Some integral inequalities for approximately h -convex functions and their applications

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Abstract:

In this paper, by applying the new and improved form of Hölder's integral inequality called Hölder-İşcan integral inequality three inequalities of Hermite-Hadamard and Hadamard integral type for (h, d) -convex functions have been established. Various special cases including classes for instance, h -convex, s -convex function of Breckner and Godunova-Levin-Dragomir and strong versions of the aforementioned types of convex functions have been identified. Some applications to error estimations of presented results have been analyzed. At the end, a briefly conclusion is given.

Keywords: Hermite-Hadamard inequality; Hölder-İşcan inequality; Convexity, Numerical integration.

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1. Introduction

Convex functions and its generalizations have various applications in the fields of pure and applied sciences. Due to these applications, it is the most attractive area for researchers now a days. The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1.1. Let I be an interval in $\mathbf{R}$. A function $\varrho : I \rightarrow \mathbf{R}$, is said to be convex on I if the inequality

$$(1.1) \quad \varrho(ua_1 + (1-u)a_2) \leq u\varrho(a_1) + (1-u)\varrho(a_2)$$

holds for all $a_1, a_2 \in I$ and $u \in [0, 1]$. Also, we say that ϱ is concave, if the inequality in (1.1) holds in the reverse direction.

Theorem 1.2. Let $\varrho : I \subset \mathbf{R} \rightarrow \mathbf{R}$ be a convex function, $a_1, a_2 \in I$ and $a_1 < a_2$. Then

$$(1.2) \quad \varrho\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(u) du \leq \frac{\varrho(a_1) + \varrho(a_2)}{2}.$$

Authors of recent decades have studied the inequality (1.2), known as Hermite–Hadamard inequality (or trapezium inequality) in the premises of newly invented definitions due to motivation of convex function. Interested readers can see the references [2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31].

The stability theory for functional inequalities has its origin in the paper of Hyers and Ulam [13], in which the notion of δ -convex functions is introduced. A function $\varrho : D \rightarrow \mathbf{R}$ is known as δ -convex function, if

$$(1.3) \quad \varrho(ua_1 + (1-u)a_2) \leq u\varrho(a_1) + (1-u)\varrho(a_2) + \delta$$

for all $a_1, a_2 \in D, u \in [0, 1]$, where D is a convex subset of a real linear space X and δ is a non-negative real number. Roughly speaking, the Hyers–Ulam theorem says that a function ϱ is indeed a perturbation of a convex function by a bounded function if and only if there exists a non negative real number δ which satisfies (1.3). Now the error term is now refer to a bounded perturbation. Páles in [24], extended the notion to (ϵ, δ) -convex functions while investigating the stability constants as follows:

A real valued function ϱ defined on a real interval I is called (ϵ, δ) -convex function if it satisfies the following inequality:

(1.4) $\varrho(ua_1 + (1-u)a_2) \leq u\varrho(a_1) + (1-u)\varrho(a_2) + \epsilon u(1-u) \|a_1 - a_2\| + \delta,$
 $\forall a_1, a_2 \in I$ and $u \in [0, 1]$. In (1.4), there are two error terms. Páles in [24], shown that the error terms $\epsilon u(1-u) \|a_1 - a_2\|$ and δ , respectively, corresponds to perturbation of a convex function by a Lipschitz function and by a bounded function.

Burai and Hazy [5] considered a function $d : X \times X \rightarrow \mathbf{R}$ a function on normed space X and D a non-empty open, convex subset of X . They generalized the notion of convex function by introducing (h, d) -convex function as follows:

Definition 1.3. ((h, d) -convex function) A function $\varrho : D \rightarrow \mathbf{R}$, is called (h, d) -convex function, if

$$\varrho(ua_1 + (1-u)a_2) \leq h(u)\varrho(a_1) + h(1-u)\varrho(a_2) + d(a_1, a_2), \quad \forall a_1, a_2 \in D, u \in (0, 1). \quad (1.5)$$

the class of (h, d) -convex functions includes the class of classical convex functions, δ -convex functions, (ϵ, δ) -convex functions, h -convex functions, s -convex functions of Breckner type, s -convex functions of Godunova-Levin-Drăgomir type etc., and classes of strongly convex versions of aforementioned convex functions.

It is worth mentioning that the convexity of the function ϱ depends on the value of the function $d(\cdot, \cdot)$ and the function $d(\cdot, \cdot)$ entirely depends on the convex combination. Therefore it will be convenient to modify this definition. We now propose a modification to (h, d) -convex function.

Definition 1.4. (**Modified** (h, d) -convex function) A function $\varrho : D \rightarrow \mathbf{R}$, is called modified (h, d) -convex function, if

$$\varrho(ua_1 + (1-u)a_2) \leq h(u)\varrho(a_1) + h(1-u)\varrho(a_2) + u(1-u)d(a_1, a_2) \quad (1.6)$$

$$\forall a_1, a_2 \in D, u \in (0, 1).$$

Now we discuss some special cases, by restricting ourselves to the only well studied notions of convex functions by assigning different values to the function d .

I. If $d(a_1, a_2) = \frac{\delta}{u(1-u)}$, then we have following extension of well known δ -convex functions given by inequality (1.3).

Definition 1.5. (δ - h -convex function) A function $\varrho : D \rightarrow \mathbf{R}$ is called δ - h -convex function, if ϱ satisfies

$$\varrho(ua_1 + (1-u)a_2) \leq h(u)\varrho(a_1) + h(1-u)\varrho(a_2) + \delta, \quad \forall a_1, a_2 \in D, u \in (0, 1). \quad (1.7)$$

Moreover we have class of δ -convex functions, δ -Godunova-Levin functions, δ - P -functions, δ - tgs -functions, δ - MT -convex functions, Breckner δ - s -convex functions and δ - s -Godunova-Levin functions by using $h(u) = u, u^{-1}, 1, u(1-u), \frac{\sqrt{u}}{2\sqrt{1-u}}, u^s$ and $h(u) = u^{-s}$, respectively.

