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On the mixed multifractal formalism for vector-valued measures

Bilel Selmi
University of Monastir, Tunisia
and
Anouar Ben Mabrouk
University of Monastir, Tunisia
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Abstract

The multifractal formalism for vector-valued measures holds whenever the existence of corresponding Gibbs-like measures, supported on the singularities sets holds. We tried through this article to improve a result developed by Menceur et al. in [29] and to suggest a new sufficient condition for a valid mixed multifractal formalism for vector-valued measures. We describe a necessary condition of validity for the formalism which is very close to the sufficient one.

Keyword: *Hausdorff dimension, Packing dimension, Multifractal analysis.*

MSC-2010: *28A20, 28A80.*

1. Introduction

Recently, multifractal analysis has taken an enormous interest in the mathematical literature. Many authors were interested in such analysis and its applications especially in financial time series, econo-physical data, where the most used models are based on the past and for a long period are linear and multi-linear models. These models have shown some inefficiencies especially by the discovery and the inclusion of stochastic, chaotic and fractal factors in the mathematical models. This was one strong cause and motivation that have led researchers to develop more sophisticated approaches. Multifractal analysis has appeared firstly and has shown some success in overcoming many problems. However, in some others, the efforts have to be more and more developed especially for simultaneous time series behavior. One simple example is the financial crisis that appears. Such a crisis like the recent American one did not affect only the local national market but was spread and thus affected the worldwide markets, leading thus to a simultaneous or a worldwide crisis. See [3, 4, 5, 6, 7, 11, 12, 14, 15, 16, 17, 18, 19, 24, 25, 29, 30, 32, 33, 44].

Mixed multifractal analysis as in the case of single one studies both functions (signals, time series, images, ...) and measures. Indeed, some geometric sets are essentially known by means of measures that are supported on, i.e., given a set X and a measure μ, the quantity $\mu(X)$ may be computed as the maximum value $\mu(F)$ for all subsets $F \subset X$. This means that, we somehow forget the geometric structure of X and focus instead on the properties of the measure μ. The set X is thus partitioned into α-level sets $X_\mu(\alpha)$ relatively to the regularity exponent of μ (see for example [1, 2, 9, 21, 22, 23, 27, 31, 39, 40, 42, 43]). In the present work, we will be interested to the development of a mixed multifractal analysis of finitely many measures. So, many natural fractal-like objects that one wants to understand do not come always from simultaneous functions, but from simultaneous measures. This is why, in [29, 30], a mixed multifractal formalism associated with the mixed multifractal generalizations of Hausdorff and packing measures and dimensions is proved, in some cases, based on a generalization of the well known large deviation formalism. Furthermore, a mixed multifractal formalism has been proved for the Gibbs-like measures. In general, one needs some degree of similarity to prove the existence of Gibbs-like measures. For example, in dynamic contexts, the existence of such measures are often natural.

As we have noticed, previously, only the scaling behavior of a single measure by means of its Hölder exponent

$$\alpha_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

has been investigated (see for example [1, 2, 9, 10, 31]). However, the mixed multifractal analysis of measures on $\mathbf{R}^n$ investigates the simultaneous scaling behavior

$$\alpha_\mu(x) = (\lim_{r \to 0} \frac{\log \mu_1(B(x,r))}{\log r},\ \lim_{r \to 0} \frac{\log \mu_2(B(x,r))}{\log r}, \dots, \lim_{r \to 0} \frac{\log \mu_k(B(x,r))}{\log r}).$$

of finitely many measures $\mu_1, \mu_2, \dots, \mu_k$, and thus μ is considered as a vector valued measure $\mu = (\mu_1, \mu_2, \dots, \mu_k)$. It combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system and provides the basis for a significantly better understanding of the underlying dynamics. Olsen [32] conjectured a mixed multifractal formalism which links the mixed spectrum to the Legendre transform of mixed Rényi dimensions. Olsen obtained a general upper bound, he also proved that this bound is equality if both measures are self-similar with same contracting similarities. We note also that Peyrière [34] has also guessed a general vectorial multifractal formalism that is valid under some Frostman assumption. The check of this assumption proves to be very difficult. In [11], the authors conjectured a mixed wavelet multifractal formalism which links the mixed spectrum to the Legendre transform of a scaling function on the simultaneous continuous wavelet transforms. They also proved the validity of that conjecture for pairs of self-similar functions with the same contracting similarities. In [12], the authors extended the validity for pairs of some non-selfsimilar functions. In [44], Slimane conjectured mixed wavelet leaders multifractal formalism which involves mixed wavelet leaders scaling function.

