Independent form of (θ, s) -continuous functions in topological spaces

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Abstract:

We introduce a new class of almost contra- P_β -continuous functions which is a subclass of the class of almost contra-precontinuous functions [8]. This class contains the classes of regular set connected functions, perfectly continuous functions and contra- P_β -continuous functions. It is shown that almost contra- P_β -continuity is independent to (θ, s) -continuity [12] and contra-precontinuity [11]. Furthermore, we obtain basic properties and preservations theorems for almost contra- P_β -continuity.

Keywords: Almost contra-precontinuous functions; (θ, s) -continuous functions; Almost contra- P_β -continuous functions; P_β -open, topological space.

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1. Introduction

In 1996, Dontchev introduced the notion of contra-continuous functions [6]. Baker introduced almost contra-continuous and contra almost β -continuous functions [4]. Dontchev, Ganster and Reilly introduced a new class of functions called regular set connected functions [7] and Jafari and Noiri introduced contra-precontinuous functions [11]. Ekici introduced and studied about almost contra-precontinuous functions [8]. Joseph and Kwack introduced the notion of (θ, s) -continuous functions [12]. Noiri and Popa [17] pointed out that almost contra-continuity is equivalent to (θ, s) -continuity. Recently, the new class of functions called contra- P_β -continuous functions [30] was studied also (see [[26]-[30]]). In this paper, we introduced the notion of almost contra- P_β -continuity which is a strong form of almost contra-precontinuity and weaker than the notion of regular set connected functions, perfectly continuous functions, and contra- P_β -continuous functions. Also, it is shown that almost contra- P_β -continuity is independent to (θ, s) -continuity and contra-precontinuity.

This paper is organized as follows. Section-2 develops the necessary preliminaries. In Section-3, the concept of almost contra- P_β -continuity is introduced, and we obtain characterization and basic properties of this notion. In Section-4 axioms are studied in relation to almost contra- P_β -continuity. In section-5, P_β -closed, P_β -regular and contra- P_β -closed graphs of an almost contra- P_β -continuous functions are defined. Sections-6 gives sufficient conditions for a space to be connected and hyperconnected.

2. Preliminaries

Let (X, τ) or X (resp. (Y, σ) or Y , (Z, v) or Z) be a topological space or space. We will denote by $Cl(A)$ and $Int(A)$, the closure of A and the interior of A , respectively for a subset A of a space.

Definition 2.1. A subset A of a space is said to be

1. α -open [15] if $A \subseteq Int(Cl(Int(A)))$;
2. pre-open [19] if $A \subseteq Int(Cl(A))$;
3. semi-open [14] if $A \subseteq Cl(Int(A))$;
4. β -open [1] if $A \subseteq Cl(Int(Cl(A)))$;
5. regular open [23] if $A = Int(Cl(A))$;

The complement of pre-open (resp. semi-open, α -open, β -open, regular open) set is said to be *pre-closed* (resp. *semi-closed*, *α -closed*, *β -closed*, *regular closed*).

Definition 2.2. [26] A pre-open subset A of a topological space X is said to be P_β -open, if for each $x \in A$ there exists a β -closed set F such that $x \in F \subseteq A$.

That is, the pre-open sets is expressed as a union of β -closed sets.

The family of all pre-open (resp. semi-open, α -open, β -open, regular open, P_β -open) subsets of a topological space (X, τ) is denoted by $PO(X)$ (resp. $SO(X)$, $\alpha O(X)$, $\beta O(X)$, $RO(X)$, $P_\beta O(X)$). The collections $SO(X)$, $PO(X)$, $\beta O(X)$ and $P_\beta O(X)$ are closed under arbitrary unions. The family of all closed (resp. pre-closed, semi-closed, α -closed, β -closed, regular closed, P_β -closed) subsets of a topological space (X, τ) is denoted by $C(X)$ (resp. $PC(X)$, $SC(X)$, $\alpha C(X)$, $\beta C(X)$, $RC(X)$, $P_\beta C(X)$). Set $C(X, x) = \{V \in C(X) | x \in V\}$ for $x \in X$ and similarly $PO(X, x)$, $SO(X, x)$, $\alpha O(X, x)$, $\beta O(X, x)$, $P_\beta O(X, x)$.

Definition 2.3. A point $x \in X$ is said to be θ -semi-cluster point [12] of a subset A of X if $Cl(U) \cap A \neq \emptyset$ for every $U \in SO(X, x)$. The set of all θ -semi-cluster points of A is called the θ -semi-closure of A and is denoted by $\theta sCl(A)$. A subset is called θ -semi-closed if $A = \theta sCl(A)$. The complement of a θ -semi-closed set is called θ -semi-open.

Definition 2.4. A function $f : X \rightarrow Y$ is called

1. *contra- P_β -continuous* [30] if $f^{-1}(V)$ is P_β -closed for each open set V of Y .
2. *almost- P_β -continuous* [30] if $f^{-1}(V)$ is P_β -open for each regular open set V of Y .
3. *P_β -continuous* [26] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a $U \in P_\beta O(X, x)$ such that $f(U) \subset V$.
4. *precontinuous* [19] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a $U \in PO(X, x)$ such that $f(U) \subset V$.
5. *regular set connected* [7] if $f^{-1}(V)$ is clopen for each regular open set V of Y .

6. almost contra-continuous [4] ((θ, s) -continuous [12]) if $f^{-1}(V)$ is closed for each regular open set V of Y .
7. almost contra-precontinuous [8] if $f^{-1}(V)$ is preclosed for each regular open set V of Y .
8. almost contra- β -continuous [4] if $f^{-1}(V)$ is β -closed for each regular open set V of Y .