II. If $d(a_1, a_2) = \epsilon \|a_1 - a_2\| + \frac{\delta}{u(1-u)}$, then we have following new type of convex functions.

Definition 1.6. ((ϵ, δ) - h -convex function) A function $\varrho : D \rightarrow \mathbf{R}$ is called (ϵ, δ) - h -convex function if

$$(1.8) \quad \varrho(ua_1 + (1-u)a_2) \leq h(u)\varrho(a_1) + h(1-u)\varrho(a_2) + \epsilon u(1-u) \|a_1 - a_2\| + \delta, \\ \forall a_1, a_2 \in D \text{ and } u \in (0, 1).$$

Moreover we have class of (ϵ, δ) -convex functions, (ϵ, δ) -Godunova-Levin functions, (ϵ, δ) - P -functions, (ϵ, δ) - tgs -functions, (ϵ, δ) - MT -convex functions, Breckner (ϵ, δ) - s -convex functions and (ϵ, δ) - s -Godunova-Levin functions by using $h(u) = u, u^{-1}, 1, u(1-u), \frac{\sqrt{u}}{2\sqrt{1-u}}, u^s$ and $h(u) = u^{-s}$ respectively.

III. If $d(a_1, a_2) = -\epsilon \|a_1 - a_2\|^2$, then we have following new type of convex functions.

Definition 1.7. (**Strongly- h -convex function**) A function $\varrho : D \rightarrow \mathbf{R}$ is called strongly h -convex function, if

$$(1.9) \quad \varrho(ua_1 + (1-u)a_2) \leq h(u)\varrho(a_1) + h(1-u)\varrho(a_2) - \epsilon u(1-u) \|a_1 - a_2\|^2, \\ \forall a_1, a_2 \in D \text{ and } u \in (0, 1).$$

Moreover, classes of strongly convex functions, strongly Godunova-Levin functions, strongly P -functions, strongly tgs -functions, strongly MT -convex functions, Breckner strongly s -convex functions and strongly s -Godunova-Levin functions are obtained by using $h(u) = u, u^{-1}, 1, u(1-u), \frac{\sqrt{u}}{2\sqrt{1-u}}, u^s$ and $h(u) = u^{-s}$, respectively.

Now we give some preliminaries which will be helpful for further study.

Sarikaya et al. [25] prove the following Hadamard inequality for h -convex function.

Theorem 1.8. Let $\varrho : I \rightarrow \mathbf{R}$ be an h -convex function, $a_1, a_2 \in I$ with $a_1 < a_2$ and $\varrho \in L_1[a_1, a_2]$. Then

$$(1.10) \quad \frac{1}{2h\left(\frac{1}{2}\right)}\varphi\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{a_2-a_1} \int_{a_1}^{a_2} \varrho(x)dx \leq [\varrho(a_1) + \varrho(a_2)] \int_0^1 h(r)dr,$$

which was proved in [1].

In [10], following identity for differentiable mappings is proved.

Lemma 1.9. Let $\varrho : I \subset \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable function on I^0 where $a_1, a_2 \in I$ with $a_1 < a_2$. If $\varrho' \in L[a_1, a_2]$, then the following equality holds:

$$(1.11) \quad \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x)dx = \left(\frac{a_2 - a_1}{2}\right) \int_0^1 (1-2u)\varrho'(ua_1 + (1-u)a_2)du.$$

The well-known Hölder inequality and power-mean inequalities are given as follows:

Theorem 1.10. (Hölder integral inequality) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ϕ and ψ are real valued functions on $[a_1, a_2]$ and if $|\phi|^p$ and $|\psi|^q$ are integrable on $[a_1, a_2]$, then

$$(1.12) \quad \int_{a_1}^{a_2} |\phi(x)\psi(x)| dx \leq \left(\int_{a_1}^{a_2} |\phi(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{a_1}^{a_2} |\psi(x)|^q dx \right)^{\frac{1}{q}},$$

where equality holds if and only if $A|\phi(x)|^p = B|\psi(x)|^q$ almost everywhere, where A and B are constants.

Theorem 1.11. (Power-mean integral inequality) Let $q \geq 1$. If ϕ and ψ are real valued functions on $[a_1, a_2]$ and if $|\phi|$ and $|\phi||\psi|^q$ are integrable on $[a_1, a_2]$, then

$$(1.13) \quad \int_{a_1}^{a_2} |\phi(x)\psi(x)| dx \leq \left(\int_{a_1}^{a_2} |\phi(x)| dx \right)^{1-\frac{1}{q}} \left(\int_{a_1}^{a_2} |\phi(x)||\psi(x)|^q dx \right)^{\frac{1}{q}}.$$

Recently, in [14], İşcan gave the following inequality which gives the refinement form of Hölder's integral inequality as follows:

Theorem 1.12. (Hölder–İşcan integral inequality) Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If ϕ and ψ are real valued functions on $[a_1, a_2]$ and if $|\phi|^p$ and $|\psi|^q$ are integrable on $[a_1, a_2]$, then

$$(1.14) \quad \begin{aligned} & \int_{a_1}^{a_2} |\phi(x)\psi(x)| dx \\ & \leq \frac{1}{a_2 - a_1} \left\{ \left(\int_{a_1}^{a_2} (a_2 - x) |\phi(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{a_1}^{a_2} (a_2 - x) |\psi(x)|^q dx \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{a_1}^{a_2} (x - a_1) |\phi(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{a_1}^{a_2} (x - a_1) |\psi(x)|^q dx \right)^{\frac{1}{q}} \right\} \end{aligned}$$

$$(1.15) \quad \leq \left(\int_{a_1}^{a_2} |\phi(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{a_1}^{a_2} |\psi(x)|^q dx \right)^{\frac{1}{q}},$$

where equality holds if and only if $A|\phi(x)|^p = B|\psi(x)|^q$ almost everywhere, where A and B are constants.

