



Relating centralities in graphs and the principal eigenvector of its distance matrix

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Abstract:

In this work a new centrality measure of graphs is presented, based on the principal eigenvector of the distance matrix: spectral closeness. Using spectral graph theory, we show some of its properties and we compare the results of this new centrality with closeness centrality. In particular, we prove that for threshold graphs these two centralities always coincide. In addition we construct an infinity family of graphs for which these centralities never coincide.

Keywords: Centrality; Distance matrix; Principal eigenvector; Spectral closeness.

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1. Introduction

Graph Theory is an important tool for analysing different types of networks. In this sense, the concept of centrality, introduced by [1] in the context of social networks, is used to measure the relevance of actors (vertices) appearing in the network. Some applications of this concept in different areas can be seen in [11,16,17,9].

Due to the diversity of characteristics that networks modelling real situations can present, a large number of centrality measures have already been defined and its properties have been studied [4,7,12]. Among them, we can mention degree centrality, eigenvector centrality and closeness centrality. In this work we introduce a new centrality measure: spectral closeness centrality. Two important concepts in network analysis are behind these measures: number of connections and distance between nodes. The first two measures, degree and eigenvector centralities, concern the amount of connections. The degree centrality of a vertex considers only the connections of this vertex, whereas the eigenvector centrality takes into account the degree of the vertex and that of its neighbours, being, in many applications, a more accurate measurement. Likewise, the closeness centrality of a vertex considers only the distances of this vertex to all the others in the network. The measure proposed here, as well as the eigenvector in relation to degree centrality, also considers the relative position of the other vertices in the network.

The spectral content of the proposed measure implies that discussing its properties, necessarily, involve Spectral Graph Theory. In particular, we use statements about the values assumed by the entries of the eigenvector associated with the largest eigenvalue of the distance matrix of a graph.

The article is structured in three sections besides this introduction. In the next section we present the definitions and basic concepts, necessary for the development of this work.

In Section 3 we introduce the new centrality, the spectral closeness and, from a computational search, we establish comparisons with the closeness centrality. Also in this section we provide some properties about this measure as a consequence of results from literature on the principal eigenvector of the distance matrix.

In the last section we present a sufficient condition for a graph to have the same vertex as more central, according to the centralities of closeness and spectral closeness. Similar problems were considered in the context of degree and eigenvector centralities [7,8].

Concluding, we analyze the behavior of this new centrality in certain families of graphs, comparing the order of the vertices obtained by it and by the closeness. We obtain results in the class of thresholds and for some subfamilies of cographs. In addition, we have introduced an infinite family of graphs for which the ordering by the two centralities is always different.

2. Preliminaries

In this work, $G(V, E)$, or simply G , denotes a simple and undirected graph on n vertices. We denote the set of neighbours of a vertex $v \in V(G)$ by $N(v)$. For the degree of v , $|N(v)|$, we use $\deg(v)$. If $N(v) = V(G) \setminus \{v\}$, v is called a dominant vertex. If it has exactly 1 neighbour, it is called a pendant vertex and if it does not have neighbours, it is called an isolated vertex.

The adjacency matrix of a graph G , $A(G) = [a_{i,j}]$, is the square matrix of order n , such that $a_{i,j} = 1$, if v_i and v_j are adjacent and $a_{i,j} = 0$, otherwise.

The distance matrix of a connected graph G , $\mathcal{D}(G) = [d_{i,j}]$, is the square matrix of order n , such that $d_{i,j} = d(v_i, v_j)$, where $d(v_i, v_j)$ is the distance (the length of a shortest path) between vertices v_i and v_j of G . For $1 \leq i \leq n$, the sum of the distances from v_i to all other vertices in G is known as the transmission of the vertex v_i and is denoted by $Tr(v_i)$. If all the vertices of the graph have the same transmission then the graph is said to be regular transmission.

From now on, all the considered graphs are connected. We will refer to the largest eigenvalue of $A(G)$ and $\mathcal{D}(G)$, respectively, as adjacency index and distance index.

