_\mu^m x &= \Phi_\lambda^{N-1} \Psi_\mu^m x = \dots = \Phi_\lambda^1 \Psi_\mu^m x = \Psi_\mu^m x = \Psi_\mu^{m-1} x \\ &= \Psi_\mu^{m-2} x = \dots = \Psi_\mu^2 x = \Psi_\mu^1 x = x. \end{aligned}$$

From (3.8), we obtain

$$(3.9) \quad x = J_\mu^1 x.$$

From (3.8) and (3.9), we obtain

$$(3.10) \quad x = \Psi_\mu^2 x = J_\mu^2(J_\mu^1 x) = J_\mu^2 x.$$

Continuing in this manner, we obtain

$$(3.11) \quad x = J_\mu^1 x = J_\mu^2 x = \dots = J_\mu^{m-1} x = J_\mu^m x = J_\lambda^1 x = \dots = J_\lambda^{N-1} x = J_\lambda^N x.$$

Thus, it follows from (3.11) that

$$F(\Phi_\lambda^N \circ \Psi_\mu^m) \subseteq \left(\cap_{i=1}^N A_i^{-1}(0) \right) \cap \left(\cap_{j=1}^m \arg \min_{y \in X} f_j(y) \right),$$

which completes the proof. $\square$

Let X be a Hadamard space and X^* be its dual space. Let $A_i : X \rightarrow 2^{X^*}$, $i = 1, 2, \dots, N$ be multivalued monotone mappings that satisfy the range condition and $f_j : X \rightarrow (-\infty, \infty]$, $j = 1, 2, \dots, m$ be proper convex and lower semi continuous functions. Let $T : X \rightarrow X$ be a uniformly L -Lipschitzian and asymptotically demicontractive mapping with constant $k \in (0, 1)$ and sequence $\{k_n\}_{n=1}^\infty \subseteq [1, \infty)$ such that $\sum_{n=1}^\infty (k_n^2 - 1) < \infty$.

Suppose that $\Gamma := F(T) \cap \left(\bigcap_{i=1}^N A_i^{-1}(0)\right) \cap \left(\bigcap_{j=1}^m \arg \min_{y \in X} f_j(y)\right) \neq \emptyset$.
Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by

$$(3.12) \quad \begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = \Phi_\lambda^N \circ \Psi_\mu^m(v_n), \\ y_n = (1 - \beta_n)u_n \oplus \beta_n T^n u_n, \\ x_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{cases}$$

where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \dots \circ J_\lambda^2 \circ J_\lambda^1$, $\Phi_\lambda^0 = I$, $\Psi_\mu^m = J_\mu^N \circ J_\mu^{m-1} \circ \dots \circ J_\mu^2 \circ J_\mu^1$, $\Psi_\mu^0 = I$, $\lambda, \mu \in (0, \infty)$ and $\{t_n\}, \{\beta_n\}, \{\alpha_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

C1: $n \rightarrow \infty \lim t_n = 0$,

C2: $\sum_{n=1}^\infty t_n = \infty$,

C3: $0 < \epsilon \leq \alpha_n \leq k\beta_n < \beta_n \leq b < \frac{2}{\left(\frac{k+\delta}{k}\right) + \sqrt{\left(\frac{k+\delta}{k}\right)^2 + 4L^2}}$, where $\delta := n \geq 1 \sup k_n$,

C4: $n \rightarrow \infty \lim \frac{k_n^2 - 1}{t_n} = 0$.

Then $\{x_n\}$ converges strongly to an element of Γ .

Proof. We first observe that since T is uniformly L -Lipschitzian, we obtain from (3.12) that

$$(3.13) \quad \begin{aligned} d(T^n u_n, T^n y_n) &\leq Ld(u_n, y_n) \\ &= Ld(u_n, (1 - \beta_n)u_n \oplus \beta_n T^n u_n) \\ &\leq L\beta_n d(u_n, T^n u_n). \end{aligned}$$

