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Exponential stability and instability in nonlinear differential equations with multiple delays

Adusei-Poku Afful
University of Cape Coast, Ghana
and
Ernest Yankson
University of Cape Coast, Ghana
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Abstract

Inequalities regarding the solutions of the nonlinear differential equation with multiple delays

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(x(t-h_i)),$$

are obtained by means of Lyapunov functionals. These inequalities are then used to obtain sufficient conditions that guarantee exponential decay of solutions to zero of the multi delay nonlinear differential equation.

In addition, we obtain a criterion for the instability of the zero solution. The results generalizes some results in the literature.

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1. Introduction

The study of the stability properties of differential equations has attracted the attention of many mathematicians lately, see [1], [4], [5], [6], [7], [8], [9],[10], [11] and the references cited therein. In particular, Wang in [11] considered the scalar linear functional differential equation

$$x'(t) = a(t)x(t) + b(t)x(t-h). \tag{1.1}$$

The author obtained sufficient conditions for asymptotic stability if $a(t) \geq 0$ and instability if $a(t) \leq 0$ for (1.1).

Moreover, Cable and Raffoul in [1] used Lyapunov functionals to obtain sufficient conditions that guarantee exponential decay of solutions to zero for the multi delay linear differential equation

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i). \tag{1.2}$$

They also obtained a criterion for instability of the zero solution of (1.2). The highlight of the paper by Cable and Raffoul is that $a(t)$ is allowed to change signs. Motivated by [1] and [11] we consider the scalar nonlinear differential equation with multi delay

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(x(t-h_i)), \tag{1.3}$$

where a, b are continuous with $0 < h_i \leq h^*$ for $i = 1, ..., n$ for some positive constant h^* and $f : \mathbf{R} \to \mathbf{R}$ with $f(0) = 0$ is continuous.

Let

$$f_1(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

Then equation (1.3) is equivalent to

$$x'(t) = Q(t)f_1(x(t))x(t) - \sum_{i=1}^{n} \frac{d}{dt}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds, \tag{1.4}$$

where

$$Q(t) = \Big(a(t) + \sum_{i=1}^{n} b_i(t+h_i)\Big).$$

Let $\psi : [-h^*, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$||\psi|| = \max_{-h^* \le s \le 0} |\psi(s)|.$$

We also denote the norm of a function $\varphi : [-h^*, \infty) \to (-\infty, \infty)$ by

$$||\varphi|| = \sup_{-h^* \le s \le \infty} |\varphi(s)|.$$

We say that $x(t) \equiv x(t, t_0, \psi)$ is a solution of (1.3) if $x(t)$ satisfies (1.3) for $t \ge t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s)$, $s \in [-h^*, 0]$. We end this section by stating a Lemma which contains the Lyapunov functionals we propose to use to obtain our inequalities in this paper without proof.

Lemma 1.1. *Let δ and H be constants such that $\delta > 0$ and $H > 0$. Then the functionals $V(t)$ and $V_1(t)$ defined by*

$$\begin{aligned} V(t) &= \Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\Big]^2 \\ &\quad + \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} V_1(t) &= \Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\Big]^2 \\ &\quad - H \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds, \end{aligned} \tag{1.6}$$

respectively are Lyapunov functionals.

The rest of the paper is organized as follows. In the next section, we use a Lyapunov functional to obtain inequalities that can guarantee exponential stability and in the final section inequalities that lead to instability are obtained by a Lyapunov functional for (1.3).

2. Exponential Stability

In this section we obtain inequalities that can be used to deduce the exponential stability of (1.3).

Lemma 2.1. *Suppose that*

$$\frac{\delta}{-(\delta+1)h^*} \leq Q(t)f_1(x(t)) \leq -\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t)) \tag{2.1}$$

hold for $\delta > 0$. *If* $f_1(x) \geq 1$ *then*

$$V'(t) \leq Q(t)V(t) \tag{2.2}$$

where $V(t)$ *is given by (1.5).*

Proof.
Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.3) with $V(t)$ defined by (1.5). It must be noted that $Q(t) < 0$ for all $t \geq 0$ in view of condition (2.1) and the fact that $f_1(x) \geq 1$. Then along the solutions of (1.4) we have

$$\begin{aligned}
V'(t) &= 2\left[x(t) + \sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]Q(t)f_1(x(t))x(t) \\
&\quad + \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+h_i)f^2(x(t))ds \\
&\quad - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds \\
&\leq 2\left[x(t) + \sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]Q(t)f_1(x(t))x(t) \\
&\quad + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t)) \\
&\quad - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds \\
&\leq Q(t)f_1(x(t))\left[x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right] \\
&\quad + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t)
\end{aligned}$$