Remark 2.1. *If G is a connected graph, both $A(G)$ and $\mathcal{D}(G)$ are symmetric, irreducible and nonnegative. By Perron-Fronebius Theorem (see [10], for example), the adjacency and the distance indices of G have multiplicity equal to 1, each of these indices being associated with a single positive unitary eigenvector (for a fixed norm), oftenly called the principal eigenvector of the matrix.*

Let G be a connected graph and $v_i \in V(G)$. The simplest of the centrality measures is the degree, proposed by [21]. We present below this definition and also the definition of eigenvector centrality, introduced by [2]:

- Degree centrality of v_i is defined by $c_d(v_i) = \deg(v_i)$, that is, the degree of vertex v_i .
- Eigenvector Centrality of v_i is defined by $c_{ev}(v_i) = x_i = \frac{1}{\lambda} \sum_{k \in N(v_i)} x_k$, where x_i is the i -th coordinate of the unit positive eigenvector associated to the adjacency index of the graph.

While the degree centrality takes into account the vertex neighbours, the eigenvector centrality considers not only the number of neighbours of the vertex but also its relevance. In this sense, the eigenvector centrality is considered as an extension of degree centrality.

In many networks the distance between the vertices is an essential information. In this sense, [19] introduced the closeness centrality:

- Closeness centrality of v_i is defined by $c_c(v_i) = \frac{1}{\sum_{k=1}^n d_{v_i, v_k}} = \frac{1}{Tr(v_i)}$, that is, the inverse of the sum of the distances from v_i to all other vertices of the graph.

3. Spectral closeness centrality

Motivated by the fact that the eigenvector centrality extends, in some sense, the degree centrality, by giving more information about the structure of the graph, we propose a new centrality measure, called *spectral closeness*, that extends the closeness centrality.

Definition 3.1. *The spectral closeness of v_i is defined by $c_{sc}(v_i) = \frac{1}{x_i}$, where x_i is the i -th coordinate of the unit positive eigenvector associated with the distance index of the graph.*

It follows from Remark 2.1 that the spectral closeness centrality is well defined. Moreover, if ∂ and $\mathbf{x} = (x_1, \dots, x_n)$ are, respectively, the index and the unit positive eigenvector associated with the matrix $\mathcal{D}(G)$ then, from $\mathcal{D}\mathbf{x} = \partial\mathbf{x}$, we conclude that

$$(3.1) \quad x_i = \frac{1}{\partial} \sum_{k=1}^n d_{v_i, v_k} x_k \iff c_{sc}(v_i) = \frac{1}{x_i} = \frac{\partial}{\sum_{k=1}^n d_{v_i, v_k} x_k}.$$

Thus, from the Equation 1, a vertex v_i becomes more central with respect to spectral closeness if and only if $\sum_{k=1}^n d_{v_i, v_k} x_k$ is minimum among all the vertices of G , which means that the centrality of v_i depends on the sum of the centralities of all vertices of G , weighted by its entries in the principal eigenvector. So, reduced value for its transmission (only condition that was present in closeness centrality) is still important. In addition, this centrality also attributes importance to v_i according to the number of vertices distant from v_i that have small values associated in the principal eigenvector. That is, if v_i has a small transmission and furthermore v_i is far away from other central vertices (therefore, close to less central vertices), then v_i tends to become more important.

From the interpretation above, a natural question to be made is how the ordering of vertices relevance according to closeness and spectral closeness are related. Comparing the behavior of these two centralities in an example, we verify that these two measures differ from each other.

Example 3.2. For the graph of Figure 3.1, we have the followings values for closeness centrality and spectral closeness centrality.

$c_c(v_1)$	$c_c(v_6)$	$c_c(v_7)$	$c_c(v_2)$	$c_c(v_5)$	$c_c(v_8)$	$c_c(v_9)$	$c_c(v_3)$	$c_c(v_4)$
0,067	0,067	0,067	0,059	0,059	0,059	0,055	0,053	0,053

$c_{sc}(v_1)$	$c_{sc}(v_6)$	$c_{sc}(v_7)$	$c_{sc}(v_2)$	$c_{sc}(v_5)$	$c_{sc}(v_8)$	$c_{sc}(v_9)$	$c_{sc}(v_3)$	$c_{sc}(v_4)$
3,340	3,332	3,291	3,021	3,014	2,947	2,806	2,744	2,737

Note that there is a lot of ties in ordering considering closeness centrality, the only vertex that does not admit another one with the same centrality is v_9 , while using the spectral closeness, we get a strict ordering of the vertices.