Lemma 2.5. [16] For a subset A of a topological space Y , the following statements hold:

1. $\alpha Cl(A) = Cl(A)$ for every $A \in \beta O(Y)$;
2. $pCl(A) = Cl(A)$ for every $A \in SO(Y)$;
3. $sCl(A) = Int(Cl(A))$ for every $A \in PO(Y)$.

Definition 2.6. Let A be a subset of a space (X, τ) .

1. The set $\cap\{U \in RO(X) \mid A \subset U\}$ is called the r -kernel [9] of A and is denoted by $rker(A)$.
2. The set $\cap\{F \in P_\beta C(X) \mid A \subset F\}$ is called the P_β -closure [26] of A and is denoted by $P_\beta Cl(A)$.
3. The set $\cup\{U \in P_\beta O(X) \mid U \subset A\}$ is called the P_β -Interior [26] of A and is denoted by $P_\beta Int(A)$.

Lemma 2.7. [9] The following statements hold for subsets A and B of a space X :

1. $x \in rker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in RC(X, x)$.
2. $A \subset ker(A)$ and $A = ker(A)$ if A is regular open in X .
3. if $A \subset B$, then $rker(A) \subset rker(B)$.

3. Almost contra- P_β -continuity

Definition 3.1. A function $f : X \rightarrow Y$ is called almost contra- P_β -continuous if $f^{-1}(V)$ is P_β -closed in X for each regular open set V of Y .

Theorem 3.2. Every regular set connected function is almost contra- P_β -continuous.

Proof. Suppose that $f : X \rightarrow Y$ is a regular set connected function. Then for each regular open set V of Y , $f^{-1}(V)$ is clopen, which implies $f^{-1}(V)$ is P_β -closed.

Theorem 3.3. Every almost contra- P_β -continuous function is almost contra-precontinuous.

Proof. Suppose that $f : X \rightarrow Y$ is a almost contra- P_β -continuous function. Then for each regular open set V of Y , $f^{-1}(V)$ is P_β -closed and hence preclosed.

Theorem 3.4. Every contra- P_β -continuous function is almost contra- P_β -continuous.

Proof. Suppose that $f : X \rightarrow Y$ is a contra- P_β -continuous function. Then for each regular open set V of Y , $f^{-1}(V)$ is P_β -closed. The following diagram summarizes the obvious implications, $A \rightarrow B$ denotes A implies B but not conversely:

Example 3.5. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Define a function $f : (X, \tau) \rightarrow (X, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is almost contra- P_β -continuous but not regular set-connected, not contra- P_β -continuous not contra pre-continuous and not almost contra-continuous.

Example 3.6. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Define a function $f : (X, \tau) \rightarrow (X, \sigma)$ by $f(d) = b$ and $f(b) = a$. Then f is contra-precontinuous, almost contra-pre-continuous and almost contra-continuous but not almost contra- P_β -continuous.

The other implications not reversible are shown in [4, 8, 30, 6]

Remark 3.7. From Example 3.5 and Example 3.6 note that almost contra- P_β -continuity is independent to both almost contra-continuity and contra-precontinuity. Hence almost contra- P_β -continuity is independent to both (θ, s) -continuity and contra-precontinuity

Theorem 3.8. The following statements are equivalent for a function $f : X \rightarrow Y$:

1. f is almost contra- P_β -continuous;
2. the inverse image of each regular closed set in Y is P_β -open in X ;
3. the inverse image of a θ -semi-open set of Y is P_β -open;
4. the inverse image of a θ -semi-closed set of Y is P_β -closed;
5. for each point x in X and each $V \in RC(Y, f(x))$, there is $U \in P_\beta O(X, x)$ such that $f(U) \subset V$;
6. for each point x in X and each regular open set V in Y non-containing $f(x)$, there exist a P_β -closed set K in X not containing x such that $f^{-1}(V) \subset K$;
7. for each point x in X and each $V \in SO(Y, f(x))$, there is $U \in P_\beta O(X, x)$ such that $f(U) \subset Cl(V)$;
8. $f^{-1}(V) \subset P_\beta Int(f^{-1}(Cl(V)))$ for every $V \in SO(Y)$;
9. $f(P_\beta Cl(A)) \subset rker(f(A))$ for every subset A of X ;
10. $P_\beta Cl(f^{-1}(B)) \subset f^{-1}(rker(B))$ for every subset B of Y ;
11. $f^{-1}(Cl(V))$ is P_β -open for every $V \in \beta O(Y)$;
12. $f^{-1}(Cl(V))$ is P_β -open for every $V \in SO(Y)$;
13. $f^{-1}(Int(Cl(V)))$ is P_β -closed for every subset V of Y ;
14. $f^{-1}(Cl(Int(V)))$ is P_β -open for every subset V of Y ;
15. $f(P_\beta Cl(A)) \subset \theta sCl(f(A))$ for every subset A of X ;
16. $P_\beta Cl(f^{-1}(B)) \subset f^{-1}(\theta sCl(B))$ for every subset B of Y ;

17. $P_\beta Cl(f^{-1}(V)) \subset f^{-1}(\theta sCl(V))$ for every open subset V of Y ;
 18. $P_\beta Cl(f^{-1}(V)) \subset f^{-1}(sCl(V))$ for every open subset V of Y ;
 19. $P_\beta Cl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V)))$ for every open subset V of Y .

Proof. (1) $\Leftrightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). Since any θ -semi-open set is a union of regular closed sets, by (2), (3) holds.

(3) $\Rightarrow$ (7). Let $x \in X$ and $V \in SO(Y, f(x))$. Since $Cl(V)$ is θ -semi-open in Y , there exist a $U \in P_\beta O(X, x)$ such that $x \in U \subset f^{-1}(Cl(V))$. Hence $f(U) \subset Cl(V)$.