In another context, and to overcome the problem of being non doubling, non-hölderian measure, the authors in [13, 20, 38] introduced and studied a relative multifractal analysis by comparing the original measure μ to an appropriate other ν. The singularity decomposition sets $X_\mu(\alpha)$ are replaced by two-parameters ones

$$X_{\mu,\nu}(\alpha,\beta) = \left\{ x \middle| \lim_{r \downarrow 0} \frac{\log(\mu(B(x,r)))}{\log\ r} = \alpha \text{ and } \lim_{r \downarrow 0} \frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r)))} = \beta \right\}.$$

In [4], a mixed multifractal analysis has been developed by considering pressure-like quantities instead of Hausdorff measures. Besides, instead of evaluating or studying the local behavior of measures $(\mu(B(x,r)))$ by means of diameter power lows r^α, the decomposition sets $X_\mu(\alpha)$ is replaced by

$$X(\alpha) = \left\{ x \left| \frac{\displaystyle\int_X \varphi d\mu}{\displaystyle\int_X \psi d\mu} = \alpha \right. \right\},$$

which yields a mixed analysis of general density-like measures. A similar example will be explicitly developed next. See also [26], [27], [35], [36], [37], [38].

In this paper, we have applied the techniques and results from the validity theory for multifractal formalism developed in [29, 30, 31, 41], especially in [9] to give a systematic and detailed account of the substantially more complicated problem of computing the mixed multifractal spectra. The purpose of this paper is to improve the result of Menceur et al. developed in [29] and to suggest a new sufficient condition that gives the lower bound of the validity for the mixed multifractal formalism for vector-valued measures. We have also observed that this sufficient condition is very close to being a necessary and sufficient one.

2. Main results

In [31], the author studied some variants of Hausdorff and packing measures relatively to Borel probability measures on Euclidean space based on some special set/dimension functions. More precisely, for a Borel probability measure μ on $\mathbf{R}^n$, and $(q,t) \in \mathbf{R}^2$, let

$$h_{q,t}(r) = \mu(B(x,r))^q r^t, \quad r > 0,$$

Hausdorff and packing measures relatively to $h_{q,t}$ have been studied. One motivation behind this construction is the fact that Hausdorff and packing measures associated with this function are in many important cases supported on the so-called multifractal decomposition sets

$$X_\mu(\alpha) = \left\{ x \left| \lim_{r\downarrow 0} \frac{\log(\mu(B(x,r)))}{\log r} = \alpha \right. \right\} \tag{2.1}$$

for a suitable choice of α and suitable measure μ such as doubling, Holderian, Gibbs, etc, and provide powerful tools for computing the Hausdorff and packing dimensions of this set. Olsen applied a large deviation formalism to construct Gibbs measures ν on these sets and used the Billingsley theorem to deduce the dimension of $X_\mu(\alpha)$ from modifid sets $X_\nu(f(\alpha))$ for some suitable function f.

There are several existing extensions of [31] which resemble to the present one (and thus constitutes motivations also) where the authors considered the dimension function

$$h_{q,t}(r) = \mu(B(x,r))^q r^t, \quad r > 0,$$

which in some sense means that we compare again the measure μ to the powers of the diameter.

In the present paper, one aim is to investigate Hausdorff and packing measures based on the following more general dimension function

$$H_{q_1,\ldots,q_k,t}(r) = \mu_1(B(x,r))^{q_1} \ldots \mu_k(B(x,r))^{q_k} r^t, \quad r > 0,$$

for $q_1, \ldots, q_k, t \in \mathbf{R}$ and $\mu_1, \ldots, \mu_k$ are Borel probability measures on $\mathbf{R}^n$. Associated Hausdorff and packing measures are introduced based on the dimension function $H_{q_1,\ldots,q_k,t}$. Observe that if $k = 1$, we obtain

$$H_{q,t}(r) = h_{q,t}(r)$$

for all $q, t \in \mathbf{R}$, and the Hausdorff and packing measures based on $H_{q_1,\ldots,q_k,t}$ are therefore extensions of the Hausdorff and packing measures based on $h_{q,t}$ in [31].