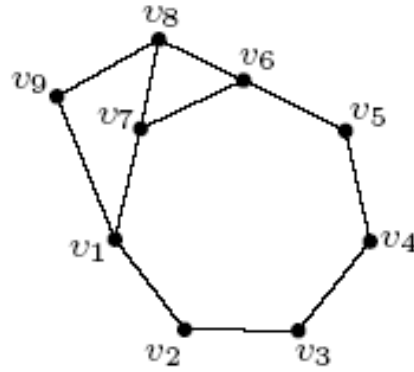


Figure 3.1: Only v_9 does not admits another vertex with same closeness centrality.

Computationally, by using the softwares nauty and Traces,[15], and SageMath[20] we could compare the sets of more central vertices according to these measures. The results, concerning all connected graphs on n vertices, $5 \leq n \leq 10$, and all trees up to 21 vertices are exhibit in Table 3.1 and Table 3.2, respectively. For a connected graph $G(V, E)$ on n vertices with principal distance eigenvector $\mathbf{x} = (x_1, x_2, \dots, x_n)$, we denote by T the set of vertices with greatest closeness centrality, $T = \left\{ v \in V : Tr(v) = \min_{1 \leq k \leq n} Tr(x_k) \right\}$, and by W the set of vertices with greatest spectral closeness centrality, $W = \left\{ v \in V : x_v = \min_{1 \leq k \leq n} x_k \right\}$.

The computational experiments reinforce the idea that the transmissions and the entries of the main distance eigenvector are related, which, in the context of centralities, indicates that the spectral closeness could refine the closeness since the first one consider more elements of the structure of the graph and, in most of the cases, we get $W \subseteq T$. More than this, in all tested cases, $|W| \leq |T|$.

In Figure 3.2 we exhibit all 7 connected graphs with 10 vertices with no intersection between sets T and W . Those are the smallest graphs with such property. This is also the smallest number of vertices so that a graph has just one vertex with the largest closeness centrality, just one vertex with

n	Graphs	$W \subseteq T$	$W = T$	$ W < T $	$ W = T $
5	21	21	19	2	19
6	112	112	93	19	93
7	853	853	634	219	634
8	11117	11117	7560	3557	7560
9	261080	261080	166573	94507	166573
10	11716571	11716564	7209084	4507486	7209085

Table 3.1: Comparing centralities among connected graphs.

n	Trees	$W \subseteq T$	$W = T$	$ W < T $	$ W = T $
11	235	235	235	0	235
12	551	551	361	190	361
13	1301	1301	1301	0	1301
14	3159	3159	2031	1128	2031
15	7741	7741	7741	0	7741
16	19320	19320	12765	6555	12765
17	48629	48620	48620	0	48629
18	123867	123867	83112	40755	83112
19	317955	317641	317641	0	317955
20	823065	823065	564945	258120	564945
21	2144505	2139619	2139619	0	2144505

Table 3.2: Comparing centralities among trees.

the largest spectral closeness centrality, and they are different. Restricting this search to trees, the smallest one has order 17. In Figure 3.3 we show one of the 9 trees with this property.