(7) $\Rightarrow$ (2). Let V be any regular closed set of Y and $x \in f^{-1}(V)$. Then V is a semiopen set containing $f(x)$ and there exists a $U \in P_\beta O(X, x)$ such that $U \subset f^{-1}(Cl(V)) = f^{-1}(V)$. Therefore, $x \in U \subset f^{-1}(V)$ and hence $x \in U \subset P_\beta Int(f^{-1}(V))$ and $f^{-1}(V) \subset P_\beta Int(f^{-1}(V))$. Therefore, $f^{-1}(V) = P_\beta Int(f^{-1}(V))$. Hence, $f^{-1}(V)$ is P_β -open.

(7) $\Rightarrow$ (8). Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. By (7), there exists a $U \in P_\beta O(X, x)$ such that $f(U) \subset Cl(V)$. It follows that $x \in U \subset f^{-1}(Cl(V))$. Hence, $x \in P_\beta Int(f^{-1}(Cl(V)))$. Thus, $f^{-1}(V) \subset P_\beta Int(f^{-1}(Cl(V)))$.

(8) $\Rightarrow$ (1). Let F be any regular closed of Y . Since $F \in SO(Y)$, by (8), $f^{-1}(F) \subset P_\beta Int(f^{-1}(F))$. This shows that $f^{-1}(F)$ is P_β -open.

(7) $\Rightarrow$ (16). Let A be any subset of X . Suppose that $x \in P_\beta Cl(A)$ and G is any semiopen set of Y containing $f(x)$. By (7), there exists a $U \in P_\beta O(X, x)$ such that $f(U) \subset Cl(G)$. Since $x \in P_\beta Cl(A)$, $U \cap A \neq \phi$ and hence $\phi \neq f(U) \cap f(A) \subset Cl(G) \cap f(A)$. Therefore, $f(x) \in \theta sCl(f(A))$.

(16) $\Rightarrow$ (17). Let B be any subset of Y . Then $f(P_\beta Cl(f^{-1}(B))) \subset \theta sCl(f(f^{-1}(B))) \subset \theta sCl(B)$.

(17) $\Rightarrow$ (7). Let V be any semiopen set of Y containing $f(x)$. Since $Cl(V) \cap (Y - Cl(V)) = \phi$, $f(x) \notin \theta sCl((Y - Cl(V)))$ and $x \notin f^{-1}(\theta sCl(Y - Cl(V)))$. By (17) $x \notin P_\beta Cl(f^{-1}(Y - Cl(V)))$. Hence, there exists a $U \in P_\beta O(X, x)$ such that $U \cap f^{-1}(Y - Cl(V)) = \phi$ so that $f(U) \cap (Y - Cl(V)) = \phi$.

(3) $\Leftrightarrow$ (4). Obvious.

(2) $\Rightarrow$ (5). Let $x \in X$ and V be a regular closed set of Y containing $f(x)$. Then $x \in f^{-1}(V)$. By (2), $f^{-1}(V)$ is P_β -ope. So there exists a $U \in P_\beta O(X, x)$ such that $U \subset f^{-1}(V)$.

(5) $\Rightarrow$ (9). Let A be any subset of X . Suppose that $y \notin rker(f(A))$. Then, by Lemma 2.7, there exists a $V \in RC(Y, y)$ such that $f(A) \cap V = \phi$. For any $x \in f^{-1}(V)$, by (5), there exists a $U_x \in P_\beta O(X, x)$ such that

$f(U_x) \subset V$. Hence, $f(A \cap U_x) \subset f(A) \cap f(U_x) \subset f(A) \cap V = \phi$ so that $A \cap U_x = \phi$. This shows that $x \notin P_\beta Cl(A)$ for any $x \in f^{-1}(V)$. Therefore, $f^{-1}(V) \cap P_\beta Cl(A) = \phi$ and therefore, $V \cap f(P_\beta Cl(A)) = \phi$. Thus, $y \notin f(P_\beta Cl(A))$.

(9) $\Leftrightarrow$ (10). Let B be any subset of Y . By (9) and Lemma 2.7, $f(P_\beta Cl(f^{-1}(B))) \subset rker(f(f^{-1}(B))) \subset rker(B)$ and $P_\beta Cl(f^{-1}(B)) \subset f^{-1}(rker(A))$. Conversely, let $B = f(A)$, where A is a subset of X . Then $P_\beta Cl(A) \subset P_\beta Cl(f^{-1}(B)) \subset f^{-1}(rker(f(A)))$. Therefore, $f(P_\beta Cl(A)) \subset rker(f(A))$.

(10) $\Rightarrow$ (1). Let V be any regular open set of Y . Then, by (10) and Lemma 2.7, $P_\beta Cl(f^{-1}(V)) \subset f^{-1}(rker(V)) = f^{-1}(V)$ and $P_\beta Cl(f^{-1}(V)) = f^{-1}(V)$.

(5) $\Leftrightarrow$ (6). Let V be any regular open set in Y not containing $f(x)$. Then $Y - V$ is a regular closed set containing $f(x)$. By (5), there exists a P_β -open set U in X containing x such that $f(U) \subset Y - V$. Hence, $U \subset X - f^{-1}(V)$ and then $f^{-1}(V) \subset X - U$, converse follows similarly.

(2) $\Rightarrow$ (11). Let V be any β -open set of Y . It follows from ([2], Theorem.2.4) that $Cl(V)$ is regular closed. By (2), $f^{-1}(Cl(V))$ is P_β -open in X .

(11) $\Rightarrow$ (12). This is clear since every semiopen set is β -open.