We now introduce our main results. Let $\mu_1, \mu_2, ..., \mu_k$ be the probability measures on $\mathbf{R}^d$ with a common support equal to K. Let also $\mathbf{q} = (q_1, q_2, ..., q_k) \in \mathbf{R}^k$, $t \in \mathbf{R}$, $E \subseteq \mathbf{R}^d$ be a nonempty set and $\delta > 0$. The mixed generalized multifractal Hausdorff measure and the mixed generalized multifractal packing one are defined as follows. We denote

$$\mu(B(x,r)) = \Big(\mu_1(B(x,r)), \mu_2(B(x,r)), \ldots, \mu_k(B(x,r))\Big)$$

and the product is

$$\mu(B(x,r))^{\mathbf{q}} = \mu_1(B(x,r))^{q_1} \times \mu_2(B(x,r))^{q_2} \times \ldots \times \mu_k(B(x,r))^{q_k}.$$

We define

$$\overline{\mathcal{P}}^{\mathbf{q},t}_{\mu,\delta}(E) = \sup\left\{\sum_i \mu(B(x_i,r_i))^{\mathbf{q}} (2r_i)^t\right\},$$

where the supremum is taken over all centered δ-packing of E. The mixed multifractal packing pre-measure is then given by

$$\overline{\mathcal{P}}_{\mu}^{\mathbf{q},t}(E) = \inf_{\delta>0} \overline{\mathcal{P}}_{\mu,\delta}^{\mathbf{q},t}(E).$$

In a similar way, we define

$$\overline{\mathcal{H}}_{\mu,\delta}^{\mathbf{q},t}(E) = \inf \left\{ \sum_i \mu(B(x_i, r_i))^{\mathbf{q}} (2r_i)^t \right\},$$

where the infinimum is taken over all centered δ-coverings of E. The mixed multifractal Hausdorff pre-measure is defined by

$$\overline{\mathcal{H}}_{\mu}^{\mathbf{q},t}(E) = \sup_{\delta>0} \overline{\mathcal{H}}_{\mu,\delta}^{\mathbf{q},t}(E).$$

$\overline{\mathcal{H}}_{\mu}^{\mathbf{q},t}$ is σ-subadditive but not increasing and $\overline{\mathcal{P}}_{\mu}^{\mathbf{q},t}$ is increasing but not σ-subadditive. That's why Menceur et al. introduced the following modifications on the mixed generalized Hausdorff and packing measures $\mathcal{H}_{\mu}^{\mathbf{q},t}$ and $\mathcal{P}_{\mu}^{\mathbf{q},t}$,

$$\mathcal{H}_{\mu}^{\mathbf{q},t}(E) = \sup_{F\subseteq E} \overline{\mathcal{H}}_{\mu}^{\mathbf{q},t}(F) \quad \text{and} \quad \mathcal{P}_{\mu}^{\mathbf{q},t}(E) = \inf_{E\subseteq\bigcup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu}^{\mathbf{q},t}(E_i).$$

The functions $\mathcal{H}_{\mu}^{\mathbf{q},t}$ and $\mathcal{P}_{\mu}^{\mathbf{q},t}$ are metric outer measures and thus they are measures on the family of Borel subsets of $\mathbf{R}^d$. An important feature of the mixed multifractal Hausdorff and packing measures is that,

$$\mathcal{H}_{\mu}^{\mathbf{q},t} \leq \xi \mathcal{P}_{\mu}^{\mathbf{q},t} \leq \xi \overline{\mathcal{P}}_{\mu}^{\mathbf{q},t}, \tag{2.2}$$

where ξ is the number related to the Besicovitch covering theorem.

The measures $\mathcal{H}_{\mu}^{\mathbf{q},t}$ and $\mathcal{P}_{\mu}^{\mathbf{q},t}$ and the pre-measure $\overline{\mathcal{P}}_{\mu}^{q,t}$ assign in the usual way a mixed multifractal dimension to each subset E of $\mathbf{R}^d$. They are respectively denoted by $\dim_{\mu}^{\mathbf{q}}(E)$, $Dim_{\mu}^{\mathbf{q}}(E)$ and $\Delta_{\mu}^{\mathbf{q}}(E)$ (see [29]).

1. There exists a unique number $dim_{\mu}^{\mathbf{q}}(E) \in [-\infty, +\infty]$, such that

$$\mathcal{H}_{\mu}^{\mathbf{q},t}(E) = \begin{cases} \infty & \text{if} \quad t < dim_{\mu}^{\mathbf{q}}(E), \\ 0 & \text{if} \quad \dim_{\mu}^{\mathbf{q}}(E) < t. \end{cases}$$

2. There exists a unique number $Dim_{\mu}^{\mathbf{q}}(E) \in [-\infty, +\infty]$, such that

$$\mathcal{P}_{\mu}^{\mathbf{q},t}(E) = \begin{cases} \infty & \text{if} \quad t < Dim_{\mu}^{\mathbf{q}}(E), \\ 0 & \text{if} \quad Dim_{\mu}^{\mathbf{q}}(E) < t. \end{cases}$$