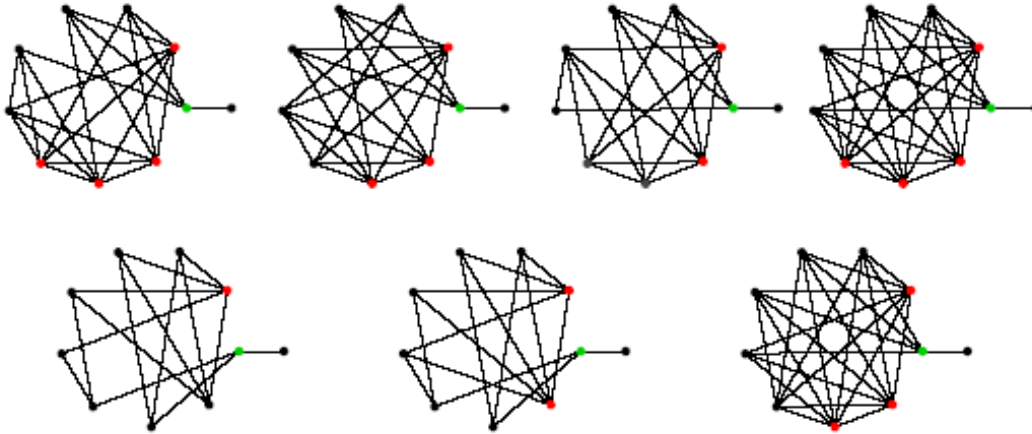


Figure 3.2: Smallest graphs with $W \cap T = \emptyset$. Red and green vertices represent, respectively, vertices with greatest closeness and greatest spectral closeness measures.

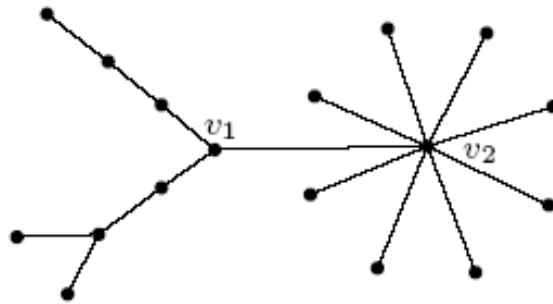


Figure 3.3: Tree on 17 vertices where v_1 has the largest spectral closeness centrality and v_2 the largest closeness centrality.

In the remainder of this section, we present some theoretical results on this new centrality, based on already known properties concerning the entries of the principal eigenvector for the distance matrix of a graph. We shall now present some immediate consequences, adapted to the concept of spectral closeness centrality. For this purpose, it is enough to note that if $v_i, v_j \in V(G)$ with corresponding entries in the distance principal eigenvector x_i, x_j then

$$x_i > x_j \Leftrightarrow c_{sc}(v_i) = \frac{1}{x_i} < \frac{1}{x_j} = c_{sc}(v_j).$$

In [22] it was established a relationship between the entries of the principal eigenvector of the distance matrix when there is an automorphism between the corresponding vertices. In the context of spectral closeness centrality, we have the following relation:

Proposition 3.3. *Let G be a connected graph and $v_i, v_j \in V(G)$. If there is an automorphism ϕ of G such that $\phi(v_i) = v_j$ then $c_{sc}(v_i) = c_{sc}(v_j)$.*

Note that if G is a regular transmission graph, then all its vertices have the same spectral closeness centrality. In fact, since the sum of entries in each line of $D(G)$ is constant, then this matrix has as its principal eigenvector $\mathbf{1} = (1, 1, \dots, 1)^T$. So, regular transmission graphs play an analogous role for the spectral closeness centrality to that performed by regular graphs with respect to the eigenvector centrality since, in the second case, the principal eigenvector for $A(G)$ is also $\mathbf{1}$ and all vertices have the same eigenvector centrality. Anyway, we point out that the converse of Proposition 3.3 is not true, in general. As an example, we can consider the non regular graph presented in Figure 3.4. As $\deg(v_1) = \deg(v_2)$, there is no automorphism of G mapping v_1 in v_2 . But, as all vertices have transmission equal 14, all vertices have the same spectral closeness centrality.

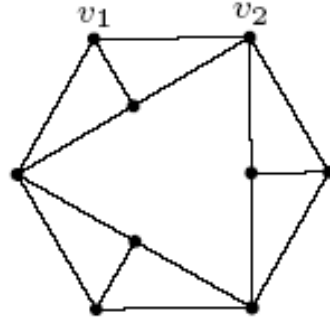


Figure 3.4: Graph with all vertices having the same spectral closeness centrality, without automorphism between v_1 and v_2

The next result was presented firstly, in the context of the principal eigenvector for the distance matrix, in [18].

Proposition 3.4. *Let G be a connected graph with $n \geq 3$ vertices. If v_j is a pendant vertex adjacent to v_i in G , then $c_{sc}(v_i) > c_{sc}(v_j)$.*

Still in this work, the authors stated other properties concerning trees.