(12) $\Rightarrow$ (13). Let V be any subset of Y . Then $Int(Cl(V))$ is regular open. Therefore, $Y - Int(Cl(V))$ is regular closed and hence it is semiopen. By (12), $X - f^{-1}(Int(Cl(V))) = f^{-1}(Y - Int(Cl(V))) = f^{-1}(Cl(Y - Cl(V)))$ is P_β -open. Hence, $f^{-1}(Int(Cl(V)))$ is P_β -closed.

(13) $\Rightarrow$ (1). Let V be any regular open set of Y . By (13), $f^{-1}(V) = f^{-1}(Int(Cl(V)))$ is P_β -closed. It show that f is almost contra- P_β -continuous.

(2) $\Leftrightarrow$ (14). Let V be any subset of Y . Since $Cl(Int(V))$ is regular closed, $f^{-1}(Cl(Int(V)))$ is P_β -open. The converse follows similarly.

(16) $\Rightarrow$ (17). Obvious.

(17) $\Rightarrow$ (18). It follows from [12] the fact that $\theta sCl(V) = sCl(V)$ for an open set V .

(18) $\Rightarrow$ (19). Obvious from Lemma 2.5.

(19) $\Rightarrow$ (1). Let $V \in RO(Y)$. Then, by (20), $P_\beta Cl(f^{-1}(V)) \subset f^{-1}(Int(Cl(V))) = f^{-1}(V)$.

The statement (5) is the pointwise definition of a almost contra P_β -continuous function.

Corollary 3.9. For a function $f : X \rightarrow Y$, the following are equivalent:

1. f is almost contra P_β -continuous.
2. $f^{-1}(\alpha Cl(A))$ is P_β -open for every $A \in \beta O(Y)$;
3. $f^{-1}(pCl(A))$ is P_β -open for every $A \in SO(Y)$;

4. $f^{-1}(sCl(A))$ is P_β -open for every $A \in PO(Y)$;
5. $P_\beta Cl(f^{-1}(B)) \subset f^{-1}(\theta sCl(B))$ for every subset $B \in SO(Y)$;
6. $P_\beta Cl(f^{-1}(B)) \subset f^{-1}(\theta sCl(B))$ for every subset $B \in PO(Y)$;
7. $P_\beta Cl(f^{-1}(B)) \subset f^{-1}(\theta sCl(B))$ for every subset $B \in \beta O(Y)$.

Proof. The equivalences of (1), (2), (3) and (4) follow from Lemma 2.5, and the equivalences of (1), (5), (6) and (7) follow from Theorem 3.8 ((1) $\Leftrightarrow$ (16)).

Theorem 3.10. If a function $f : X \rightarrow Y$ is an almost contra- P_β -continuous function which satisfies the property $P_\beta Int(f^{-1}(Cl(V))) \subset f^{-1}(V)$ for each open set V of Y , then f is P_β -continuous.

Proof. Let V be any open set of Y . Then $Cl(V)$ is a semi-open set in Y . Since f is almost contra- P_β -continuous, by Theorem 3.8(8), $f^{-1}(V) \subset f^{-1}(Cl(V)) = P_\beta Int(P_\beta Int(f^{-1}(Cl(V)))) \subset (P_\beta Int(f^{-1}(V)) \subset f^{-1}(V)$.

Definition 3.11. [31] A topological space is said to be P_Σ if for any open set V of X and each $x \in V$, there exists a regular closed set F of X containing x such that $x \in F \subset V$.

Theorem 3.12. If $f : X \rightarrow Y$ is an almost contra- P_β -continuous function and Y is P_Σ , then f is P_β -continuous.

Proof. Suppose that V is any open set of Y . By the fact that Y is P_Σ , there exists a subfamily Ω of regular closed sets of Y such that $V = \cup\{F \mid F \in \Omega\}$. Since f is almost contra- P_β -continuous, then $f^{-1}(F)$ is P_β -open in X for each $F \in \Omega$. Therefore, $f^{-1}(V)$ is P_β -open in X .

Definition 3.13. [13] A topological space (X, τ) is said to be extremally disconnected if the closure of every open set of X is open in X .

Theorem 3.14. If (Y, σ) is said to be extremally disconnected, then the following properties are equivalent for a function $f : X \rightarrow Y$:

1. f is almost contra P_β -continuous;
2. f is almost P_β -continuous.

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and U be any regular open set of Y containing $f(x)$. Since Y is extremally disconnected, U is clopen and hence U is regular closed. Thus, $f^{-1}(U)$ is P_β -open in X . (2) $\Rightarrow$ (1). Let B be any regular closed set of Y . since Y is extremally disconnected, B is regular open and $f^{-1}(B)$ is P_β -open in X .

We cannot relax the condition extremally disconnected in Theorem 3.14.

Example 3.15. The set of the real number $\mathbf{R}$ with usual topology τ , is not a extremally disconnected space. Define the identity function $f : (\mathbf{R}, \tau) \rightarrow (\mathbf{R}, \tau)$ is almost P_β -continuous but not almost contra- P_β -continuous.

Definition 3.16. [30] The P_β -frontier of a set A in a space X , denoted $P_\beta Fr(A)$, is defined as $P_\beta Fr(A) = P_\beta Cl(A) \cap P_\beta Cl(X - A)$.

Theorem 3.17. The set of all points x of X at which $f : X \rightarrow Y$ is not almost contra- P_β -continuous is identical with the union of the P_β -frontier points of the inverse images of regular closed sets of Y containing $f(x)$.