A different version named as improved power-mean integral inequality can be given by:

Theorem 1.13. (Improved Power-mean integral inequality) Let $q \geq 1$. If ϕ and ψ are real valued functions on $[a_1, a_2]$ and if $|\phi|$ and $|\phi| |\psi|^q$ are integrable on $[a_1, a_2]$, then

$$\begin{aligned}
 & \int_{a_1}^{a_2} |\phi(x)\psi(x)| dx \\
 & \leq \frac{1}{a_2 - a_1} \left\{ \left(\int_{a_1}^{a_2} (a_2 - x) |\phi(x)| dx \right)^{1-\frac{1}{q}} \left(\int_{a_1}^{a_2} (a_2 - x) |\phi(x)| |\psi(x)|^q dx \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(\int_{a_1}^{a_2} (x - a_1) |\phi(x)| dx \right)^{1-\frac{1}{q}} \left(\int_{a_1}^{a_2} (x - a_1) |\phi(x)| |\psi(x)|^q dx \right)^{\frac{1}{q}} \right\} \\
 (1.16) \quad & \leq \left(\int_{a_1}^{a_2} |\phi(x)| dx \right)^{1-\frac{1}{q}} \left(\int_{a_1}^{a_2} |\phi(x)| |\psi(x)|^q dx \right)^{\frac{1}{q}}.
 \end{aligned}$$

The main objective of this paper is to discover in Section 2, three inequalities of Hermite-Hadamard and Hadamard integral type for (h, d) -convex functions by applying Hölder-İşcan integral inequalities. Various special cases including classes for instance, h -convex, s -convex function of Breckner and Godunova-Levin-Dragomir and strong versions of the aforementioned types of convex functions will be identified. In Section 3, some applications to special means and error estimations will be provided. In Section 4, a briefly conclusion is given as well.

2. Main Results

We have the following results:

Theorem 2.1. Let ϱ be an integrable function defined on $[a_1, a_2]$. If ϱ is a modified (h, d) -convex function on $[a_1, a_2]$, then the following inequality holds:

$$\begin{aligned}
& \frac{1}{2h\left(\frac{1}{2}\right)} \left[\varrho\left(\frac{a_1+a_2}{2}\right) - \frac{1}{4} \int_0^1 d(sa_1 + (1-s)a_2, (1-s)a_1 + sa_2) ds \right] \\
& \leq \int_0^1 \varrho(sa_1 + (1-s)a_2) ds \\
(2.1) \quad & \leq [\varrho(a_1) + \varrho(a_2)] \int_0^1 h(s) ds + \frac{1}{6} d(a_1, a_2).
\end{aligned}$$

Proof. By the notion of modified (h, d) -convex functions, we have

$$(2.2) \quad \varrho\left(\frac{x_1+x_2}{2}\right) \leq h\left(\frac{1}{2}\right) [\varrho(x_1) + \varrho(x_2)] + \frac{1}{4} d(x_1, x_2)$$

for all $x_1, x_2 \in [a_1, a_2]$. Now, by taking $x_1 = sa_1 + (1-s)a_2$ and $x_2 = sa_2 + (1-s)a_1$, $s \in [0, 1]$, from (2.2), we get

$$\begin{aligned}
& \frac{1}{h\left(\frac{1}{2}\right)} \left[\varrho\left(\frac{a_1+a_2}{2}\right) - \frac{1}{4} d(sa_1 + (1-s)a_2, (1-s)a_1 + sa_2) \right] \\
(2.3) \quad & \leq \varrho(sa_1 + (1-s)a_2) + \varrho((1-s)a_1 + sa_2).
\end{aligned}$$

Now integrating (2.3) over $(0, 1)$ with respect to s , we obtain

$$\begin{aligned}
& \frac{1}{2h\left(\frac{1}{2}\right)} \left[\varrho\left(\frac{a_1+a_2}{2}\right) - \frac{1}{4} \int_0^1 d(sa_1 + (1-s)a_2, (1-s)a_1 + sa_2) dx \right] \\
(2.4) \quad & \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx.
\end{aligned}$$

On the other hand, since ϱ is an (h, d) -convex function, one has

$$\begin{aligned}
& \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx = \int_0^1 \varrho(sa_1 + (1-s)a_2) ds \leq [\varrho(a_1) + \varrho(a_2)] \\
(2.5) \quad & \int_0^1 h(s) ds + \frac{1}{6} d(a_1, a_2).
\end{aligned}$$

Required double inequality (2.1) is obtained by combining (2.4) and (2.5).
□

Remark 2.2. If $d(x, y) = 0$, then we obtain [25, Theorem 6].

Corollary 2.3. If ϱ is an δ - h -convex function on $[a_1, a_2]$, then

$$\frac{1}{2h\left(\frac{1}{2}\right)} \left[\varrho\left(\frac{a_1 + a_2}{2}\right) - \delta \right] \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \leq [\varrho(a_1) + \varrho(a_2)] \int_0^1 h(s) ds + \delta.$$

Corollary 2.4. If ϱ is (ϵ, δ) - h -convex function on $[a_1, a_2]$, then

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} \left[\varrho\left(\frac{a_1 + a_2}{2}\right) - \delta \right] &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \\ &\leq [\varrho(a_1) + \varrho(a_2)] \int_0^1 h(s) ds + \frac{\epsilon}{6}(a_2 - a_1) + \delta. \end{aligned}$$

Corollary 2.5. If ϱ is (ϵ, δ) - s -convex function of Breckner type on $[a_1, a_2]$, then

$$\begin{aligned} 2^{s-1} \left[\varrho\left(\frac{a_1 + a_2}{2}\right) - \delta \right] &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \\ &\leq \frac{\varrho(a_1) + \varrho(a_2)}{s+1} + \frac{\epsilon}{6}(a_2 - a_1) + \delta. \end{aligned}$$