Proposition 3.5. *Let T be a tree with $n \geq 3$ vertices.*

- *The vertex of largest spectral closeness centrality occurs at an inner vertex. Moreover, the maximum spectral closeness centrality can occur in at most two vertices and, in this case, they are adjacent.*
- *The smallest spectral closeness centrality occurs at a pendant vertex and it may occur at several vertices.*

In [13] and [14] it was proven a relation between the vertices transmissions and also a relation between the entries of the principal eigenvector for the distance matrix, respectively, when there is an inclusion between the set of neighbours of the corresponding vertices, implying that:

Proposition 3.6. *Let G be a connected graph and $v_i, v_j \in V(G)$. If $N(v_j) \setminus \{v_i\} \subset N(v_i) \setminus \{v_j\}$, then $c_c(v_i) > c_c(v_j)$. Moreover, if $N(v_i) \setminus \{v_j\} = N(v_j) \setminus \{v_i\}$, then $c_c(v_i) = c_c(v_j)$.*

Proposition 3.7. *Let G be a connected graph and $v_i, v_j \in V(G)$. If $N(v_j) \setminus \{v_i\} \cap N(v_i) \setminus \{v_j\} \neq \emptyset$, then $c_{sc}(v_i) > c_{sc}(v_j)$. Moreover, if $N(v_i) \setminus \{v_j\} = N(v_j) \setminus \{v_i\}$, then $c_{sc}(v_i) = c_{sc}(v_j)$.*

As a consequence, we can enunciate:

Proposition 3.8. *Let G be a connected graph and $S \subset V(G)$, $S \neq \emptyset$, the set of dominant vertices of G . The more central vertices according to closeness and spectral closeness centralities are exactly the vertices of S .*

Proof. If v_1 is a dominant vertex and $v_2 \notin S$, then $N(v_2) \setminus \{v_1\} \cap N(v_1) \setminus \{v_2\} \neq \emptyset$ and the result follows from Proposition 3.6 and Proposition 3.7. \square

4. Ordering vertices by closeness and spectral closeness centralities

As shown before, the vertices of a graph can have different ordering according to closeness and spectral closeness centralities. In this section we discuss conditions and families of graphs where it is possible to guarantee that the more central vertices, according to these centralities, coincide and cases where it does not coincide.

4.1. Conditions on the distance index

Here, we obtain a similar condition to that presented by [8] comparing degree and eigenvector centralities, but now considering closeness and spectral closeness centralities.

Throughout this subsection, we consider a connected graph $G(V, E)$ on n vertices, with transmissions $Tr_1 \leq Tr_2 \leq \dots \leq Tr_n$, and unit positive eigenvector $\mathbf{x} = (x_1, x_2, \dots, x_n)$, associated with the distance index ∂ . Also, we denote

$$U = \left\{ v \in V(G) : x_v = \max_{1 \leq k \leq n} x_k \right\},$$

$W = \left\{ v \in V(G) : x_v = \min_{1 \leq k \leq n} x_k \right\}$, $T_{\min}(U) = \min \{Tr(v) : v \in U\}$ and $T_{\max}(W) = \max \{Tr(v) : v \in W\}$. We remember that $x_v = \min \{x_k : 1 \leq k \leq n\}$ means that the largest spectral closeness centrality is attained in this vertex. Similarly, if $Tr(v) = Tr_1$ then v has the largest closeness centrality in G .

Initially we get a new bound for the distance index of a connected graph.

Theorem 4.1. *Let G be a connected graph. $\mathcal{D}(G)$. Then, $T_{\max}(W) \leq \partial \leq T_{\min}(U)$, with equality occurring if, and only if, G is regular transmission.*

Proof. We will prove just the second inequality, since the other can be get analagously. Let $v \in U$ such that $Tr(v) = T_{\min}(U)$. So,

$$(4.1) \quad \partial = \frac{1}{x_v} \sum_{k=1}^n d_{k,v} x_k \leq \sum_{k=1}^n d_{k,v} = Tr(v) = T_{\min}(U).$$