$$- \delta \sum_{i=1}^{n} \int_{-h_i}^{0} b_i^2(t+s+h_i) f^2(x(t+s)) ds + Q(t) f_1(x(t)) x^2(t)$$

$$\begin{aligned}
&\leq Q(t)f_1(x(t))V(t) - Q(t)f_1(x(t))\delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds \\
&\quad - Q(t)f_1(x(t)) \left(\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds \right)^2 \\
&\quad - \delta \sum_{i=1}^{n} \int_{-h_i}^{0} b_i^2(t+s+h_i) f^2(x(t+s)) ds \\
&\quad + \left(\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i) f_1^2(x(t)) + Q(t) f_1(x(t)) \right) x^2(t). \qquad (2.3)
\end{aligned}$$

If we let $u = t + s$, then

$$\delta \sum_{i=1}^{n} \int_{-h_i}^{0} b_i^2(t+s+h_i) f^2(x(t+s)) ds = \delta \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(u+h_i) f^2(x(u)) ds. \qquad (2.4)$$

By the Holder's inequality we obtain

$$\begin{aligned}
&-Q(t)f_1(x(t)) \left(\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds \right)^2 \\
&\leq -Q(t) f_1(x(t)) h^* \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds. \qquad (2.5)
\end{aligned}$$

We observe that

$$\delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds \leq \delta h^* \int_{t-h_i}^{t} \sum_{i=1}^{n} b_i^2(s+h_i) f^2(x(s)) ds \qquad (2.6)$$

Substituting (2.4), (2.5), and (2.6) into (2.3) and making use of (2.1) we obtain

$$V'(t) \leq Q(t) f_1(x(t)) V(t) + \Big[- Q(t) f_1(x(t)) h^* (\delta + 1)$$

$$
\begin{aligned}
&\quad - \delta\Big] \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\
&\quad + \left(\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i) f_1^2(x(t)) + Q(t) f_1(x(t)) \right) x^2(t) \\
&\leq Q(t) f_1(x(t)) V(t) \\
&\leq Q(t) V(t).
\end{aligned}
$$

Theorem 2.2. *Suppose that condition (2.1) hold. Then any solution $x(t) = x(t, t_0, \psi)$ satisfies the exponential inequality*

$$
|x(t)| \leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big) V(t_0) e^{-\frac{\delta h^*}{2} \int_{t_0}^{t-\frac{h_i}{2}} [\sum_{i=1}^{n} b_i^2(s+h_i)] ds}} \tag{2.7}
$$

for $t \geq t_0 + h^*/2$.

Proof.
By changing the order of integration of the second term in $V(t)$ given by (1.5) and using the fact that $t - \frac{h_i}{2} \leq z \leq t$ implies that $\frac{h_i}{2} \leq z - t + h_i \leq h_i$ we obtain

$$
\begin{aligned}
&\quad \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds \\
&= \delta \sum_{i=1}^{n} \int_{t-h_i}^{t} \int_{-h_i}^{z-t} b_i^2(z+h_i) f^2(x(z)) ds dz \\
&= \delta \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(z+h_i) f^2(x(z))(z-t+h_i) dz \\
&= \delta \sum_{i=1}^{n} \int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(z+h_i) f^2(x(z))(z-t+h_i) dz \\
&\quad + \delta \sum_{i=1}^{n} \int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i) f^2(x(z))(z-t+h_i) dz
\end{aligned}
$$

$$
\begin{aligned}
&\geq \delta \sum_{i=1}^{n} \int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i) f^2(x(z))(z-t+h_i) dz \\
&\geq \delta \sum_{i=1}^{n} \frac{h_i}{2} \int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i) f^2(x(z)) dz \qquad (2.8)
\end{aligned}
$$

Thus, in view of (2.8) we have

$$\begin{aligned} V(t) &\geq \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds \\ &\geq \delta \sum_{i=1}^{n} \frac{h_i}{2} \int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i) f^2(x(z)) dz \end{aligned}$$

Consequently

$$V\Big(t - \frac{h_i}{2}\Big) \geq \delta \sum_{i=1}^{n} \frac{h_i}{2} \int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(z+h_i) f^2(x(z)) dz. \tag{2.9}$$