Let $p \in \Gamma$, then from (3.12) and (3.13), we have

$$\begin{aligned} d^2(T^n y_n, p) &\leq k_n d^2(y_n, p) + k d^2(y_n, T^n y_n) \\ &= k_n d^2((1 - \beta_n)u_n \oplus \beta_n T^n u_n, p) + k d^2(y_n, T^n y_n) \\ &\leq k_n(1 - \beta_n) d^2(u_n, p) + k_n \beta_n d^2(T^n u_n, p) - k_n \beta_n(1 - \beta_n) d^2(u_n, T^n u_n) \\ &\quad + k d^2((1 - \beta_n)u_n \oplus \beta_n T^n u_n, T^n y_n) \\ &\leq k_n(1 - \beta_n) d^2(u_n, p) + k_n \beta_n d^2(T^n u_n, p) - k_n \beta_n(1 - \beta_n) d^2(u_n, T^n u_n) \end{aligned}$$

$$\begin{aligned}
& +k(1-\beta_n)d^2(u_n, T^n y_n) + k\beta_n d^2(T^n u_n, T^n y_n) - k\beta_n(1-\beta_n)d^2(T^n u_n, u_n) \\
\leq & k_n(1-\beta_n)d^2(u_n, p) + k_n\beta_n d^2(T^n u_n, p) - k_n\beta_n(1-\beta_n)d^2(u_n, T^n u_n) \\
& +k(1-\beta_n)d^2(u_n, T^n y_n) + kL^2\beta_n^3 d^2(u_n, T^n u_n) - k\beta_n(1-\beta_n)d^2(T^n u_n, u_n) \\
\leq & k_n(1-\beta_n)d^2(u_n, p) + k_n\beta_n \left[k_n d^2(u_n, p) + k d^2(u_n, T^n u_n) \right] - k_n\beta_n(1-\beta_n)d^2(u_n, T^n u_n) \\
& +k(1-\beta_n)d^2(u_n, T^n y_n) + kL^2\beta_n^3 d^2(u_n, T^n u_n) - k\beta_n(1-\beta_n)d^2(T^n u_n, u_n) \\
= & (k_n - k_n\beta_n + k_n^2\beta_n)d^2(u_n, p) - \beta_n \left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n \right] d^2(u_n, T^n u_n) \\
& +k(1-\beta_n)d^2(u_n, T^n y_n) \\
\leq & \left[k_n^2(1-\beta_n) + k_n^2\beta_n \right] d^2(u_n, p) + k(1-\beta_n)d^2(u_n, T^n y_n) \\
& -\beta_n \left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n \right] d^2(u_n, T^n u_n) \\
= & k_n^2 d^2(u_n, p) + k(3.14)\beta_n d^2(u_n, T^n y_n) - \beta_n \left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n \right] d^2(u_n, T^n u_n)
\end{aligned}$$

Also, from (3.12), (3.14) and condition C3 , we obtain

$$\begin{aligned}
d^2(x_{n+1}, p) & \leq (1-\alpha_n)d^2(u_n, p) + \alpha_n d^2(T^n y_n, p) - \alpha_n(1-\alpha_n)d^2(u_n, T^n y_n) \\
& \leq (1-\alpha_n)d^2(u_n, p) + \alpha_n k_n^2 d^2(u_n, p) + k\alpha_n(1-\beta_n)d^2(u_n, T^n y_n) \\
& \quad -\alpha_n\beta_n \left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n \right] d^2(u_n, T^n u_n) - \alpha_n(1-\alpha_n)d^2(u_n, T^n y_n) \\
& \leq k_n^2 d^2(u_n, p) - \alpha_n \left[(1-k) + (k\beta_n - \alpha_n) \right] d^2(u_n, T^n y_n) \\
& \quad -\alpha_n\beta_n \left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n \right] d^2(u_n, T^n u_n) \\
& \leq k_n^2 d^2(u_n, p) - \alpha_n\beta_n \left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n \right] d^2(u_n, T^n u_n) \\
& \leq \left[1 + (k_n^2 - 1) \right] d^2(u_n, p) \\
& = \left[1 + (k_n^2 - 1) \right] d^2(\Phi_\lambda^N \Psi_\mu^m v_n, p) \\
& \leq \left[1 + (k_n^2 - 1) \right] d^2(\Phi_\lambda^{N-1} \Psi_\mu^m v_n, p) \\
& \vdots \\
& \leq \left[1 + (k_n^2 - 1) \right] d^2(\Psi_\mu^m v_n, p) \\
& \leq \left[1 + (k_n^2 - 1) \right] d^2(\Psi_\mu^{m-1} v_n, p) \\
& \vdots \\
& \leq \left[1 + (k_n^2 - 1) \right] d^2(v_n, p) \\
& \leq \left[1 + (k_n^2 - 1) \right] \left[(1-t_n)d^2(x_n, p) + t_n d^2(u, p) \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \left[1 + (k_n^2 - 1)\right] \max\{d^2(x_n, p), d^2(u, p)\} \\
&\vdots \\
&\leq \prod_{r=1}^n \left[1 + (k_r^2 - 1)\right] \max\{d^2(x_1, p), d^2(u, p)\}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$, then we have that $\{x_n\}$ is bounded. Consequently, $\{u_n\}, \{v_n\}$ are all bounded.