If equality holds in Equation 4.1, then, $\sum_{k=1}^n d_{k,v} x_k = \sum_{k=1}^n d_{k,v} x_v$. As $0 < x_k \leq x_v$, it implies $x_k = x_v \forall k \in V(G)$ and G is regular transmission. \square

It is well known (see [5], for instance) the following bounds for the distance index: $Tr_1 \leq \partial \leq Tr_n$. Actually, whenever the sets W and $T = \left\{ v \in V : Tr(v) = \min_{1 \leq k \leq n} Tr(x_k) \right\}$, are disjoint, the lower bound from Theorem 4.1 improves this bound. It happens for all graphs presented in Figure 3.2 and Figure 3.3. In the last case, for example, the lower bound is improved from 31 to 32. Analogously, the same improvement can be applied to the upper bound.

The next results are essential to ensure a sufficient condition for the vertex with the largest spectral closeness centrality has also the largest closeness centrality.

Corollary 4.2. *Let G be a connected graph and $v \in V(G)$ such that $x_v = \min_{1 \leq k \leq n} x_k$. If $w \in V$ is such that $Tr(w) > \partial$ then $x_w > x_v$.*

Proof. If $x_w = x_v$, it would imply $w \in W$ and $Tr(w) \leq T_{\max}(W) \leq \partial$. Contradiction. \square

Proposition 4.3. *Let G be a connected graph and $v \in V(G)$ such that $x_v = \min_{1 \leq k \leq n} x_k$. If $Tr_1 = Tr_2 = \dots = Tr_k \leq \partial < Tr_{k+1} \leq \dots Tr_n$, for some $1 \leq k \leq n-1$, then $Tr(v) = Tr_1$.*

Proof. If $v \in V(G)$ is such that $x_v = \min x_i$, by Corollary 4.2, $Tr(v) \leq \partial$ and, then, $Tr(v) = Tr_1$. \square

Finally, in the next theorem we have a sufficient condition for a graph to have exactly one vertex with the largest closeness centrality and spectral closeness centrality.

Theorem 4.4. *Let G be a connected graph. If $Tr_2 > \partial$ then there is a unique $v \in V(G)$ such that $x_v < \min_{i \neq v} x_i$ and $Tr(v) = Tr_1 < Tr_2$.*

Proof. Let $v \in V(G)$ such that $Tr(v) = Tr_1$. Then, $Tr(v) \leq \partial < Tr_2$. Moreover, if $w \in V$ is such that $x_w = \min x_i$, from Proposition 4.3, $Tr(w) = Tr(v)$, and $w = v$. \square

Example 4.5. *Let S_n be a star on $n \geq 3$ vertices. Its values for transmissions are $Tr_1 = n - 1$ and $Tr_2 = Tr_3 = \dots = Tr_n = 2n - 3$. More than this, as $\rho(S_n) = n - 2 + \sqrt{(n - 2)^2 + (n - 1)}$, it follows that $Tr_1 < \rho(S_n) < Tr_2 = \dots = Tr_n$. So, the dominant vertex of the star is the only one that attains the largest value for closeness and closeness spectral centralities.*

It is easy to check that the vertices of the star are ordered in the same way by considering the closeness and the spectral closeness centralities. In fact, the star is a threshold graph. We prove in the sequence that, for this family of graphs, both spectral closeness and closeness centralities provide the same ordering of vertices.

4.2. Threshold Graphs and Cographs

Threshold graphs were introduced in [6] and they have applications in several areas. We remind that a *threshold* graph G of order n can be obtained through an iterative process that begins with an isolated vertex and, at each step, either a new isolated vertex is added, or a vertex adjacent to all previous one is added. Thus, a threshold graph can be represented by a binary sequence (b_1, b_2, \dots, b_n) , where $b_i = 0$ means the addition of an isolated vertex and $b_i = 1$ the addition of a dominant vertex. Considering a connected threshold graph, we have $b_n = 1$.

We determine how to order the vertices of a threshold graph according to the closeness centrality and spectral closeness centrality.