Due to the fact that $V'(t) \leq 0$ we have for $t \geq t_0 + h^*/2$ that

$$0 \leq V(t) + V\Big(t - \frac{h_i}{2}\Big) \leq 2V\Big(t - \frac{h_i}{2}\Big).$$

Using (2.8) and (2.9) we obtain

$$\begin{aligned} & V(t) + V\Big(t - \frac{h_i}{2}\Big) \\ = & \Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\Big]^2 \\ & + \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds + V\Big(t - \frac{h_i}{2}\Big) \\ \geq & \Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\Big]^2 + \delta \sum_{i=1}^{n} \frac{h_i}{2} \int_{t-\frac{h_i}{2}}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\ & + \delta \sum_{i=1}^{n} \frac{h_i}{2} \int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(s+h_i) f^2(x(s)) ds \\ \geq & \Big[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\Big]^2 + \frac{\delta}{2} \Big(\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\Big)^2 \\ = & \frac{\delta}{2+\delta} x^2(t) + \Big[\frac{1}{\sqrt{1+\frac{\delta}{2}}} x(t) + \sqrt{1+\frac{\delta}{2}} \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\Big]^2 \\ \geq & \frac{\delta}{2+\delta} x^2(t). \end{aligned}$$

Consequently,

$$\frac{\delta}{2+\delta}x^2(t) \le V(t) + V\Big(t - \frac{h_i}{2}\Big) \le 2V\Big(t - \frac{h_i}{2}\Big) \tag{2.10}$$

An integration of (2.2) from t_0 to t yields $V(t) \le V(t_0)e^{\int_{t_0}^{t}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds}$.
This implies that $V\Big(t - \frac{h_i}{2}\Big) \le V(t_0)e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds}$.
It follows from (2.11) that

$$x^2(t) \le 2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds}.$$

Hence

$$\begin{aligned} |x(t)| &\le \sqrt{2\Big(\tfrac{2+\delta}{\delta}\Big)V(t_0)}e^{\frac{1}{2}\int_{t_0}^{t-\frac{h_i}{2}}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds} \\ &\le \sqrt{2\Big(\tfrac{2+\delta}{\delta}\Big)V(t_0)}e^{-\frac{\delta h^*}{2}\int_{t_0}^{t-\frac{h_i}{2}}[\sum_{i=1}^{n} b_i^2(s+h_i)]ds} \end{aligned}$$

for $t \ge t_0 + h^*/2$. This completes the proof.

Corollary 2.3. *Suppose condition (2.1) hold. If $f_1(x) \ge 1$ and*

$$\sum_{i=1}^{n} b_i^2(s+h_i) \ge \gamma \tag{2.11}$$

for some positive constant γ for all $t \ge t_0$ then the zero solution of (1.3) is exponentially stable.

Proof.
It follows from (2.7) that

$$\begin{aligned} |x(t)| &\le \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)}e^{-\frac{\delta h^*}{2}\int_{t_0}^{t-\frac{h_i}{2}}[\sum_{i=1}^{n} b_i^2(s+h_i)]ds} \\ &\le \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)}e^{-\frac{\delta h^*}{2}\gamma(t-t_0)}. \end{aligned}$$

This completes the proof.

3. Instability Criteria

In this section, a Lyapunov functional is used to obtain a criterion that can be applied to test for instability of the zero solution of (1.1).

Lemma 3.1. *Assume that there exists a positive constant $H > h^*$ such that*

$$Q(t)f_1(x(t)) - H\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t)) \geq 0. \tag{3.1}$$

If $f_1(x) \geq 1$ and $V_1(t)$ is given by (1.6), then

$$V_1'(t) \geq Q(t)V_1(t). \tag{3.2}$$

Proof.
In view of condition (3.1 and the fact that $f_1(x) \geq 1$ it is clear that $Q(t) > 0$ for all $t \geq 0$. Let $x(t) = x(t, t_0, \psi)$ be a solution of (1.3) with $V_1(t)$ defined by (1.6). Taking the time derivative of the functional $V_1(t)$ along the solution of (1.3) we have

$$\begin{aligned}
V_1'(t) &= 2\left[x(t) + \sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]Q(t)f_1(x(t))x(t) \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t)) + H\sum_{i=1}^{n} b_i^2(t)f^2(x(t-h_i)) \\
&= Q(t)f_1(x(t))\left[x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right] \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t) + H\sum_{i=1}^{n} b_i^2(t)f_1^2(x(t-h_i))x^2(t-h_i) \\
&\quad + Q(t)f_1(x(t))x^2(t) \\
&= Q(t)f_1(x(t))V(t) + Q(t)f_1(x(t))H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
&\quad - Q(t)f_1(x(t))\Big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big)^2 \\
&\quad + H\sum_{i=1}^{n} b_i^2(t)f_i^2(x(t-h_i))x^2(t-h_i)
\end{aligned}$$