Corollary 2.6. If ϱ is (ϵ, δ) - s -convex function of Godunova-Levin-Dragomir type on $[a_1, a_2]$, then

$$\begin{aligned} 2^{-(s+1)} \left[\varrho\left(\frac{a_1 + a_2}{2}\right) - \delta \right] &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \\ &\leq \frac{\varrho(a_1) + \varrho(a_2)}{1-s} + \frac{\epsilon}{6}(a_2 - a_1) + \delta. \end{aligned}$$

Corollary 2.7. If ϱ is (ϵ, δ) - P -convex function on $[a_1, a_2]$, then

$$\begin{aligned} \frac{1}{2} \left[\varrho\left(\frac{a_1 + a_2}{2}\right) - \delta \right] &\leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \\ &\leq [\varrho(a_1) + \varrho(a_2)] + \frac{\epsilon}{6}(a_2 - a_1) + \delta. \end{aligned}$$

Corollary 2.8. *If ϱ is strongly- h -convex function on $[a_1, a_2]$, then*

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} \left[\varrho\left(\frac{a_1+a_2}{2}\right) + \frac{\epsilon}{12}(a_2-a_1)^2 \right] &\leq \frac{1}{a_2-a_1} \int_{a_1}^{a_2} \varrho(x) dx \\ &\leq [\varrho(a_1) + \varrho(a_2)] \int_0^1 h(s) ds - \frac{\epsilon}{6}(a_2-a_1)^2, \end{aligned}$$

which was proved in [1].

Corollary 2.9. *If ϱ is strongly s -convex function of Breckner type on $[a_1, a_2]$, then*

$$\begin{aligned} 2^{s-1} \left[\varrho\left(\frac{a_1+a_2}{2}\right) + \frac{\epsilon}{12}(a_2-a_1)^2 \right] &\leq \frac{1}{a_2-a_1} \int_{a_1}^{a_2} \varrho(x) dx \\ &\leq \frac{\varrho(a_1) + \varrho(a_2)}{s+1} + \frac{\epsilon}{6}(a_2-a_1)^2. \end{aligned}$$

Corollary 2.10. *If ϱ is strongly δ - s -convex function of Godunova-Levin-Dragomir type on $[a_1, a_2]$, then*

$$\begin{aligned} 2^{-(s+1)} \left[\varrho\left(\frac{a_1+a_2}{2}\right) + \frac{\epsilon}{12}(a_2-a_1)^2 \right] &\leq \frac{1}{a_2-a_1} \int_{a_1}^{a_2} \varrho(x) dx \\ &\leq \frac{\varrho(a_1) + \varrho(a_2)}{1-s} + \frac{\epsilon}{6}(a_2-a_1)^2. \end{aligned}$$

Corollary 2.11. *If ϱ is strongly P -convex function on $[a_1, a_2]$, then*

$$\begin{aligned} \frac{1}{2} \left[\varrho\left(\frac{a_1+a_2}{2}\right) + \frac{\epsilon}{12}(a_2-a_1)^2 \right] &\leq \frac{1}{a_2-a_1} \int_{a_1}^{a_2} \varrho(x) dx \leq [\varrho(a_1) + \varrho(a_2)] \\ &\quad + \frac{\epsilon}{6}(a_2-a_1)^2. \end{aligned}$$

Theorem 2.12. *Let ϱ be a differentiable function defined on $[a_1, a_2]$. If $|\varrho'|^q$ is a modified (h, d) -convex function on $[a_1, a_2]$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$(2.6) \quad \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2 \sqrt[p]{2(p+1)}} \\ \times \left\{ \left[H_1 |\varrho'(a_1)|^q + H_2 |\varrho'(a_2)|^q + \frac{1}{12} d(a_1, a_2) \right]^{\frac{1}{q}} \right. \\ \left. + \left[H_1 |\varrho'(a_2)|^q + H_2 |\varrho'(a_1)|^q + \frac{1}{12} d(a_1, a_2) \right]^{\frac{1}{q}} \right\},$$

where

$$H_1 = \int_0^1 u h(u) du, \quad H_2 = \int_0^1 u h(1-u) du.$$

Proof. Taking the modulus on both sides of identity (1.11) and using Hölder–İşcan integral inequality, we have

$$\left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ \leq \left(\frac{a_2 - a_1}{2} \right) \int_0^1 |1 - 2u| |\varrho'(ua_1 + (1-u)a_2)| du \\ \leq \left(\frac{a_2 - a_1}{2} \right) \left\{ \left(\int_0^1 (1-u) |1 - 2u|^p du \right)^{\frac{1}{p}} \left(\int_0^1 (1-u) |\varrho'(ua_1 + (1-u)a_2)|^q du \right)^{\frac{1}{q}} \right. \\ (2.7) \quad \left. + \left(\int_0^1 u |1 - 2u|^p du \right)^{\frac{1}{p}} \left(\int_0^1 u |\varrho'(ua_1 + (1-u)a_2)|^q du \right)^{\frac{1}{q}} \right\}.$$

Since $|\varrho'|^q$ is modified (h, d) -convex function on $[a_1, a_2]$, we get

$$\int_0^1 u |\varrho'(ua_1 + (1-u)a_2)|^q du \\ \leq \int_0^1 u [h(u) |\varrho'(a_1)|^q + h(1-u) |\varrho'(a_2)|^q + u(1-u) d(a_1, a_2)] du \\ = \int_0^1 u h(u) |\varrho'(a_1)|^q du + \int_0^1 u h(1-u) |\varrho'(a_2)|^q du + \frac{1}{12} d(a_1, a_2) \\ (2.8)$$

and

$$\begin{aligned} \int_0^1 (1-u) |\varrho'(ua_1 + (1-u)a_2)|^q du &\leq \int_0^1 uh(1-u) |\varrho'(a_1)|^q du \\ &+ \int_0^1 uh(u) |\varrho'(a_2)|^q du + \frac{1}{12} d(a_1, a_2). \end{aligned}$$

On the other hand,

$$(2.9) \quad \int_0^1 u |1-2u|^p du = \int_0^1 (1-u) |1-2u|^p du = \frac{1}{2(p+1)}.$$

Required inequality (2.6) is obtained by using (2.8), (2.9) and (2.9) in (2.7).