Theorem 4.6. *Let G be a connected threshold graph with binary sequence $(0, b_2, \dots, b_{n-1}, 1)$. Then, the ordering of the vertices of G , according to the spectral closeness centrality is $c_{sc}(v_1) \geq c_{sc}(v_2) \geq \dots \geq c_{sc}(v_m) \geq c_{sc}(w_1) \geq c_{sc}(w_2) \geq \dots \geq c_{sc}(w_k)$, $m + k = n$, where:*

- All vertices v_i , $1 \leq i \leq m$, correspond to numbers 1 in the binary sequence of G ; all vertices w_j , $1 \leq j \leq k$, correspond to numbers 0 in the binary sequence of G ;

- $c_{sc}(v_m) = c_{sc}(w_1)$ if and only if $b_2=1$;
- $c_{sc}(v_i) \geq c_{sc}(v_j)$ if and only if v_i is associated with an entry in the binary sequence of G after that associated with v_j ; $c_{sc}(v_i) = c_{sc}(v_j)$ if and only if v_i and v_j are vertices associated with consecutive entries in the binary sequence;
- $c_{sc}(w_i) \geq c_{sc}(w_j)$ if and only if w_i is associated with an entry in the binary sequence of G previous to that associated with w_j ; $c_{sc}(w_i) = c_{sc}(w_j)$ if and only if w_i e w_j are vertices associated with consecutive entries in the binary sequence.

Proof. Let G be a threshold graph with binary sequence $(0, b_2, \dots, b_{n-1}, 1)$. Note that the vertices associated with number 1 in the binary sequence determine, from left to right, an increasing sequence of “nesting neighbourhood sets” in the sense that, if $b_i = b_j = 1$ and $i < j$, then $N(v_i) \setminus \{v_j\} \subset N(v_j) \setminus \{v_i\}$. Moreover, $N(v_i) \setminus \{v_j\} = N(v_j) \setminus \{v_i\}$ if, and only if, $j = i + 1$. In this way, it follows from Theorem 3.7 that, if $b_i = b_j = 1$ and $i < j$ then $c_{sc}(v_i) \leq c_{sc}(v_j)$, with equality holding if, and only if, $j = i + 1$. Still, by analogous argument, the vertices related with number 0 in the binary sequence determine, from left to right, a decreasing sequence of “nesting neighbourhood sets”. Again by Theorem 3.7, if $b_i = b_j$ and $i < j$ then $c_{sc}(v_i) \geq c_{sc}(v_j)$, with equality holding if, and only if, $j = i + 1$. For finishing the proof, it remains to note that if $i_0 \in \mathbf{N}$ is the smallest index such that $b_{i_0} = 1$ and $j_0 \in \mathbf{N}$ is the smallest index such that $b_{j_0} = 0$, then neste caso, que $N(v_j) \setminus \{v_{j_0}\} \subset N(v_{i_0}) \setminus \{v_{j_0}\}$, with equality holding if, and only if, $j_0 = 1$ and $i_0 = 2$. \square

By Proposition 3.6, it is possible to obtain an analogous result to Theorem 4.6 replacing, in its statement, *spectral closeness centrality* by *closeness centrality*. Therefore, both measures order the vertices of a connected threshold graph in the same way.

Theorem 4.7. *Every connected threshold graph presents the same ordering of vertices according to closeness and spectral closeness centralities.*

From now on, as usual, we denote by P_n, C_n and K_n , respectively, the path, cycle and complete graph on n vertices. For graphs G and H and $p \in \mathbf{N}$, pG denotes the disjoint union of p copies of G and $G \vee H$ denotes the join of the graphs G and H .

A cograph is a graph free of P_4 . As thresholds are graphs free of P_4, C_4 and $2K_2$, a natural question would be to check whether the above result

can be extended to cographs. The answer is negative, as shown in the graph presented in Figure 4.1. In this case, the central vertices according to closeness and spectral closeness centralities are not the same. For closeness, the largest centrality is attained for $\{v_3, v_4, v_7, v_8, v_9, v_{10}\}$, while for spectral closeness, the largest centrality is attained just for $\{v_3, v_4\}$.