Proof. Suppose that f is not almost contra- P_β -continuous at a $x \in X$. Then there exists a $F \in RC(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \phi$ for every $U \in P_\beta O(X, x)$. Then $U \cap f^{-1}(Y - F) \neq \phi$. Therefore, $x \in P_\beta Cl(f^{-1}(Y - F)) = P_\beta Cl(X - f^{-1}(F))$. Since $x \in f^{-1}(F)$, $x \in P_\beta Cl(f^{-1}(F))$. Therefore, $x \in P_\beta Fr(f^{-1}(F))$. Now suppose $x \in P_\beta Fr(f^{-1}(F))$ for some $F \in RC(Y, f(x))$. Assume that f is almost contra- P_β -continuous at $x \in X$. Then there exists a $U \in P_\beta O(X, x)$ such that $f(U) \subset F$. Therefore, $x \in U \subset f^{-1}(F)$, so that $x \in P_\beta Int(f^{-1}(F)) \subset X - P_\beta Fr(f^{-1}(F))$, a contradiction. This means that f is not almost contra- P_β -continuous at x .

Definition 3.18. A function $f : X \rightarrow Y$ is said to be:

1. R -map [5] if $f^{-1}(V)$ is regular closed in X for every regular closed set V of Y .
2. contra R -map [9] if $f^{-1}(V)$ is regular closed in X for every regular open set V of Y .
3. P_β -resolute [26] if $f(V)$ is P_β -closed in Y for every P_β -closed set V of X .
4. P_β -irresolute [26] if $f^{-1}(V)$ is P_β -open in X for every P_β -open set V of Y .

Theorem 3.19. Let $\{X_i \mid i \in \Delta\}$ be a family of topological spaces. If a function $f : X \rightarrow \prod X_i$ is almost contra P_β -continuous, then $P_{r_i} \circ f : X \rightarrow X_i$ is almost contra P_β -continuous for each $i \in \Delta$, where P_{r_i} is the projection of $\prod X_i$ onto X_i .

Proof. For a fixed $i \in \Delta$, let V_i be any regular open sets of X_i . Since P_{r_i} is continuous, $P_{r_i}^{-1}(V_i)$ is open in $\prod X_i$, it is an R-map and hence $P_{r_i}^{-1}(V_i)$ is regular open in $\prod X_i$. Since f is almost contra P_β -continuous $f^{-1}(P_{r_i}^{-1}(V_i)) = (P_{r_i} \circ f)^{-1}(V_i)$ is P_β -closed in X . Therefore, $P_{r_i} \circ f$ is almost contra P_β -continuous for each $i \in \Delta$.

Theorem 3.20. Let $f : X \rightarrow Y$ is a P_β -resolute surjection and $g : Y \rightarrow Z$ is a function such that $g \circ f$ is almost contra P_β -continuous then, g is almost contra P_β -continuous.

Proof. Let V be any regular open set in Z . Since $g \circ f$ is almost contra P_β -continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is P_β -closed. Since f is P_β -resolute surjection, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is P_β -closed.

Theorem 3.21. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Then the following statements hold:

1. If f is almost contra P_β -continuous and g is R-map, then $g \circ f$ is almost contra- P_β -continuous.
2. If f is almost P_β -continuous and g is contra R-map, then $g \circ f$ is almost contra- P_β -continuous.
3. If f is P_β -irresolute and g is almost contra P_β -continuous., then $g \circ f$ is almost contra- P_β -continuous.

Proof. The proof follows from the fact $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ and the definitions.

Definition 3.22. A topological space is said to be

1. P_β c-closed [30] (resp. nearly compact [20]) if every P_β -closed (resp. regular open) cover of X has a finite subcover.
2. countably P_β c-closed [30] (resp. nearly countably compact [10, 21]) if every countable cover of X by P_β -closed (resp. regular open) sets has a finite subcover.

3. P_β c-Lindelöf [30] (resp. nearly Lindelöf [8]) if every cover of X by P_β -closed (resp. regular open) sets has a countable subcover.

Theorem 3.23. Let $f : X \rightarrow Y$ be an almost contra P_β -continuous surjection. Then, the following statements hold:

1. If X is P_β c-closed, then Y is nearly compact;
2. If X is countably P_β c-closed, then Y is nearly countably compact;
3. If X is P_β c-Lindelöf, then Y is nearly Lindelöf.

Proof. (1). Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y . Since f is almost contra P_β -continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a P_β -closed cover of X . Since X is P_β c-closed, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is nearly compact. The proofs of (2) and (3) are similar to the proof of (1).

Definition 3.24. A topological space is said to be

1. P_β -compact [30] (resp. S -closed [25]) if every P_β -open (resp. regular closed) cover of X has a finite subcover.
2. countably P_β -compact [30] (resp. countably S -closed [1]) if every countable cover of X by P_β -open (resp. regular closed) sets has a finite subcover.
3. P_β -Lindelöf [30] (resp. S -Lindelöf [18]) if every cover of X by P_β -open (resp. regular closed) sets has a countable subcover.

Theorem 3.25. Let $f : X \rightarrow Y$ be an almost contra P_β -continuous surjection. Then the following statements hold:

1. If X is P_β -compact, then Y is S -closed;
2. If X is countably P_β -compact, then Y is countably S -closed;
3. If X is P_β -Lindelöf, then Y is S -Lindelöf.

Proof. (1). Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y . Since f is almost contra P_β -continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a P_β -open cover of X . Since X is P_β -compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is S -closed.

4. Separation Axioms

Definition 4.1. A topological space (X, τ) is said to be:

1. $P_\beta - T_0$ [30] (resp. $P_\beta - T_1$ [30]) if for any distinct points x and y in X , there is a P_β -open U in X containing x but not y or (resp. and) a P_β -open set V in X containing y but not x .
2. $P_\beta - T_2$ [30] if for every pair of distinct points x and y , there exist two P_β -open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

From above definitions, we have following implications:

$$P_\beta - T_2 \Rightarrow P_\beta - T_1 \Rightarrow P_\beta - T_0.$$

Example 4.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a, b\}, X\}$. Then (X, τ) is $P_\beta - T_1$ but not $P_\beta - T_2$ space.