3. There exists a unique number $\Delta_{\mu}^{\mathbf{q}}(E) \in [-\infty, +\infty]$, such that

$$\overline{\mathcal{P}}_{\mu}^{\mathbf{q},t}(E) = \begin{cases} \infty & \text{if} \quad t < \Delta_{\mu}^{\mathbf{q}}(E), \\ 0 & \text{if} \quad \Delta_{\mu}^{\mathbf{q}}(E) < t. \end{cases}$$

Next, we define the separator functions b_{μ}, B_{μ} and $\Lambda_{\mu} \,:\, R^k \to [-\infty, +\infty]$ by

$$b_{\mu} :\ \mathbf{q} \to \dim_{\mu}^{\mathbf{q}}(K), \quad B_{\mu} :\ \mathbf{q} \to Dim_{\mu}^{\mathbf{q}}(K) \quad \text{and} \quad \Lambda_{\mu} :\ \mathbf{q} \to \Delta_{\mu}^{\mathbf{q}}(K).$$

From (2.2) it follows that

$$b_{\mu}(\mathbf{q}) \le B_{\mu}(\mathbf{q}) \le \Lambda_{\mu}(\mathbf{q}).$$

For $x \in \mathbf{R}^d$ and $j = 1, 2, \ldots, k$, we denote

$$\underline{\alpha}_{\mu_j}(x) = \liminf_{r \to 0} \frac{\log \mu_j(B(x,r))}{\log r} \qquad \text{and} \qquad \overline{\alpha}_{\mu_j}(x) = \limsup_{r \to 0} \frac{\log \mu_j(B(x,r))}{\log r}.$$

respectively the local lower dimension and the local upper dimension of μ_j at the point x and as usually the local dimension $\alpha_{\mu_j}(x)$ of μ_j at x will be the common value when these are equal. Next for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbf{R}_+^k$, let us introduce the fractal sets

$$\underline{E}_{\alpha} = \left\{ x \in \mathbf{R}^d \;\middle|\; \underline{\alpha}_{\mu_j}(x) \ge \alpha_j, \ \ \forall j = 1, 2, \ldots, k \right\}$$

$$\overline{E}^{\alpha} = \left\{ x \in \mathbf{R}^d \;\middle|\; \overline{\alpha}_{\mu_j}(x) \le \alpha_j, \ \ \forall j = 1, 2, \ldots, k \right\}$$

and

$$E(\alpha) = \underline{E}_{\alpha} \bigcap \overline{E}_{\alpha}.$$

The mixed multifractal spectrum of the vector-valued measure μ is defined by $\alpha \mapsto \dim_H E(\alpha)$ where dim stands for the Hausdorff dimension. Our purpose in the following theorem is to improve [29, Theorem 6.1] and to propose a new sufficient condition that gives the lower bound of the validity for the mixed multifractal formalism for vector-valued measures. To state Theorem 2.1, we need the notion of the Legendre transform of a function defined on $\mathbf{R}^k$. For a function $\varphi : \mathbf{R}^k \to \mathbf{R}$ we define the Legendre transform $\varphi^* : \mathbf{R}^k \to \mathbf{R}$ by

$$\varphi^*(\mathbf{x}) = \inf_{\mathbf{y}} \Big(\langle \mathbf{x},\ \mathbf{y}\rangle + \varphi(\mathbf{y})\Big),$$

where $\langle .,.\rangle$ denotes the usual inner product in $\mathbf{R}^k$. We need also the following notions: $\nabla_+\varphi$ and $\nabla_-\varphi$ denote the left and right hand sides derivative of φ for vectors in $\mathbf{R}^k$, respectively. We denote the derivative of the function φ for vectors in $\mathbf{R}^k$ by $\nabla\varphi$,

$$\mathbf{R}^k_- = (-\infty, 0] \times (-\infty, 0] \times \ldots \times (-\infty, 0]$$

and

$$\mathbf{R}^k_+ = [0, +\infty) \times [0, +\infty) \times \ldots \times [0, +\infty).$$

Theorem 2.1. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector-valued probability measure on $\mathbf{R}^d$ and $\mathbf{q} \in \mathbf{R}^k$ such that $\mathcal{H}_\mu^{\mathbf{q},B_\mu(\mathbf{q})}(K) > 0$. Then,*

$$\dim_H \Big(\underline{E}_{\,-\nabla_+ B_\mu(\mathbf{q})} \cap \overline{E}^{\,-\nabla_- B_\mu(\mathbf{q})}\Big) \geq \begin{cases} \langle -\nabla_- B_\mu(\mathbf{q}), \mathbf{q}\rangle + B_\mu(\mathbf{q}), & \textit{for} \quad \mathbf{q} \in \mathbf{R}^k_-, \\ \langle -\nabla_+ B_\mu(\mathbf{q}), \mathbf{q}\rangle + B_\mu(\mathbf{q}), & \textit{for} \quad \mathbf{q} \in \mathbf{R}^k_+. \end{cases}$$

In particular, if B_μ is differentiable at $\mathbf{q}$, one has

$$\dim_H E\Big(- \nabla B_\mu(\mathbf{q})\Big) = \dim_P E\Big(- \nabla B_\mu(\mathbf{q})\Big) = B_\mu^*(-\nabla B_\mu(\mathbf{q})).$$

The following result proves that the condition $\mathcal{H}_\mu^{\mathbf{q},B_\mu(\mathbf{q})}(K) > 0$ is very close to being a necessary and sufficient condition for the validity of the mixed multifractal formalism for vector-valued measures.