$$
\begin{aligned}
& + Q(t)f_1(x(t))x^2(t) - H\sum_{i=1}^{n} b_i^2(t+h_i)f_i^2(x(t))x^2(t) \\
= \quad & Q(t)f_1(x(t))V(t) + Q(t)f_1(x(t))\Big[H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
& -\Big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big)^2\Big] \qquad (3.3)\\
& + \Big[Q(t)f_1(x(t)) - H\sum_{i=1}^{n} b_i^2(t+h_i)f_i^2(x(t))\Big]x^2(t) \\
& + H\sum_{i=1}^{n} b_i^2(t)f_i^2(x(t-h_i))x^2(t-h_i) \\
& \geq Q(t)f_1(x(t))V_1(t) \\
& \geq Q(t)V_1(t), \qquad (3.4)
\end{aligned}
$$

where we have used (3.1) and the fact that by Holder's inequality

$$
\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b(s+h_i)f^2(x(s))ds\right)^2 \leq h^*\sum_{i=1}^{n}\int_{t-h_i}^{t} b^2(s+h_i)f^2(x(s))ds.
$$

This completes the proof.

Theorem 3.2. *Suppose that condition (3.1) hold and* $f_1(x) \geq 1$. *Then the zero solution of (1.3) is unstable, provided that*

$$
\sum_{i=1}^{n}\int_{t_0}^{\infty} b_i^2(s+h_i)ds = \infty.
$$

Proof.
Integrating (3.5) from t_0 to t we obtain

$$
V_1(t) \quad \geq \quad V(t_0)e^{\int_{t_0}^{t}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds}. \qquad (3.5)
$$

With $V(t)$ given by (1.6) we have

$$
V_1(t) \quad = \quad x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds
$$

$$+ \left(\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds \right)^2$$

$$- H \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \tag{3.6}$$

Let $\beta = H - h^*$. Then it follows from
$\left(\frac{\sqrt{h}}{\sqrt{\beta}} a - \frac{\sqrt{\beta}}{\sqrt{h}} b \right)^2 \geq 0$
that

$$2ab \leq \frac{h}{\beta} a^2 + \frac{\beta}{h} b^2 \tag{3.7}$$

Using inequality (3.7) we obtain

$$\begin{aligned}
2x(t) \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds &\leq 2|x(t)| |\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds| \\
&\leq \frac{h_i}{\beta} x^2(t) + \frac{\beta}{h_i} \left(\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds \right)^2 \\
&\leq \frac{h^*}{\beta} x^2(t) + \beta \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \qquad (3.8)
\end{aligned}$$

Substituting (3.8) into (3.6) we obtain

$$\begin{aligned}
V_1(t) &\leq x^2(t) + \frac{h^*}{\beta} x^2(t) + (\beta + h^* - H) \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\
&= x^2(t) + \frac{h^*}{\beta} x^2(t) \\
&= \frac{\beta + h^*}{\beta} x^2(t) \\
&= \frac{H}{H - h^*} x^2(t) \qquad (3.9)
\end{aligned}$$

Using inequalities (3.1), (3.5) and (3.9) we obtain

$$|x(t)| \geq \sqrt{\frac{H - h^*}{H}} V_1^{1/2}(t_0) e^{\frac{1}{2} \int_{t_0}^{t} [a(s) + \sum_{i=1}^{n} b_i(s+h_i)] ds}$$

$$
\begin{aligned}
&= \sqrt{\frac{H-h^*}{H}} V_1^{1/2}(t_0) e^{\frac{1}{2}\int_{t_0}^{t}[a(s)+\sum_{i=1}^{n} b_i(s+h_i)]ds} \\
&\geq \sqrt{\frac{H-h^*}{H}} V_1^{1/2}(t_0) e^{\frac{H}{2}\int_{t_0}^{t}\sum_{i=1}^{n} b_i^2(s+h_i)ds}.
\end{aligned}
$$

This completes the proof.

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Adusei-Poku Afful
Department of Mathematics
University of Cape Coast
Ghana
e-mail: opokuafful2@gmail.com

and

Ernest Yankson
Department of Mathematics
University of Cape Coast
Ghana
e-mail: ernestoyank@gmail.com
Corresponding author