From (3.12) and condition C1, we have that

$$(3.17) \quad \lim_{n \rightarrow \infty} d(v_n, x_n) \leq \lim_{n \rightarrow \infty} t_n d(u, x_n) = 0.$$

We now divide our proof into two cases.

Case 1: Suppose that $\{d^2(x_n, p)\}$ is monotonically non-increasing, then $n \rightarrow \infty \lim\{d^2(x_n, p)\}$ exists. Consequently,

$$(3.18) \quad \lim_{n \rightarrow \infty} \left[d^2(x_{n+1}, p) - d^2(x_n, p) \right] = 0.$$

From Remark 3.1 and (3.15), we obtain

$$\begin{aligned}
d^2(u_n, \Phi_\lambda^{N-1} \Psi^m v_n) &= d^2(J_\lambda^N(\Phi_\lambda^{N-1} \Psi^m v_n), \Phi_\lambda^{N-1} \Psi^m v_n) \\
&\leq d^2(\Phi_\lambda^{N-1} \Psi_\mu^m v_n, p) - d^2(J_\lambda^N(\Phi_\lambda^{N-1} \Psi^m v_n), p) \\
&\vdots \\
&\leq d^2(v_n, p) - \frac{1}{k_n^2} d^2(x_{n+1}, p) \\
&\leq d^2(v_n, x_n) + d^2(x_n, p) + 2d(v_n, x_n)d(x_n, p) - \frac{1}{k_n^2} d^2(x_{n+1}, p) \\
(3.19) \quad &= d^2(v_n, x_n) + 2d(v_n, x_n)d(x_n, p) + \left[d^2(x_n, p) - \frac{1}{k_n^2} d^2(x_{n+1}, p) \right].
\end{aligned}$$

Since $n \rightarrow \infty \lim k_n = 1$ and $n \rightarrow \infty \lim d(x_n, p)$ exists, we have from (3.17) and (3.19) that

$$(3.20) \quad \lim_{n \rightarrow \infty} d^2(u_n, \Phi_\lambda^{N-1} \Psi^m v_n) = 0.$$

Again, from Lemma 2.1 and Remark 3.1, we obtain

$$\begin{aligned}
d^2(\Phi_\lambda^{N-1}\Psi_\mu^m v_n, \Phi_\lambda^{N-2}\Psi_\mu^m v_n) &\leq d^2(\Phi_\lambda^{N-2}\Psi_\mu^m v_n, p) - d^2(\Phi_\lambda^{N-1}\Psi_\mu^m v_n, p) \\
&\vdots \\
&\leq d^2(v_n, p) - d^2(u_n, p) \\
&\leq d^2(v_n, p) - \frac{1}{k_n^2}d^2(x_{n+1}, p) \\
&\leq d^2(v_n, x_n) + 2d(v_n, x_n)d(x_n, p) \\
(3.21) \qquad &+ \left[d^2(x_n, p) - \frac{1}{k_n^2}d^2(x_{n+1}, p) \right] \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

Continuing in the same manner, we have that

$$\lim_{n \rightarrow \infty} d^2(\Phi_\lambda^{N-2}\Psi_\mu^m v_n, \Phi_\lambda^{N-3}\Psi_\mu^m v_n) = \dots = \lim_{n \rightarrow \infty} d^2(\Phi_\lambda^1\Psi_\mu^m v_n, \Psi_\mu^m v_n) = 0.$$

Also, from Lemma 2.10, we have

$$\frac{1}{2\mu}d^2(\Psi_\mu^m v_n, p) - \frac{1}{2\mu}d^2(\Psi_\mu^{m-1}v_n, p) + \frac{1}{2\mu}d^2(\Psi_\mu^{m-1}v_n, \Psi_\mu^m p) + f(\Psi_\mu^m v_n) \leq f(p).$$

Since $f(p) \leq f(\Psi_\mu^m v_n)$, we have that

$$\begin{aligned}
d^2(\Psi_\mu^m v_n, \Psi_\mu^{m-1}v_n) &\leq d^2(\Psi_\mu^{m-1}v_n, p) - d^2(\Psi_\mu^m v_n, p) \\
&\vdots \\
&\leq d^2(v_n, p) - d^2(u_n, p) \\
&\leq d^2(v_n, p) - \frac{1}{k_n^2}d^2(x_{n+1}, p) \\
&\leq d^2(v_n, x_n) + 2d(v_n, x_n)d(x_n, p) \\
(3.23) \qquad &+ \left[d^2(x_n, p) - \frac{1}{k_n^2}d^2(x_{n+1}, p) \right] \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