Theorem 2.2. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector valued probability measure on $\mathbf{R}^d$ and $\mathbf{q} \in \mathbf{R}^k$. Now, suppose that one of the following hypotheses is satisfied,*

1. $\dim_H \Big(\underline{E}_{\,-\nabla_+ B_\mu(\mathbf{q})} \cap \overline{E}^{\,-\nabla_- B_\mu(\mathbf{q})}\Big) \geq \langle -\nabla_+ B_\mu(\mathbf{q}), \mathbf{q}\rangle + B_\mu(\mathbf{q})$, *for* $\mathbf{q} \in \mathbf{R}^k_-$,

2. $\dim_H \left(\underline{E}_{-\nabla_+ B_\mu(\mathbf{q})} \cap \overline{E}^{-\nabla_- B_\mu(\mathbf{q})} \right) \geq \langle -\nabla_- B_\mu(\mathbf{q}), \mathbf{q} \rangle + B_\mu(\mathbf{q})$, *for* $\mathbf{q} \in \mathbf{R}_+^k$.

Then, $b_\mu(\mathbf{q}) = B_\mu(\mathbf{q})$. *That is,* $\mathcal{H}_\mu^{\mathbf{q},t}(K) > 0$ *for every* $t < B_\mu(\mathbf{q})$.

Remark 2.1. *Let* $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ *be a vector valued probability measure on* $\mathbf{R}^d$ *and* $\mathbf{q} \in \mathbf{R}^k$. *We suppose that* B_μ *is differentiable at* $\mathbf{q}$. *Then we have*

1. *if* $\mathcal{H}_\mu^{\mathbf{q},B_\mu(\mathbf{q})}(K) > 0$, *then*

$$\dim_H E\Big(-\nabla B_\mu(\mathbf{q})\Big) = \dim_P E\Big(-\nabla B_\mu(\mathbf{q})\Big) = B_\mu^*(-\nabla B_\mu(\mathbf{q})) = b_\mu^*(-\nabla B_\mu(\mathbf{q})).$$

2. $\dim_H E\Big(-\nabla B_\mu(\mathbf{q})\Big) \geq B_\mu^*(-\nabla B_\mu(\mathbf{q}))$, *then* $\mathcal{H}_\mu^{\mathbf{q},t}(K) > 0$ *for every* $t < B_\mu(\mathbf{q})$.

We apply the techniques of Ben Nasr et al. especially in [8] and [9] with the necessary modifications to study the existence of an auxiliary Radon (nontrivial) measure. More specifically, for $\mathbf{q} \in \mathbf{R}_-^k$ by using the hypothesis $\mathcal{H}_\mu^{\mathbf{q},B_\mu(\mathbf{q})}(K) > 0$ and Frostman's technique, there exists an auxiliary Radon (nontrivial) measure $\nu_\mathbf{q}$ satisfying

$$\nu_\mathbf{q}\Big(B(x,r)\Big) \leq C\mu\Big(B(x,r)\Big)^{\mathbf{q}} \ (2r)^{B_\mu(\mathbf{q})}. \tag{2.3}$$

In the case $\mathbf{q} \in \mathbf{R}_+^k$, such a construction is only possible for doubling measures μ.

Remark 2.2. *It is clear that the existence of a nontrivial measure satisfying (2.3) implies the condition* $\mathcal{H}_\mu^{\mathbf{q},B_\mu(\mathbf{q})}(K) > 0$. *Moreover, the existence of a measure satisfying (2.3) is strictly weaker than the existence of a Gibbs-like measure.*

Theorem 2.3. *The results of Theorems 2.1, 2.2 and Remark 2.1 hold if we replace the mixed multifractal function* B_μ *by* Λ_μ.

3. Proof of the main results

3.1. Proof of Theorem 2.1.

Before proving Theorem 2.1 we need the preliminary lemma.