□

Corollary 2.13. *If ϱ is h -convex function on $[a_1, a_2]$, then*

$$\begin{aligned} \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| &\leq \frac{(a_2 - a_1)}{2 \sqrt[p]{2(p+1)}} \\ &\times \left\{ [H_1 |\varrho'(a_1)|^q + H_2 |\varrho'(a_2)|^q]^{\frac{1}{q}} + [H_1 |\varrho'(a_2)|^q + H_2 |\varrho'(a_1)|^q]^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.14. *If ϱ is s -convex function of Breckner type, then we get [15, Theorem 2.2].*

Corollary 2.15. *If ϱ is s -convex function of Godunova-Levin-Dragomir type on $[a_1, a_2]$, then*

$$\begin{aligned} \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| &\leq \frac{(a_2 - a_1)}{2 \sqrt[p]{2(p+1)}} \left[\left(\frac{(1-s) |\varrho'(a_1)|^q + |\varrho'(a_2)|^q}{(1-s)(2-s)} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\frac{(1-s) |\varrho'(a_2)|^q + |\varrho'(a_1)|^q}{(1-s)(2-s)} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

(2.10)

Corollary 2.16. *If ϱ is P -convex function on $[a_1, a_2]$, then*

$$\left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2\sqrt[p]{p+1}} [|\varrho'(a_1)|^q + |\varrho'(a_2)|^q].$$

Remark 2.17. *We skip the cases of δ - h -convex functions as they gives similar estimates to h -convex functions as discussed earlier.*

Corollary 2.18. *If ϱ is (ϵ, δ) - h -convex function on $[a_1, a_2]$, then*

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2\sqrt[p]{2(p+1)}} \left\{ \left[H_1 |\varrho'(a_1)|^q + H_2 |\varrho'(a_2)|^q + \frac{1}{2} \left[\frac{\epsilon}{6}(a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} \right. \\ (2.11) \quad & \left. + \left[H_1 |\varrho'(a_2)|^q + H_2 |\varrho'(a_1)|^q + \frac{1}{2} \left[\frac{\epsilon}{6}(a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 2.19. *If ϱ is (ϵ, δ) - s -convex function of Breckner type on $[a_1, a_2]$, then*

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2\sqrt[p]{2(p+1)}} \left[\left(\frac{(1+s)|\varrho'(a_1)|^q + |\varrho'(a_2)|^q}{(1+s)(2+s)} + \frac{1}{2} \left[\frac{\epsilon}{6}(a_2 - a_1) + \delta \right] \right)^{\frac{1}{q}} \right. \\ (2.12) \quad & \left. + \left(\frac{(1+s)|\varrho'(a_2)|^q + |\varrho'(a_1)|^q}{(1+s)(2+s)} + \frac{1}{2} \left[\frac{\epsilon}{6}(a_2 - a_1) + \delta \right] \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Corollary 2.20. *If ϱ is (ϵ, δ) - s -convex function of Godunova-Levin-Dragomir type on $[a_1, a_2]$, then*

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2\sqrt[p]{2(p+1)}} \left[\left(\frac{(1-s)|\varrho'(a_1)|^q + |\varrho'(a_2)|^q}{(1-s)(2-s)} + \frac{1}{2} \left[\frac{\epsilon}{6}(a_2 - a_1) + \delta \right] \right)^{\frac{1}{q}} \right. \\ (2.13) \quad & \left. + \left(\frac{(1-s)|\varrho'(a_2)|^q + |\varrho'(a_1)|^q}{(1-s)(2-s)} + \frac{1}{2} \left[\frac{\epsilon}{6}(a_2 - a_1) + \delta \right] \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Corollary 2.21. *If ϱ is (ϵ, δ) - P -convex function on $[a_1, a_2]$, then following inequality hold:*

$$\left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2 \sqrt[p+1]{p+1}} [| \varrho'(a_1) |^q + | \varrho'(a_2) |^q + [\frac{\epsilon}{6}(a_2 - a_1) + \delta]^{\frac{1}{q}}].$$

Corollary 2.22. *If ϱ is strongly h -convex function on $[a_1, a_2]$, then*

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2 \sqrt[p+1]{2(p+1)}} \left[(H_1 | \varrho'(a_1) |^q + H_2 | \varrho'(a_2) |^q - \frac{\epsilon}{12} (a_2 - a_1)^2)^{\frac{1}{q}} \right. \\ (2.14) \quad & \left. + (H_1 | \varrho'(a_2) |^q + H_2 | \varrho'(a_1) |^q - \frac{\epsilon}{12} (a_2 - a_1)^2)^{\frac{1}{q}} \right]. \end{aligned}$$

Corollary 2.23. *If ϱ is strongly s -convex function of Breckner type on $[a_1, a_2]$, then*

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2 \sqrt[p+1]{2(p+1)}} \left[\left(\frac{(1+s) | \varrho'(a_1) |^q + | \varrho'(a_2) |^q}{(1+s)(2+s)} - \frac{\epsilon}{12} (a_2 - a_1)^2 \right)^{\frac{1}{q}} \right. \\ (2.15) \quad & \left. + \left(\frac{(1+s) | \varrho'(a_2) |^q + | \varrho'(a_1) |^q}{(1+s)(2+s)} - \frac{\epsilon}{12} (a_2 - a_1)^2 \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Corollary 2.24. *If ϱ is strongly s -convex function of Godunova-Levin- Dragomir type on $[a_1, a_2]$, then*