Example 4.3. Let $\mathbf{R}$ be the set of real numbers with the right order topology [23] generated by the basis $S_a = \{x : x > a, a \in \mathbf{Q}\}$, where $\mathbf{Q}$ is the set of rational numbers. Then (X, τ) is $P_\beta - T_0$ but not $P_\beta - T_1$ space.

Theorem 4.4. Let X be a topological space. Suppose that for each pair of distinct points x_1 and x_2 in X , there exists a function f of X into a Uryshon space Y such that $f(x_1) \neq f(x_2)$. Moreover, let f be almost contra- P_β -continuous at x_1 and x_2 . Then X is $P_\beta - T_2$.

Proof. Let x_1 and x_2 be any distinct points in X . Suppose that there exist a Uryshon space Y and a function $f : X \rightarrow Y$ such that $f(x_1) \neq f(x_2)$ and f is almost contra- P_β -continuous at x_1 and x_2 . Let $w_1 = f(x_1)$ and $w_2 = f(x_2)$. Then $w_1 \neq w_2$, since Y is Uryshon, there exist open sets U and V containing w_1 and w_2 , respectively such that $Cl(U) \cap Cl(V) = \phi$. Since f is almost contra- P_β -continuous at x_1 and x_2 , there exist P_β -open sets A and B containing x_1 and x_2 respectively, such that $f(A) \subset Cl(U)$ and $f(B) \subset Cl(V)$. So $A \cap B = \phi$ since $Cl(U) \cap Cl(V) = \phi$.

Corollary 4.5. If f is almost contra- P_β -continuous injection of a topological space X into a Uryshon space Y , then X is $P_\beta - T_2$.

Definition 4.6. A topological space (X, τ) is said to be:

1. P_β -normal [30] if for each pair of nonempty disjoint closed sets F_1 and F_2 , there exist disjoint P_β -open sets U and V containing F_1 and F_2 , respectively.
2. Weakly Hausdorff [22] (briefly weakly- T_2) if every point of X is an intersection of regular closed sets of X .
3. s -Urysohn [3] if for each pair of distinct points x and y in X , there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $Cl(U) \cap Cl(V) = \phi$.
4. Ultra Hausdorff [24] if for each pair of distinct points x and y in X , there exist disjoint clopen sets U and V containing x and y , respectively.

Theorem 4.7. *If f is an almost contra- P_β -continuous injection of a topological space X into a s -Urysohn space Y , then X is $P_\beta - T_2$.*

Proof. Since f is injective, we have $f(x) \neq f(y)$ for any distinct points x and y in X . Since Y is s -Urysohn, there exist $U_1 \in SO(X, x)$ and $U_2 \in SO(X, y)$ such that $Cl(U_1) \cap Cl(U_2) = \phi$. Since f is almost contra- P_β -continuous, there exist P_β -open sets V_1 and V_2 in X containing x and y , respectively such that $f(V_1) \subset Cl(U_1)$ and $f(V_2) \subset Cl(U_2)$. Therefore, $V_1 \cap V_2 = \phi$.

Theorem 4.8. *If f is an almost contra- P_β -continuous injection of a topological space X into a Ultra Hausdorff space Y , then X is $P_\beta - T_2$.*

Proof. Let x and y be distinct points in X . Since f is injective and Y is Ultra Hausdorff, there exist disjoint clopen sets U_1 and U_2 containing $f(x)$ and $f(y)$, respectively. Then U_1 and U_2 are regular closed set in Y . Since f is almost contra- P_β -continuous, there exist P_β -open sets V_1 and V_2 in X containing x and y respectively, such that $f(V_1) \subset Cl(U_1)$ and $f(V_2) \subset Cl(U_2)$. Therefore, $V_1 \cap V_2 = \phi$.

Theorem 4.9. *If f is an almost contra- P_β -continuous injection of a topological space X into a weakly Hausdorff space Y , then X is $P_\beta - T_1$.*

Proof. Since Y is weakly Hausdorff and f is injective, for any distinct points x_1 and x_2 of X , there exist $V_1, V_2 \in RC(X)$ such that $f(x_1) \in V_1$, $f(x_2) \notin V_1$, $f(x_2) \in V_2$, and $f(x_1) \notin V_2$. Since f is almost contra- P_β -continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are P_β -open sets and $x_1 \in f^{-1}(V_1)$, $x_2 \notin f^{-1}(V_1)$, $x_2 \in f^{-1}(V_2)$, and $x_1 \notin f^{-1}(V_2)$.

Theorem 4.10. *If $f : X \rightarrow Y$ is a almost contra- P_β -continuous closed injection and Y is normal, then X is P_β -normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of X . Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y . Since Y is normal, there exist disjoint open sets V_1 and V_2 containing $f(F_1)$ and $f(F_2)$, respectively. Then there exist open sets U_i such that $f(F_i) \subset U_i \subset Cl(U_i) \subset V_i$. Hence, $F_i \subset f^{-1}(Cl(U_i))$, $f^{-1}(Cl(U_i)) \in P_\beta O(X)$ for $i=1,2$ and $f^{-1}(Cl(U_1)) \cap f^{-1}(Cl(U_2)) = \phi$.

Definition 4.11. *A topological space is called a $P_\beta c$ -space if the union of any two P_β -closed sets is P_β -closed set.*

From Definition 4.11, it follows that in a $P_\beta c$ -space, P_β -open sets form a topology.