Lemma 3.1. *For all $\eta_j > 0$, $\forall j = 1, 2, \ldots, k$, $t \in \mathbf{R}$ and $\mathbf{q} \in \mathbf{R}^k$, $\alpha \in \mathbf{R}_+^k$ such that $\langle \alpha, \mathbf{q}\rangle + t \geq 0$, we have*

1. *if $E \subset \overline{E}^\alpha$, is Borel then $\mathcal{H}^{\langle \alpha,\mathbf{q}\rangle+t-\eta}(E) \geq 2^{\langle \alpha,\mathbf{q}\rangle-\eta}\, \mathcal{H}_\mu^{\mathbf{q},t}(E)$ for $\mathbf{q} \in \mathbf{R}_-^k$,*
2. *if $E \subset \underline{E}_\alpha$, is Borel then $\mathcal{H}^{\langle \alpha,\mathbf{q}\rangle+t-\eta}(E) \geq 2^{\langle \alpha,\mathbf{q}\rangle-\eta}\, \mathcal{H}_\mu^{\mathbf{q},t}(E)$ for $\mathbf{q} \in \mathbf{R}_+^k$, where $\eta = \sum_{j=1}^k \eta_j$,*

here $\mathcal{H}^\alpha$ denotes the α-dimensional centered Hausdorff measure.

Proof. We treat the case $\mathbf{q} \in \mathbf{R}_-^k$. The other case is proved similarly. The result is true for $\mathbf{q} = 0$, so we may assume that $\mathbf{q} \in \mathbf{R}_-^k \setminus \{(0, 0, ..., 0)\}$. For $m \in \mathbf{N}^*$, write

$$E_m = \left\{ x \in E \;\middle|\; \frac{\log\Big(\mu_j(B(x,r))\Big)}{\log r} \leq \alpha_j - \frac{\eta_j}{q_j} \text{ for } 0 < r < \frac{1}{m},\ \ \forall\, 1 \leq j \leq k \right\}.$$

Given $F \subseteq E_m$, $0 < \delta < \dfrac{1}{m}$ and $\Big(B(x_i, r_i)\Big)_i$ a centered δ-covering of F, we have

$$\frac{\log \mu_j(B(x_i, r_i))}{\log r_i} \leq \alpha_j - \frac{\eta_j}{q_{\ j}}.$$

This implies that

$$\mu_j(B(x_i, r_i))^{q_j} \leq {r_i}^{\alpha_j q_j - \eta_j}.$$

However, it follows that

$$\mu(B(x_i, r_i))^{\mathbf{q}}(2r_i)^t \leq 2^t {r_i}^{\langle \alpha,\mathbf{q}\rangle + t - \sum_{j=1}^k \eta_j}.$$

We now have

$$\begin{aligned}\overline{\mathcal{H}}^{\mathbf{q},t}_{\mu,\delta}(F) &\leq \sum_i \mu(B(x_i,r_i))^{\mathbf{q}}(2r_i)^t \\ &\leq 2^{-\langle\alpha,\mathbf{q}\rangle+\eta}\sum_i (2r_i)^{\langle\alpha,\mathbf{q}\rangle+t-\eta}.\end{aligned}$$

We can deduce that

$$\overline{\mathcal{H}}^{\mathbf{q},t}_{\mu,\delta}(F) \leq 2^{-\langle\alpha,\mathbf{q}\rangle+\eta}\overline{\mathcal{H}}^{\langle\alpha,\mathbf{q}\rangle+t-\eta}_{\delta}(F).$$

Letting $\delta \searrow 0$ gives that

$$\begin{aligned}\overline{\mathcal{H}}^{\mathbf{q},t}_{\mu}(F) &\leq 2^{-\langle\alpha,\mathbf{q}\rangle+\eta}\overline{\mathcal{H}}^{\langle\alpha,\mathbf{q}\rangle+t-\eta}(F) \\ &\leq 2^{-\langle\alpha,\mathbf{q}\rangle+\eta}\overline{\mathcal{H}}^{\langle\alpha,\mathbf{q}\rangle+t-\eta}(E_m)\end{aligned}$$

for all $F \subseteq E_m$. This clearly implies that

$$\mathcal{H}^{\mathbf{q},t}_{\mu}(E_m) \leq 2^{-\langle\alpha,\mathbf{q}\rangle+\eta}\mathcal{H}^{\langle\alpha,\mathbf{q}\rangle+t-\eta}(E_m)$$

and the result follows since $E = \bigcup_m E_m$.

Theorem 2.1 is then an easy consequence of the following lemma.