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2 \sqrt[p+1]{2(p+1)}} \left[\left(\frac{(1-s) | \varrho'(a_1) |^q + | \varrho'(a_2) |^q}{(1-s)(2-s)} - \frac{\epsilon}{12} (a_2 - a_1)^2 \right)^{\frac{1}{q}} \right. \\ (2.16) \quad & \left. + \left(\frac{(1-s) | \varrho'(a_2) |^q + | \varrho'(a_1) |^q}{(1-s)(2-s)} - \frac{\epsilon}{12} (a_2 - a_1)^2 \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Corollary 2.25. *If ϱ is strongly P -convex function on $[a_1, a_2]$, then*

$$\left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2\sqrt[p]{p+1}} [| \varrho'(a_1) |^q + | \varrho'(a_2) |^q - \frac{\epsilon}{6}(a_2 - a_1)^{\frac{1}{q}}].$$

Theorem 2.26. Let ϱ be a differentiable function defined on $[a_1, a_2]$. If $| \varrho' |^q$ for $q \geq 1$, is a modified (h, d) -convex function on $[a_1, a_2]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2\sqrt[p]{4}} \left[\left(H_3 | \varrho'(a_1) |^q + H_4 | \varrho'(a_2) |^q + \frac{1}{32} d(a_1, a_2) \right)^{\frac{1}{q}} \right. \\ (2.17) \quad & \left. + \left(H_3 | \varrho'(a_2) |^q + H_4 | \varrho'(a_1) |^q + \frac{1}{32} d(a_1, a_2) \right)^{\frac{1}{q}} \right], \end{aligned}$$

where

$$H_3 = \int_0^1 u |1 - 2u| h(u) du, \quad H_4 = \int_0^1 u |1 - 2u| h(1 - u) du.$$

Proof. Taking the modulus on both sides of identity (1.11) and applying the improved power-mean integral inequality, we have

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \left(\frac{a_2 - a_1}{2} \right) \int_0^1 |1 - 2u| | \varrho'(ua_1 + (1 - u)a_2) | du \\ & \leq \left(\frac{a_2 - a_1}{2} \right) \left\{ \left(\int_0^1 (1 - u) |1 - 2u| du \right)^{1 - \frac{1}{q}} \right. \\ & \quad \left. \left(\int_0^1 (1 - u) |1 - 2u| | \varrho'(ua_1 + (1 - u)a_2) |^q du \right)^{\frac{1}{q}} \right\} \end{aligned}$$

$$(2.18) \quad + \left(\int_0^1 u |1-2u| du \right)^{1-\frac{1}{q}} \left(\int_0^1 u |1-2u| |\varrho'(ua_1 + (1-u)a_2)|^q du \right)^{\frac{1}{q}} \Bigg\}.$$

Since $|\varrho'|^q$ is a modified (h, d) -convex function on $[a_1, a_2]$, we get

$$\begin{aligned} \int_0^1 u |1-2u| |\varrho'(ua_1 + (1-u)a_2)|^q du &\leq \int_0^1 u |1-2u| h(u) |\varrho'(a_1)|^q du \\ &\quad + \int_0^1 u |1-2u| h(1-u) |\varrho'(a_2)|^q du + \frac{1}{32} d(a_1, a_2) \end{aligned}$$

and

$$\begin{aligned} \int_0^1 (1-u) |1-2u| |\varrho'(ua_1 + (1-u)a_2)|^q du &\leq \int_0^1 u |1-2u| h(1-u) |\varrho'(a_1)|^q du \\ (2.19) \quad &\quad + \int_0^1 u |1-2u| h(u) |\varrho'(a_2)|^q du + \frac{1}{32} d(a_1, a_2). \end{aligned}$$

On the other hand,

$$(2.20) \quad \int_0^1 u |1-2u| du = \int_0^1 (1-u) |1-2u| du = \frac{1}{4}.$$

Required inequality (2.16) is obtained by using (2.18), (2.19) and (2.20) in (2.17). $\square$

Corollary 2.27. *If ϱ is h -convex function on $[a_1, a_2]$, then*

$$\begin{aligned} &\left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ (2.21) \quad &\leq \frac{(a_2 - a_1)}{2\sqrt[4]{4}} \left[(H_3 |\varrho'(a_1)|^q + H_4 |\varrho'(a_2)|^q)^{\frac{1}{q}} + (H_3 |\varrho'(a_2)|^q + H_4 |\varrho'(a_1)|^q)^{\frac{1}{q}} \right]. \end{aligned}$$

Remark 2.28. If ϱ is s -convex function of Breckner type, then we get [15, Theorem 2.4].

Corollary 2.29. If ϱ is s -convex function of Godunova-Levin-Dragomir type on $[a_1, a_2]$, then

$$(2.22) \quad \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2 \sqrt[4]{4}} \times \left\{ \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(a_1)|^q + ((5-s)2^{s-1}-s-1)|\varrho'(a_2)|^q}{(1-s)(2-s)(3-s)} \right]^{\frac{1}{q}} + \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(a_2)|^q + ((5-s)2^{s-1}-s-1)|\varrho'(a_1)|^q}{(1-s)(2-s)(3-s)} \right]^{\frac{1}{q}} \right\}.$$

Corollary 2.30. If ϱ is P -convex function on $[a_1, a_2]$, then

$$\left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{4} [|\varrho'(a_1)|^q + |\varrho'(a_2)|^q]^{\frac{1}{q}}.$$

Remark 2.31. We skip the cases of δ - h -convex functions as they gives similar estimates to h -convex functions as discussed earlier.