Using similar argument as above, we can show that

$$\lim_{n \rightarrow \infty} d^2(\Psi_\mu^{m-1}v_n, \Psi_\mu^{m-2}v_n) = \dots = \lim_{n \rightarrow \infty} d^2(\Psi_\mu^2 v_n, \Psi_\mu^1 v_n) = \lim_{n \rightarrow \infty} d^2(\Psi_\mu^1 v_n, v_n) = 0.$$

Thus,

$$\begin{aligned}
d(u_n, v_n) \leq & d(u_n, \Phi_\lambda^{N-1} \Psi_\mu^m v_n) + d(\Phi_\lambda^{N-1} \Psi_\mu^m v_n, \Phi_\lambda^{N-2} \Psi_\mu^m v_n) \\
& + \dots + d(\Phi_\lambda^1 \Psi_\mu^m v_n, \Psi_\mu^m v_n) + d(\Psi_\mu^m v_n, \Psi_\mu^{m-1} v_n) \\
& + d(\Psi_\mu^{m-1} v_n, \Psi_\mu^{m-2} v_n) + \dots + d(\Psi_\mu^1 v_n, v_n),
\end{aligned}$$

which implies from (3.20), (3.21), (3.22), (3.23) and (3.24), that

$$(3.25) \quad \lim_{n \rightarrow \infty} d(u_n, v_n) = \lim_{n \rightarrow \infty} d(\Phi_\lambda^N \Psi_\mu^m v_n, v_n) = 0.$$

From (3.17) and (3.25), we obtain

$$(3.26) \quad \lim_{n \rightarrow \infty} d(u_n, x_n) = 0.$$

Since $\{d^2(x_n, p)\}$ and $\{k_n\}$ are bounded, then there exists $M > 0$ such that

$$M := n \geq 1 \sup\{d^2(x_n, p), k_n^2\}.$$

Thus, from (3.15), we obtain

$$\begin{aligned}
& d^2(x_{n+1}, p) - d^2(x_n, p) \\
\leq & k_n^2 d^2(u_n, p) - d^2(x_n, p) \\
& - \alpha_n \beta_n \left[(1 - \beta_n)(k_n + k) - kL^2 \beta_n^2 - k k_n \right] d^2(u_n, T^n u_n) \\
\leq & k_n^2 d^2(u_n, x_n) + k_n^2 d^2(x_n, p) + 2k_n^2 d(u_n, x_n) d(x_n, p) - d^2(x_n, p) \\
& - \alpha_n \beta_n \left[(1 - \beta_n)(k_n + k) - kL^2 \beta_n^2 - k k_n \right] d^2(u_n, T^n u_n) \\
\leq & M d^2(u_n, x_n) + 2M^2 d(u_n, x_n) + (k_n^2 - 1)M \\
(3.27) \quad & - \alpha_n \beta_n \left[(1 - \beta_n)(k_n + k) - kL^2 \beta_n^2 - k k_n \right] d^2(u_n, T^n u_n).
\end{aligned}$$

Since $\sum_{n=1}^\infty (k_n^2 - 1) < \infty$, then $n \rightarrow \infty \lim(k_n^2 - 1) = 0$. Thus, letting

$$\delta_n = \alpha_n \beta_n \left[(1 - \beta_n)(k_n + k) - kL^2 \beta_n^2 - k k_n \right],$$

we obtain from (3.27) and (3.26) that

$$\begin{aligned}
\delta_n d^2(u_n, T^n u_n) & \leq M d^2(u_n, x_n) + 2M^2 d(u_n, x_n) + M(k_n^2 - 1) \\
(3.28) \quad & + d^2(x_n, p) - d^2(x_{n+1}, p) \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

From condition C3, we obtain that $2k - b(k + \delta) > kb\sqrt{\left(\frac{k+\delta}{k}\right)^2 + 4L^2}$. This implies that

$$[2k - b(k + \delta)]^2 > 4L^2k^2b^2 + b^2(k + \delta)^2,$$

which further implies that

$$k - bk - b\delta - kL^2b^2 > 0.$$

Thus,

$$\begin{aligned} \delta_n &= \alpha_n\beta_n \left[(k_n + k) - kk_n - \beta_n(k + k_n) - kL^2\beta_n^2 \right] \\ &> \epsilon^2 \left[(k_n + k) - k_n - \beta_n(k + k_n) - kL^2\beta_n^2 \right] \\ &> \epsilon^2 \left[k - bk - bk_n - kL^2b^2 \right] \\ &> \epsilon^2 \left[k - bk - b\delta - kL^2b^2 \right] > 0. \end{aligned}$$