Example 4.12. 1. Let $X = [-1, 1] \subset \mathbf{R}$, define either-or topology on $X[23]$ by $\tau = \{U : \{0\} \notin U \text{ or } (-1, 1) \subset U\}$. Here, $P_\beta O(X) = \{(-1, 1), (-1, 1], [-1, 1), [-1, 1]\} \cup \{A : 0 \notin A\}$. Then (X, τ) is a $P_\beta c$ -space.

2. Let X be the set of positive integers and $P = \{\{2k-1, 2k\} : k \in \mathbf{N}\}$ be a partition of X . Then the partition P together with the empty set ϕ is a basis for a topology τ on X , known as partition topology on $X[23]$. Every subset in this space is pre-open and hence P_β -open. Thus, (X, τ) is a $P_\beta c$ -space.

3. In Example 4.3, let $U = \{1, 2, 3, 4, 6, 8, 10, \dots\}$ and $V = \{1, 2, 3, 5, 7, 9, 11, \dots\}$. Then U and V are P_β -open sets but $U \cap V = \{1, 2, 3\}$ is not P_β -open set. Thus, (X, τ) is not a $P_\beta c$ -space.

Theorem 4.13. *If $f, g : X \rightarrow Y$ are almost contra- P_β -continuous functions, X is a $P_\beta c$ -space and Y is s -Urysohn, then $E = \{x \in X \mid f(x) = g(x)\}$ is P_β -closed in X .*

Proof. If $x \in X - E$, then $f(x) \notin g(x)$. Since Y is s-Urysohn, there exist $V_1 \in SO(Y, f(x))$ and a $V_2 \in SO(Y, g(x))$ such that $Cl(V_1) \cap Cl(V_2) = \phi$. By the fact that f and g are almost contra P_β -continuous functions, there exist P_β -open sets U_1 and U_2 in X containing x such that $f(U_1) \subset Cl(V_1)$ and $g(U_2) \subset Cl(V_2)$. We put $U = U_1 \cap U_2$. Then U is P_β -open in X . Thus, $f(U) \cap g(U) = \phi$. It follows that $x \notin P_\beta Cl(E)$. This shows that E is P_β -closed in X .

Theorem 4.14. Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow Y$ the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then f is almost contra- P_β -continuous if g is almost contra- P_β -continuous.

Proof. Let $x \in X$ and V be a regular open subset of Y containing $f(x)$. Then $X \times V$ is regular open. Since g is almost contra- P_β -continuous, $g^{-1}(X \times V) = f^{-1}(V)$ is P_β -closed.

Theorem 4.15. [26] If X is T_1 -space, then $PO(X) = P_\beta O(X)$.

Theorem 4.16. Let $f : X \rightarrow Y$ be function such that X is T_1 -space, then f is almost contra- P_β -continuous if and only if f is almost contra precontinuous.

Proof. Proof follows from Theorem 4.15.

5. P_β -closed, P_β -regular, strongly contra- P_β -closed

Recall that for a function $f : X \rightarrow Y$, the subset $G(f) = \{(x, f(x)) : x \in X\} \subset X \times Y$ is said to be the graph of f .

Definition 5.1. The graph $G(f)$ of a function $f : X \rightarrow Y$ is said to be P_β -closed (resp. P_β -regular, strongly contra- P_β -closed) if for each $(x, y) \in (X \times Y) - G(f)$, there exists a $U \in P_\beta O(X, x)$ (resp. $U \in P_\beta C(X, x)$, $U \in P_\beta O(X, x)$) and a $V \in O(Y, y)$ (resp. $V \in RO(Y, y)$, $V \in RC(Y, y)$) such that $(U \times V) \cap G(f) = \phi$.

Lemma 5.2. The graph $G(f)$ of a function $f : X \rightarrow Y$, the following statements are equivalent:

1. $G(f)$ is P_β -closed (resp. P_β -regular, strongly contra- P_β -closed);
2. for each $(x, y) \in (X \times Y) - G(f)$, there exists a $U \in P_\beta O(X, x)$ (resp. $U \in P_\beta C(X, x)$, $U \in P_\beta O(X, x)$) and a $V \in O(Y, y)$ (resp. $V \in RO(Y, y)$, $V \in RC(Y, y)$) such that $f(U) \cap V = \phi$.

Proof. The proof follow from the fact that for any subsets $A \subset X$ and $B \subset Y$, $(A \times B) \cap G(f) = \phi$ if and only if $f(A) \cap B = \phi$.

Theorem 5.3. If $f : X \rightarrow Y$ is a function with a P_β -closed graph, then for each $x \in X$, $f(x) = \cap\{Cl(f(U)) \mid U \in P_\beta O(X, x)\}$.

Proof. Suppose there exists an element $y \neq f(x)$ such that $y \in \cap\{Cl(f(U)) \mid U \in P_\beta O(X, x)\}$. Then $y \in Cl(f(U))$ for every $U \in P_\beta O(X, x)$. So $V \cap f(U) \neq \phi$, for every $V \in O(Y, y)$, which contradicts the hypothesis that f is a function with a P_β -closed graph.

Theorem 5.4. If $f : X \rightarrow Y$ is almost contra- P_β -continuous and Y is Hausdroff, then $G(f)$ is P_β -closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) - G(f)$, then $y \neq f(x)$ and there exist disjoint open sets V and W such that $y \in V$ and $f(x) \in W$. Then $f(x) \notin Y - Cl(W)$, since $Y - Cl(W)$ is a regular open set containing V , it follows that $f(x) \notin rker(V)$ and hence $x \notin f^{-1}(rker(V))$. Then, by Theorem 3.8(9), $x \notin P_\beta Cl(f^{-1}(V))$. Therefore, $(x, y) \in (X - P_\beta Cl(f^{-1}(V))) \times V$.