Corollary 2.32. If ϱ is (ϵ, δ) - h -convex function on $[a_1, a_2]$, then

$$(2.23) \quad \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2 \sqrt[4]{4}} \left\{ \left[H_3 |\varrho'(a_1)|^q + H_4 |\varrho'(a_2)|^q + \frac{1}{4} \left[\frac{\epsilon}{8} (a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} + \left[H_3 |\varrho'(a_2)|^q + H_4 |\varrho'(a_1)|^q + \frac{1}{4} \left[\frac{\epsilon}{8} (a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} \right\}.$$

Corollary 2.33. If ϱ is (ϵ, δ) - s -convex function of Breckner type on $[a_1, a_2]$, then

$$(2.24) \quad \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2 \sqrt[4]{4}} \times \left\{ \left[\frac{(s+1)(s+1+2^{-s-1})|\varrho'(a_1)|^q + ((5+s)2^{-s-1}+s-1)|\varrho'(a_2)|^q}{(s+1)(s+2)(s+3)} + \frac{1}{4} \left[\frac{\epsilon}{8} (a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} + \left[\frac{(s+1)(s+1+2^{-s-1})|\varrho'(a_2)|^q + ((5+s)2^{-s-1}+s-1)|\varrho'(a_1)|^q}{(s+1)(s+2)(s+3)} + \frac{1}{4} \left[\frac{\epsilon}{8} (a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} \right\}.$$

Corollary 2.34. If ϱ is (ϵ, δ) - s -convex function of Godunova-Levin-Dragomir type on $[a_1, a_2]$, then

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2\sqrt[4]{4}} \\ & \times \left\{ \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(a_1)|^q + ((5-s)2^{s-1}-s-1)|\varrho'(a_2)|^q}{(1-s)(2-s)(3-s)} + \frac{1}{4} \left[\frac{\epsilon}{8}(a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} \right. \\ (2.25) \quad & \left. + \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(a_2)|^q + ((5-s)2^{s-1}-s-1)|\varrho'(a_1)|^q}{(1-s)(2-s)(3-s)} + \frac{1}{4} \left[\frac{\epsilon}{8}(a_2 - a_1) + \delta \right] \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 2.35. If ϱ is (ϵ, δ) - P -convex function on $[a_1, a_2]$, then

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{4} [|\varrho'(a_1)|^q + |\varrho'(a_2)|^q] \\ & + \left[\frac{\epsilon}{8}(a_2 - a_1) + \delta \right]^{\frac{1}{q}}. \end{aligned}$$

Corollary 2.36. If ϱ is strongly h -convex function on $[a_1, a_2]$, then

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ & \leq \frac{(a_2 - a_1)}{2\sqrt[4]{4}} \left\{ [H_3 |\varrho'(a_1)|^q + H_4 |\varrho'(a_2)|^q - \frac{\epsilon}{32}(a_2 - a_1)^2]^{\frac{1}{q}} \right. \\ (2.26) \quad & \left. + [H_3 |\varrho'(a_2)|^q + H_4 |\varrho'(a_1)|^q - \frac{\epsilon}{32}(a_2 - a_1)^2]^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 2.37. If ϱ is strongly s -convex function of Breckner type on $[a_1, a_2]$, then

$$\begin{aligned} & \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2\sqrt[4]{4}} \\ & \times \left\{ \left[\frac{(s+1)(s+1+2^{-s-1})|\varrho'(a_1)|^q + ((5+s)2^{-s-1}+s-1)|\varrho'(a_2)|^q}{(s+1)(s+2)(s+3)} - \frac{\epsilon}{32}(a_2 - a_1)^2 \right]^{\frac{1}{q}} \right. \\ (2.27) \quad & \left. + \left[\frac{(s+1)(1+s+2^{-s-1})|\varrho'(a_2)|^q + ((5+s)2^{-s-1}+s-1)|\varrho'(a_1)|^q}{(s+1)(s+2)(s+3)} - \frac{\epsilon}{32}(a_2 - a_1)^2 \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Corollary 2.38. *If ϱ is strongly s -convex function of Godunova-Levin- Dragomir type on $[a_1, a_2]$, then*

$$(2.28) \quad \left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \leq \frac{(a_2 - a_1)}{2^{\frac{1}{q/4}}} \\ \times \left\{ \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(a_1)|^q + ((5-s)2^{s-1}-s-1)|\varrho'(a_2)|^q}{(1-s)(2-s)(3-s)} - \frac{\epsilon}{32}(a_2 - a_1)^2 \right]^{\frac{1}{q}} \right. \\ \left. + \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(a_2)|^q + ((5-s)2^{s-1}-s-1)|\varrho'(a_1)|^q}{(1-s)(2-s)(3-s)} - \frac{\epsilon}{32}(a_2 - a_1)^2 \right]^{\frac{1}{q}} \right\}.$$

Corollary 2.39. *If ϱ is strongly P -convex function on $[a_1, a_2]$, then*

$$\left| \frac{\varrho(a_1) + \varrho(a_2)}{2} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \varrho(x) dx \right| \\ \leq \frac{(a_2 - a_1)}{4} \left[|\varrho'(a_1)|^q + |\varrho'(a_2)|^q - \frac{\epsilon}{8}(a_2 - a_1)^2 \right]^{\frac{1}{q}}.$$

3. Applications to numerical integration

In this section, we give some new error estimates for the trapezoidal quadrature formula by using inequalities developed in the Section 2. Assume that P is a partition of the interval $[a_1, a_2]$, i.e.,

$$P : a_1 = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = a_2.$$

Trapezoidal quadrature formula is defined by

$$\int_{a_1}^{a_2} \varrho(x) dx = T(\varrho, P) + E(\varrho, P),$$

where

$$(3.1) \quad T(\varrho, P) = \sum_{i=0}^{n-1} \left[\frac{\varrho(x_i) + \varrho(x_{i+1})}{2} \right] h_i$$

is the trapezoidal version for $h_i = (x_{i+1} - x_i)$ and $i = 0, 1, \dots, n-1$, and $E(\varrho, P)$ is denote their associated approximation error.