Hence, from (3.28), we obtain

$$(3.29) \quad \lim_{n \rightarrow \infty} d(u_n, T^n u_n) = 0.$$

From (3.12) and condition C3, we obtain

$$(3.30) \quad \lim_{n \rightarrow \infty} d(y_n, u_n) \leq \lim_{n \rightarrow \infty} \beta_n d(T^n u_n, u_n) = 0.$$

Since T is uniformly L -Lipschitzian, we obtain from (3.29) and (3.30) that

$$\begin{aligned} d(x_{n+1}, y_n) &\leq (1 - \alpha_n)d(u_n, y_n) + \alpha_n d(T^n y_n, y_n) \\ &\leq (1 - \alpha_n)d(u_n, y_n) + \alpha_n d(T^n y_n, T^n u_n) \\ &\quad + \alpha_n [d(T^n u_n, u_n) + d(y_n, u_n)] \\ &\leq (1 - \alpha_n)d(u_n, y_n) + \alpha_n L d(y_n, u_n) \\ (3.31) \quad &+ \alpha_n [d(T^n u_n, u_n) + d(y_n, u_n)] \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

From (3.26), (3.30) and (3.31), we have

$$(3.32) \quad \lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0.$$

Again, since T is uniformly L -Lipschitzian, we obtain

$$\begin{aligned} d(u_n, Tu_n) &\leq d(u_n, T^n u_n) + d(T^n u_n, Tu_n) \\ &\leq d(u_n, T^n u_n) + Ld(T^{n-1} u_n, u_n) \\ &\leq d(u_n, T^n u_n) + Ld(T^{n-1} u_n, T^{n-1} u_{n-1}) \\ &\quad + Ld(T^{n-1} u_{n-1}, u_{n-1}) + Ld(u_{n-1}, u_n) \\ &\leq d(u_n, T^n u_n) + L^2 d(u_n, u_{n-1}) \\ &\quad + Ld(T^{n-1} u_{n-1}, u_{n-1}) + Ld(u_{n-1}, u_n) \\ &\leq d(u_n, T^n u_n) + Ld(T^{n-1} u_{n-1}, u_{n-1}) \\ &\quad + (L^2 + L) [d(u_n, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, u_{n-1})], \end{aligned}$$

which implies from (3.26), (3.29) and (3.32) that

$$(3.33) \quad \lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0.$$

Since $\{x_n\}$ is bounded and X is a Hadamard space, then from Lemma 2.3, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k \rightarrow \infty} x_{n_k} = z$. It follows from (3.26) that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{k \rightarrow \infty} u_{n_k} = z$. Thus, from (3.33) and Lemma 2.6, we obtain that $z \in F(T)$. Also, since $\Phi_\lambda^N \circ \Psi_\mu^m$ is the composition of nonexpansive mappings, it implies that $\Phi_\lambda^N \circ \Psi_\mu^m$ is nonexpansive. Thus, it follows from (3.25) and Lemma 2.5 that $z \in F(\Phi_\lambda^N \circ \Psi_\mu^m)$. Hence, by Lemma 3.2, we obtain that $z \in \left(\bigcap_{i=1}^N A_i^{-1}(0)\right) \cap \left(\bigcap_{j=1}^m \arg \min_{y \in X} f_j(y)\right)$. Therefore, $z \in \Gamma$. Furthermore, for arbitrary $u \in X$, we have from Lemma 2.4 that

$$(3.34) \quad \limsup_{n \rightarrow \infty} \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \leq 0,$$

which implies from condition C1 that

$$(3.35) \quad n \rightarrow \infty \limsup \left(t_n d^2(u, z) + 2(1 - t_n) \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \right) \leq 0.$$