Lemma 3.2. *We have,* $\mathcal{H}^{\mathbf{q},B_\mu(\mathbf{q})}_{\mu}\Big(K \setminus \Big(\underline{E}_{-\nabla_+ B_\mu(\mathbf{q})} \cap \overline{E}^{-\nabla_- B_\mu(\mathbf{q})}\Big)\Big) = 0.$

Proof. Let us introduce, for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_k)$ in $\mathbf{R}^k$

$$X_\alpha = K \setminus \underline{E}_\alpha \quad \text{and} \quad Y^\alpha = K \setminus \overline{E}^\beta.$$

We have just to prove that

$$\mathcal{H}^{\mathbf{q},B_\mu(\mathbf{q})}_{\mu}\Big(X_\alpha\Big) = 0, \quad \text{for all} \quad \alpha < -\nabla_+ B_\mu(\mathbf{q}) \tag{3.1}$$

and

$$\mathcal{H}^{\mathbf{q},B_\mu(\mathbf{q})}_{\mu}\Big(Y^\beta\Big) = 0, \quad \text{for all} \quad \beta > -\nabla_- B_\mu(\mathbf{q}). \tag{3.2}$$

Indeed,

$$\begin{aligned}
0 &\leq \mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}\Big(K\setminus\Big(\underline{E}_{-\nabla_+B_\mu(\mathbf{q})}\cap\overline{E}^{-\nabla_-B_\mu(\mathbf{q})}\Big)\Big)\\
&\leq \mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}\Big(K\setminus\Big(\underline{E}_{-\nabla_+B_\mu(\mathbf{q})}\Big)\Big)+\mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}\Big(K\setminus\Big(\overline{E}^{-\nabla_-B_\mu(\mathbf{q})}\Big)\Big)\\
&\leq \mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}\left(\bigcup_{\alpha<-\nabla_+B_\mu(\mathbf{q})}\underline{E}_\alpha\right)+\mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}\left(\bigcup_{\beta>-\nabla_-B_\mu(\mathbf{q})}\overline{E}^\beta\right)\\
&\leq \sum_{\alpha<-\nabla_+B_\mu(\mathbf{q})}\mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}\Big(X_\alpha\Big)+\sum_{\beta>-\nabla_-B_\mu(\mathbf{q})}\mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}\Big(Y^\beta\Big)=0.
\end{aligned}$$

We only have to prove that (3.1), the proof of (3.2) is similar. Let $\alpha<-\nabla_+B_\mu(\mathbf{q})$, then for all $1\leq j\leq k$, we can choose $t>0$ such that

$$B_\mu(q_1,q_2,\dots,q_j+t,\dots,q_k)<B_\mu(\mathbf{q})-\alpha_j t.$$

We denote $\mathbf{q}_j+t=(q_1,q_2,\dots,q_j+t,\dots,q_k)$, then

$$\mathcal{P}_{\mu}^{\mathbf{q}_j+t,B_\mu(\mathbf{q})-\alpha_j t}\Big(K\Big)=0.$$

If $x\in X_\alpha$, let $\delta>0$ we can find $j\in\{1,2,\dots,k\}$ and $0<r_x<\delta$ such that

$$\mu_j(B(x,r_x))>r_x^{\alpha_j}.$$

The family $\Big(B(x,r_x)\Big)_{x\in X_\alpha}$ is then a centered δ-covering of X_α. Using Besicovitch's Covering Theorem (see [28]), we can construct ξ that are finite or countable sub-families

$$\Big(B(x_{1p},r_{1p})\Big)_p,\dots,\Big(B(x_{\xi p},r_{\xi p})\Big)_p \text{ such that each } X_\alpha\subseteq\bigcup_{i=1}^{\xi}\bigcup_p B(x_{ip},r_{ip})$$

and $\Big(B(x_{ip},r_{ip})\Big)_p$ is a δ-packing of X_α. Observing that

$$\mu(B(x_{ip},r_{ip}))^{\mathbf{q}}(2r_{ip})^{B_\mu(\mathbf{q})}\leq 2^{\alpha_j t}\mu(B(x_{ip},r_{ip}))^{\mathbf{q}_j+t}(2r_{ip})^{B_\mu(\mathbf{q})-\alpha_j t}.$$

This clearly implies that

$$\overline{\mathcal{H}}_{\mu,\delta}^{\mathbf{q},B_\mu(\mathbf{q})}(X_\alpha)\leq\xi 2^{\alpha_j t}\overline{\mathcal{P}}_{\mu,\delta}^{\mathbf{q}_j+t,B_\mu(\mathbf{q})-\alpha_j t}(X_\alpha).$$

Letting $\delta \to 0$, we obtain

$$\overline{\mathcal{H}}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}(X_\alpha) \le \xi 2^{\alpha_j t} \overline{\mathcal{P}}_{\mu}^{\mathbf{q}_j+t,B_\mu(\mathbf{q})-\alpha_j t}(X_\alpha).$$