Proposition 3.1. Let $\varrho : [a_1, a_2] \rightarrow \mathbf{R}$ be a differentiable function on (a_1, a_2) , where $a_1 < a_2$. If $|\varrho'|^q$ is strongly s -convex of Godunova-Levin-Drăgomir type on $[a_1, a_2]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$(3.2) \quad \left| E(\varrho, P) \right| \leq \frac{1}{2^{\frac{1}{p}} \sqrt[p]{2(p+1)}} \sum_{i=0}^{n-1} h_i^2 \\ \times \left[\left(\frac{(1-s)|\varrho'(x_i)|^q + |\varrho'(x_{i+1})|^q}{(1-s)(2-s)} - \frac{\epsilon}{12} h_i^2 \right)^{\frac{1}{q}} + \left(\frac{(1-s)|\varrho'(x_{i+1})|^q + |\varrho'(x_i)|^q}{(1-s)(2-s)} - \frac{\epsilon}{12} h_i^2 \right)^{\frac{1}{q}} \right].$$

Proof. Applying inequality (3.1) in Corollary 2.24 on the subintervals $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) of the partition P , we have

$$(3.3) \quad \left| \frac{\varrho(x_i) + \varrho(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \varrho(x) dx \right| \leq \frac{(x_{i+1} - x_i)}{2^{\frac{1}{p}} \sqrt[p]{2(p+1)}} \\ \times \left[\left(\frac{(1-s)|\varrho'(x_i)|^q + |\varrho'(x_{i+1})|^q}{(1-s)(2-s)} - \frac{\epsilon}{12} h_i^2 \right)^{\frac{1}{q}} + \left(\frac{(1-s)|\varrho'(x_{i+1})|^q + |\varrho'(x_i)|^q}{(1-s)(2-s)} - \frac{\epsilon}{12} h_i^2 \right)^{\frac{1}{q}} \right].$$

Hence from (3.3), we get

$$\left| E(\varrho, P) \right| = \left| \int_{a_1}^{a_2} \varrho(x) dx - T(\varrho, P) \right| \\ \leq \left| \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \varrho(x) dx - \frac{\varrho(x_i) + \varrho(x_{i+1})}{2} h_i \right\} \right| \\ \leq \sum_{i=0}^{n-1} \left| \left\{ \int_{x_i}^{x_{i+1}} \varrho(x) dx - \frac{\varrho(x_i) + \varrho(x_{i+1})}{2} h_i \right\} \right| \\ \leq \frac{1}{2^{\frac{1}{p}} \sqrt[p]{2(p+1)}} \sum_{i=0}^{n-1} h_i^2 \\ \times \left[\left(\frac{(1-s)|\varrho'(x_i)|^q + |\varrho'(x_{i+1})|^q}{(1-s)(2-s)} - \frac{\epsilon}{12} h_i^2 \right)^{\frac{1}{q}} + \left(\frac{(1-s)|\varrho'(x_{i+1})|^q + |\varrho'(x_i)|^q}{(1-s)(2-s)} - \frac{\epsilon}{12} h_i^2 \right)^{\frac{1}{q}} \right].$$

$$+ \left(\frac{(1-s)|\varrho'(x_{i+1})|^q + |\varrho'(x_i)|^q}{(1-s)(2-s)} - \frac{\epsilon}{12} h_i^2 \right)^{\frac{1}{q}} \Bigg].$$

The proof of Proposition 3.1 is completed. $\square$

Proposition 3.2. *Let $\varrho : [a_1, a_2] \longrightarrow \mathbf{R}$ be a differentiable function on (a_1, a_2) , where $a_1 < a_2$. If $|\varrho'|^q$ is strongly s -convex of Godunova-Levin-Drăgomir type on $[a_1, a_2]$ for $q \geq 1$, then the following inequality holds:*

$$(3.4) \quad \begin{aligned} |E(f, Q)| &\leq \frac{1}{2\sqrt[p]{4}} \sum_{i=0}^{n-1} h_i^2 \\ &\times \left\{ \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(x_i)|^q + ((5-s)2^{s-1}-s-1)|\varrho'(x_{i+1})|^q}{(1-s)(2-s)(3-s)} - \frac{\epsilon}{32} h_i^2 \right]^{\frac{1}{q}} \right. \\ &\left. + \left[\frac{(1-s)(1-s+2^{s-1})|\varrho'(x_{i+1})|^q + ((5-s)2^{s-1}-s-1)|\varrho'(x_i)|^q}{(1-s)(2-s)(3-s)} - \frac{\epsilon}{32} h_i^2 \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Proof. The proof is analogous as to that of Proposition 3.1 but use inequality (2.28) in Corollary 2.38. $\square$

Remark 3.3. *By using Theorem 2.1, for appropriate choices of function ϱ and function $h(u) = u, u^{-1}, u^s, u^{-s}, u(1-u), \frac{\sqrt{u}}{2\sqrt{1-u}}$ and $h(u) = 1$, we can obtain some new integral inequalities using special means, such as arithmetic, geometric, logarithmic etc. Also, applying our Theorems 2.12 and 2.26 for above suitable functions h , we can deduce some new integral inequalities using ideas and techniques of Propositions 3.1 and 3.2. We omit their proofs and the details are left to the interested reader.*

Conclusion

In this study, by applying Hölder–İşcan integral inequality three inequalities of Hermite–Hadamard and Hadamard integral type for modified (h, d) -convex functions have been provided. Various special cases including classes for instance, h -convex, s -convex function of Breckner and Godunova–Levin–Drăgomir and strong versions of the aforementioned types of convex functions have been obtained. Also, some new applications to special means and error estimations of presented results can be analyzed by interested readers but we omit here their proofs. We hope that current work will attract the attention of researchers working in mathematical analysis, fractional calculus, (p, q) -calculus and other related fields.

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