Next, we show that $\{x_n\}$ converges strongly to z . Since $\{u_n\}$ is bounded, there exists $M' > 0$ such that $d^2(u_n, z) \leq M' \forall n \geq 1$. Thus, from (3.16) and Lemma 2.2, we obtain

$$\begin{aligned}
 d^2(x_{n+1}, z) &\leq d^2(u_n, z) + (k_n^2 - 1)d^2(u_n, z) \\
 &\leq d^2(u_n, z) + (k_n^2 - 1)M' \\
 &\leq d^2(\Phi_\lambda^{N-1}\Psi_\mu^m v_n, z) + (k_n^2 - 1)M' \\
 &\vdots \\
 &\leq d^2(\Psi_\mu^m v_n, z) + (k_n^2 - 1)M' \\
 &\vdots \\
 &\leq d^2(v_n, z) + (k_n^2 - 1)M' \\
 &\leq (1 - t_n)^2 d^2(x_n, z) + t_n^2 d^2(u, z) + 2t_n(1 - t_n)\langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle + (k_n^2 - 1)M' \\
 &\leq (1 - t_n)d^2(x_n, z) + t_n \left(t_n d^2(u, z) + 2(1 - t_n)\langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \right) + (k_n^2 - 1)M'.
 \end{aligned}$$

(3.36)

Since $\sum_{n=1}^\infty (k_n^2 - 1)M' < \infty$, it then follows from (3.35), (3.36) and Lemma 2.8 that $\{x_n\}$ converges strongly to z .

Case 2: Suppose that $\{d^2(x_n, p)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{d^2(x_{n_i}, p)\}$ of $\{d^2(x_n, p)\}$ such that $d^2(x_{n_i}, p) < d^2(x_{n_i+1}, p)$ for all $i \in \mathbf{N}$. Thus, by Lemma 2.9, there exists a nondecreasing sequence $\{m_k\} \subset \mathbf{N}$ such that $m_k \rightarrow \infty$, and

$$d^2(x_{m_k}, p) \leq d^2(x_{m_k+1}, p) \text{ and } d^2(x_k, p) \leq d^2(x_{m_k+1}, p) \forall k \in \mathbf{N}.$$

Thus, with this and (3.16), we obtain

$$\begin{aligned}
 0 &\leq \lim_{k \rightarrow \infty} \left(d^2(x_{m_k+1}, p) - d^2(x_{m_k}, p) \right) \\
 &\leq \limsup_{n \rightarrow \infty} \left(d^2(x_{n+1}, p) - d^2(x_n, p) \right) \\
 &\leq \limsup_{n \rightarrow \infty} \left(d^2(u_n, p) + (k_n^2 - 1)d^2(u_n, p) - d^2(x_n, p) \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{n \rightarrow \infty} \left((1 - t_n) d^2(x_n, p) + t_n d^2(u, p) + (k_n^2 - 1) d^2(u_n, p) - d^2(x_n, p) \right) \\
&\leq \limsup_{n \rightarrow \infty} \left[t_n \left(d^2(u, p) - d^2(x_n, p) \right) + (k_n^2 - 1) M' \right] = 0,
\end{aligned}$$

which implies

$$(3.37) \quad \lim_{k \rightarrow \infty} \left(d^2(x_{m_k+1}, p) - d^2(x_{m_k}, p) \right) = 0.$$

Following the same line of argument as in Case 1, we can verify that

$$(3.38) \quad k \rightarrow \infty \lim \left(t_{m_k} d^2(u, z) + 2(1 - t_{m_k}) \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \right) \leq 0.$$

Also from (3.36), we have

$$\begin{aligned}
d^2(x_{m_k+1}, z) &\leq (1 - t_{m_k}) d^2(x_{m_k}, z) + t_{m_k} \left(t_{m_k} d^2(u, z) + 2(1 - t_{m_k}) \langle \overrightarrow{uz}, \overrightarrow{x_{m_k} z} \rangle \right) \\
&\quad + (k_{m_k}^2 - 1) M'.
\end{aligned}$$

Since $d^2(x_{m_k}, z) \leq d^2(x_{m_k+1}, z)$, we have

$$d^2(x_{m_k}, z) \leq \left(t_{m_k} d^2(u, z) + 2(1 - t_{m_k}) \langle \overrightarrow{uz}, \overrightarrow{x_{m_k} z} \rangle \right) + \frac{(k_{m_k}^2 - 1) M'}{t_{m_k}},$$

which implies from (3.38) and condition C4 that

$$(3.39) \quad \lim_{k \rightarrow \infty} d^2(x_{m_k}, z) = 0.$$

Since $d^2(x_k, z) \leq d^2(x_{m_k+1}, z)$, we obtain from (3.39) and (3.37) that $k \rightarrow \infty \lim d^2(x_k, z) = 0$. Thus, from Case 1 and Case 2, we conclude that $\{x_n\}$ converges to $z \in \Gamma$. $\square$

If T is a uniformly L -Lipschitzian and asymptotically nonexpansive mapping defined on X , then we obtain the following result.