We can replace X_α by any arbitrary subset of X_α. Then by standard arguments we can finally conclude that

$$\mathcal{H}_{\mu}^{\mathbf{q},B_\mu(\mathbf{q})}(X_\alpha) \le \xi 2^{\alpha_j t} \mathcal{P}_{\mu}^{\mathbf{q}_j+t,B_\mu(\mathbf{q})-\alpha_j t}(K) = 0.$$

3.2. Proof of Theorem 2.2.

We have, for $\mathbf{q} \in \mathbf{R}_+^k$

$$\underline{E}_{-\nabla_+ B_\mu(\mathbf{q})} \cap \overline{E}^{-\nabla_- B_\mu(\mathbf{q})} \subseteq \overline{E}^{-\nabla_- B_\mu(\mathbf{q})},$$

this clearly implies that

$$\dim_H\left(\overline{E}^{-\nabla_- B_\mu(\mathbf{q})}\right) \ge \dim_H\left(\underline{E}_{-\nabla_+ B_\mu(\mathbf{q})} \cap \overline{E}^{-\nabla_- B_\mu(\mathbf{q})}\right) \ge \langle -\nabla_- B_\mu(\mathbf{q}), \mathbf{q}\rangle + B_\mu(\mathbf{q}).$$

Suppose that $\alpha = -\nabla_- B_\mu(\mathbf{q})$. We only prove the case where $\mathbf{q} \in \mathbf{R}_+^k$. The other one is similar. Then

$$\dim_H\left(\overline{E}^{\alpha}\right) \ge \langle \alpha, \mathbf{q}\rangle + B_\mu(\mathbf{q}).$$

For this, we have that

$$b_\mu(\mathbf{q}) \le B_\mu(\mathbf{q}).$$

It is, then, sufficient to prove $b_\mu(\mathbf{q}) \ge B_\mu(\mathbf{q})$. Let $t < B_\mu(\mathbf{q})$ and choose $\beta = (\beta_1, \dots, \beta_k)$ such that $\beta > \alpha$ and $\langle \beta, \mathbf{q}\rangle + t < \langle \alpha, \mathbf{q}\rangle + B_\mu(\mathbf{q})$. For $p \in \mathbf{N}$ we consider the set

$$F_p = \left\{ x \in \overline{E}^{\alpha} \;\middle|\; \mu_j(B(x,r)) \ge r^{\beta_j},\ 0 < r < \frac{1}{p},\ \forall\, 1 \le j \le k \right\}.$$

It is clear that $F_p \nearrow \overline{E}^{\alpha}$ as $p \to \infty$. It follows that, there exists $p > 0$, such that

$$\dim_H(F_p) > \langle \beta, \mathbf{q}\rangle + t \Rightarrow \mathcal{H}^{\langle \beta, \mathbf{q}\rangle + t}(F_p) > 0.$$

Let $0 < \delta < \frac{1}{p}$ and $\Big(B(x_i, r_i)\Big)_i$ is a centered δ-covering of F_p. Then,

$$\sum_i \mu(B(x_i,r_i))^{\mathbf{q}} (2r_i)^t \ge 2^{-\langle \beta, \mathbf{q}\rangle} \sum_i (2r_i)^{\langle \beta, \mathbf{q}\rangle + t}.$$

This shows that

$$\mathcal{H}_{\mu}^{\mathbf{q},t}(K) \ge \mathcal{H}_{\mu}^{\mathbf{q},t}(\overline{E}^{\alpha}) \ge \mathcal{H}^{\langle \beta, \mathbf{q}\rangle + t}(F_p) > 0.$$

It follows that $t \le b_\mu(\mathbf{q})$. Finally, we get $b_\mu(\mathbf{q}) = B_\mu(\mathbf{q})$.

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Bilel Selmi
Analysis, Probability & Fractals Laboratory LR18ES17
Faculty of Sciences of Monastir
Department of Mathematics
University of Monastir
5000-Monastir
Tunisia
e-mail: bilel.selmi@fsm.rnu.tn,
bilel.selmi@isetgb.rnu.tn

and

Anouar Ben Mabrouk
Algebra, Number Theory and Nonlinear Analysis Lab.
LR18ES15, Department of Mathematics,
Faculty of Sciences,
University of Monastir
5000 Monastir,
Tunisia

Department of Mathematics,
Higher Institute of Applied Mathematics and Informatics,
University of Kairouan,
3100 Kairouan,
Tunisia

Department of Mathematics,
Faculty of Sciences,
University of Tabuk,
Saudi Arabia Kindom
e-mail: anouar.benmabrouk@fsm.rnu.tn