Theorem 5.5. If a function $f : X \rightarrow Y$ has P_β -closed graph and X is P_β c-space, then $f^{-1}(K)$ is P_β -closed for every compact subset K of Y .

Proof. Let K be a compact subset of Y and $x \in X - f^{-1}(K)$. Then for each $y \in K$, $(x, y) \in (X \times Y) - G(f)$. So there exists a $U_y \in P_\beta O(X, x)$ and an open set V_y of Y containing y such that $f(U_y) \cap V_y = \phi$. The family $\{V_y \mid y \in K\}$ is an open cover of K and hence there is a finite subcover $\{V_{y_i} \mid i = 1, \dots, n\}$. Let $U = \cap\{U_{y_i} \mid i = 1, \dots, n\}$. Then $U \in P_\beta O(X, x)$ and $f(U) \cap K = \phi$. Hence, $U \cap f^{-1}(K) = \phi$, which proves that $f^{-1}(K)$ is P_β -closed in X .

Theorem 5.6. If an injective mapping $f : X \rightarrow Y$ has a P_β -regular graph $G(f)$, then X is $P_\beta - T_0$.

beginproof Let x_1 and x_2 be any two distinct points of X . Then, $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since $G(f)$ is P_β -regular, there exists a P_β -closed set U in X containing x_1 and $V \in RO(Y, f(x_2))$ such that $f(U) \cap V = \phi$. Thus, $U \cap f^{-1}(V) = \phi$. Therefore, $x_2 \notin U$.

Theorem 5.7. If $f : X \rightarrow Y$ has a P_β -regular graph $G(f)$ and f is surjective, then Y is weakly Hausdroff.

Proof. Let y_1 and y_2 be any two distinct points of Y . Since Y is surjective, there exists a point $x \in X$ such that $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. Since f has a P_β -regular graph, there exist a $U \in P_\beta C(X, x)$ and a $V \in RO(Y, y_2)$ such that $f(U) \cap V = \phi$. Thus, $y_2 \notin (Y - V) \in RC(Y)$ and $y_1 \in (Y - V)$.

Theorem 5.8. Let $f : X \rightarrow Y$ be a mapping with a P_β -closed graph.

1. if f is injective, then X is $P_\beta - T_1$;
2. if f is surjective, then Y is T_1 .

Proof. (1). Let x_1 and x_2 be any two distinct points of X . Then, $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Then there exists a P_β -open set U in X containing x_1 and $V \in O(Y, f(x_2))$ such that $f(U) \cap V = \phi$. Since $x_2 \in f^{-1}(V)$, $x_2 \notin U$. Therefore, U is a P_β -open set containing x_1 but not x_2 .
 (2). Let y_1 and y_2 be any two distinct points of Y . Since Y is surjective, there exists a $x \in X$ such that $f(x) = y_1$. Then $(x, y_2) \in (X \times Y) - G(f)$. Since f has a P_β -closed graph, there exist a $U \in P_\beta O(X, x)$ and an open set V of Y containing y_2 such that $f(U) \cap V = \phi$. Since $y_1 = f(x)$ and $x \in U$, $y_1 \notin V$.

Theorem 5.9. Let $f : X \rightarrow Y$ be a mapping with strongly contra- P_β -closed graph $G(f)$. If f is an almost contra- P_β -continuous injection, then X is $P_\beta - T_2$.

Proof. Let x and y be any distinct points of X . Since f is injective, we have $f(x) \neq f(y)$. Then, we have $(x, f(y)) \in X \times Y - G(f)$. Since $G(f)$ is strongly contra- P_β -closed, there exist a $U \in P_\beta O(X, x)$ and a regular closed set V containing $f(y)$ such that $f(U) \cap V = \phi$. Since f is almost contra- P_β -continuous, $f^{-1}(V) \in P_\beta O(X, y)$.

6. Connectedness

Definition 6.1. A space X is said to be

1. hyperconnected [13] if every nonempty open subset of X is dense in X .
2. P_β -ultra-connected [30] if every two nonempty P_β -closed subsets of X intersect.

3. P_β -connected [26] if X cannot be expressed as the union of two nonempty P_β -open sets.

Theorem 6.2. If $f : X \rightarrow Y$ is a almost contra P_β -continuous surjection and X is P_β -connected, then Y is connected.

Proof. Suppose that Y is not a connected space. Then there exists a proper clopen set U . Since f is almost contra P_β -continuous, $f^{-1}(U)$ is a proper subset of X which is both P_β -open and P_β -closed. This shows that X is not P_β -connected, a contradiction. Therefore, Y is connected.

Example 6.3. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a, b\}, X\}$. Define a function $f : (X, \tau) \rightarrow (X, \tau)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is almost contra- P_β -continuous and (X, τ) is connected but not P_β -connected.

Theorem 6.4. If X is P_β -ultra-connected and $f : X \rightarrow Y$ is an almost contra P_β -continuous and surjection, then Y is hyperconnected.

Proof. Assume that Y is not hyperconnected. Then there exists an open set V such that V is not dense in Y . Then there exist disjoint nonempty open subsets $U_1 = \text{Int}(Cl(V))$ and $U_2 = Y - Cl(V)$. Since f is almost contra P_β -continuous and surjective, $W_1 = f^{-1}(U_1)$ and $W_2 = f^{-1}(U_2)$ are disjoint nonempty P_β -closed subsets of X . Since X is P_β -ultra-connected, W_1 and W_2 must intersect, a contradiction. Thus, Y is hyperconnected.

Example 6.5. Let X be any set with cofinite topology τ is hyperconnected and P_β -ultra-connected, but an identity function $f : (X, \tau) \rightarrow (X, \tau)$ is not almost contra P_β -continuous.

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