Let X be a Hadamard space and X^* be its dual space. Let $A_i : X \rightarrow 2^{X^*}$, $i = 1, 2, \dots, N$ be multivalued monotone mappings that satisfy the range condition and $f_j : X \rightarrow (-\infty, \infty]$, $j = 1, 2, \dots, m$ be proper convex and lower semi continuous functions. Let $T : X \rightarrow X$ be a uniformly L -Lipschitzian and asymptotically nonexpansive mapping. Suppose that $\Gamma := F(T) \cap \left(\bigcap_{i=1}^N A_i^{-1}(0) \right) \cap \left(\bigcap_{j=1}^m \arg \min_{y \in X} f_j(y) \right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by

$$(3.40) \quad \begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = \Phi_\lambda^N \circ \Psi_\mu^m(v_n), \\ y_n = (1 - \beta_n)u_n \oplus \beta_n T^n u_n, \\ x_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{cases}$$

where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \dots \circ J_\lambda^2 \circ J_\lambda^1$, $\Phi_\lambda^0 = I$, $\Psi_\mu^m = J_\mu^N \circ J_\mu^{m-1} \circ \dots \circ J_\mu^2 \circ J_\mu^1$, $\Phi_\mu^0 = I$, $\lambda, \mu \in (0, \infty)$ and $\{t_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

$$\text{C1: } n \rightarrow \infty \lim t_n = 0,$$

$$\text{C2: } \sum_{n=1}^{\infty} t_n = \infty,$$

$$\text{C3: } 0 < \epsilon \leq \alpha_n \leq b < 1 \text{ and } 0 < \epsilon \leq \beta_n \leq b < 1,$$

$$\text{C4: } n \rightarrow \infty \lim \frac{k_n^2 - 1}{t_n} = 0.$$

Then $\{x_n\}$ converges strongly to an element of Γ .

By setting $N = m = 1$ in Theorem 3.3, we obtain the following result.

Let X be a Hadamard space and X^* be its dual space. Let $A : X \rightarrow 2^{X^*}$ be multivalued monotone mapping that satisfies the range condition and $f : X \rightarrow (-\infty, \infty]$ be proper convex and lower semi continuous function. Let $T : X \rightarrow X$ be a uniformly L -Lipschitzian and asymptotically demicontractive mapping with constant $k \in (0, 1)$ and sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty$. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \cap \arg \min_{y \in X} f(y) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by

$$(3.41) \quad \begin{cases} v_n &= (1 - t_n)x_n \oplus t_n u, \\ u_n &= J_\lambda \circ J_\mu(v_n), \\ y_n &= (1 - \beta_n)u_n \oplus \beta_n T^n u_n, \\ x_{n+1} &= (1 - \alpha_n)u_n \oplus \alpha_n T^n y_n, \quad n \geq 1, \end{cases}$$

where $\lambda, \mu \in (0, \infty)$ and $\{t_n\}, \{\beta_n\}, \{\alpha_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

$$\text{C1: } n \rightarrow \infty \lim t_n = 0,$$

$$\text{C2: } \sum_{n=1}^{\infty} t_n = \infty,$$

$$\text{C3: } 0 < \epsilon \leq \alpha_n \leq k\beta_n < \beta_n \leq b < \frac{2}{\left(\frac{k+\delta}{k}\right) + \sqrt{\left(\frac{k+\delta}{k}\right)^2 + 4L^2}}, \text{ where } \delta := n \geq 1 \sup k_n,$$

$$\text{C4: } n \rightarrow \infty \lim \frac{k_n^2 - 1}{t_n} = 0.$$

Then $\{x_n\}$ converges strongly to an element of Γ .

By setting $N = m = 1$ and $T = I$ (I is the identity mapping on X) in Theorem 3.3, we obtain the following result.

Let X be a Hadamard space and X^* be its dual space. Let $A : X \rightarrow 2^{X^*}$ be multivalued monotone mapping that satisfies the range condition and $f : X \rightarrow (-\infty, \infty]$ be proper convex and lower semi continuous function. Suppose that $\Gamma := A^{-1}(0) \cap \arg \min_{y \in X} f(y) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by

$$(3.42) \quad \begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = (J_\lambda \circ J_\mu)(v_n), \\ y_n = (1 - \beta_n)u_n \oplus \beta_n u_n, \\ x_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n y_n, \quad n \geq 1, \end{cases}$$

where $\lambda, \mu \in (0, \infty)$ and $\{t_n\}, \{\beta_n\}, \{\alpha_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

$$\text{C1: } n \rightarrow \infty \lim t_n = 0,$$

$$\text{C2: } \sum_{n=1}^{\infty} t_n = \infty.$$

Then $\{x_n\}$ converges strongly to an element of Γ .

4. Numerical Example

In this section, we give a numerical example of Theorem 3.3 to illustrate the applicability of our main result. Let $X = \mathbf{R}^2$ be endowed with the euclidean norm $\|\cdot\|_2$. Let $B : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be defined by $B(x) = (2x_1 + x_2, x_1 + 3x_2)$ and $b^T = (0, 0)$. Let $m = 1$ in Theorem 3.3, we define $f : \mathbf{R}^2 \rightarrow (-\infty, \infty]$ by

$$f(x) = \frac{1}{2} \|B(x) - b\|^2.$$

Then, f is a proper convex and lower semi continuous function, since B is a continuous linear mapping (see [43], [18]). Thus for $\mu = 1$, we obtain from [43] that

$$\begin{aligned} J_1(x) = \text{Prox}_f(x) &= \arg \min_{y \in \mathbf{R}^2} \left[f(y) + \frac{1}{2} \|y - x\|^2 \right] \\ &= (I + B^T B)^{-1}(x + B^T b). \end{aligned}$$

So that

$$J_1(x) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ = \left(\frac{11x_1 - 5x_2}{41}, \frac{-5x_1 + 6x_2}{41} \right).$$

Let $N = 1$ in Theorem 3.3, we define $A : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by $A(x) = (x_1, x_1 - x_2)$. Then, A is a monotone mapping.

We note that $[t\vec{ab}] \equiv t(b - a)$, for all $t \in \mathbf{R}$ and $a, b \in \mathbf{R}^2$ (see [3]). Thus, we have for each $x \in \mathbf{R}^2$ that

$$J_\lambda(x) = z \iff \frac{1}{\lambda}(x - z) \in Az \\ \iff x = (I + \lambda A)z \\ \iff z = (I + \lambda A)^{-1}x.$$

So that for $\lambda = 2$, we compute the resolvent of A as follows:

$$J_2(x) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ = \left(\frac{x_1}{5}, \frac{2x_1 - 3x_2}{5} \right).$$

Thus, for $N = m = 1$, $\lambda = 2$ and $\mu = 1$, we obtain

$$J_2(J_1x) = \left(\frac{11x_1 - 5x_2}{205}, \frac{37x_1 - 28x_2}{205} \right).$$

Let $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be defined by $T(x) = \frac{-3x}{2}$. Then T is a demicontractive mapping with $k = \frac{1}{5}$. This implies that T is an asymptotically demicontractive mapping with $k = \frac{1}{5}$ and $k_n = 1$ for all $n \geq 1$. Thus, we obtain that $\delta = 1$. It is easy to see that T is also uniformly L -Lipschitzian with $L = \frac{3}{2}$. Now, if we take $b = \frac{1}{10}$ and $\epsilon = \frac{1}{1000}$, then condition C3 becomes:

$$0 < \frac{1}{1000} \leq \alpha_n \leq \frac{\beta_n}{5} < \beta_n \leq \frac{1}{10} < \frac{2}{6 + \sqrt{45}}.$$

So that, if we choose $\beta_n = \frac{1}{10 + \frac{1}{n}}$, $\alpha_n = \frac{1}{50 + \frac{5}{n}}$ and $t_n = \frac{1}{2n+1}$, then t_n, β_n and α_n satisfy the conditions in Theorem 3.3.

Hence, for $u, x_1 \in \mathbf{R}^2$, our Algorithm (3.12) becomes:

$$(4.1) \quad \begin{cases} v_n = \frac{2n}{2n+1}x_n + \frac{u}{2n+1}, \\ u_n = J_2(J_1v_n), \\ y_n = \left(1 - \frac{1}{10+\frac{1}{n}}\right)u_n - \frac{3}{20+\frac{2}{n}}u_n, \\ x_{n+1} = \left(1 - \frac{1}{50+\frac{5}{n}}\right)u_n - \frac{3}{100+\frac{10}{n}}y_n, \quad n \geq 1. \end{cases}$$

Case I Take $x_1 = (0.1, 0.5)^T$ and $u = (0.1, 0.5)^T$.

Case II Take $x_1 = (0.1, 0.5)^T$ and $u = (1, 1.5)^T$.

Case III Take $x_1 = (-1, -0.5)^T$ and $u = (-1, 1.5)^T$.

FIGURE 1. Case I: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

FIGURE 2. Case II: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

FIGURE 3. Case III: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

The Matlab version used is R2014a and the figures are presented in Figure 1, Figure 2 and Figure 3.

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