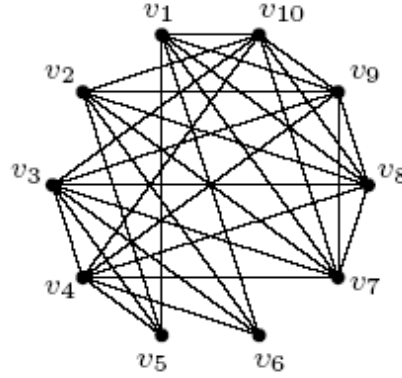


Figure 4.1: Cograph where the set of more central vertices according to spectral closeness centrality is strictly included in the set of vertices with largest closeness centrality.

However, we present below two families of non-threshold cographs where the ordering of the vertices by the two centralities considered here always coincide.

Proposition 4.8. *For $p, q \in \mathbf{N}$, let $G = pC_4 \vee K_q$. The more central vertices according to closeness centrality and spectral closeness centrality coincide.*

Proof. In fact, from Corollary 3.8, the set of vertices that induce K_q are the more central in both centralities. \square

Proposition 4.9. *For $p, q \in \mathbf{N}$, with $p < q$, let $G = pC_4 \vee qC_4$. The more central vertices according to closeness centrality and spectral closeness centrality coincide.*

Proof. Let $v_1, v_2 \in V(G)$ be vertices from the subgraph induced by pC_4 and qC_4 , respectively. From Proposition 3.3 it is enough to prove that

$c_c(v_1) > c_c(v_2)$ and that $c_{sc}(v_1) > c_{sc}(v_2)$. For the first inequality, note that $Tr(v_1) = 8p + 4q - 4$, $Tr(v_2) = 8q + 4p - 4$ and $Tr(v_1) < Tr(v_2)$.

For the second one, let M be the natural quotient matrix associated with $D(G)$. Then,

$$M = \begin{pmatrix} 8p - 4 & 4q \\ 4p & 8q - 4 \end{pmatrix},$$

is a matrix with largest eigenvalue $\partial = 4(\sqrt{p^2 - pq + q^2} + p + q - 1)$ associated with the eigenvector $\mathbf{z} = \left(\frac{p - q + \sqrt{p^2 - pq + q^2}}{p}, 1 \right)^T$. As $p - q + \sqrt{p^2 - pq + q^2} < p$, it follows that $c_{sc}(v_1) > c_{sc}(v_2)$. \square

4.3. k-bal Graphs

We finalize this work constructing a family of graphs where the more central vertices, according to closeness and spectral closeness centralities, are always different, called k -bal graphs. Moreover, we completely order the vertices according these centralities and discuss the vertices pointed as more central, through the interpretation for the spectral closeness centrality, presented in Section 3.

For $k \in \mathbf{N}$, let $G_k = K_{k+1} \cup v_2 \cup S_{k+1}$. Let's denote $V(K_{k+1}) = \{v_1, c_1, c_2, \dots, c_k\}$ and $V(S_{k+1}) = \{v_3, s_1, s_2, \dots, s_k\}$, where $d(v_3) = k$. We call a k -bal the graph on $2k + 3$ vertices obtained by adding the edges v_1v_2 and v_2v_3 in G_k . In Figure 4.2 we exhibit the 8-bal graph.

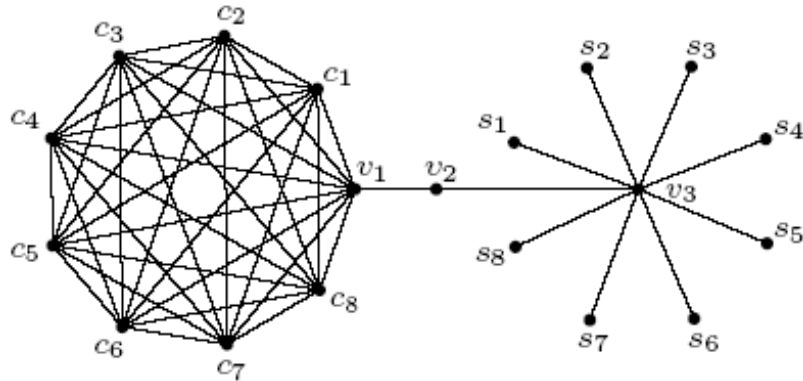


Figure 4.2: Graph 8-bal.

Theorem 4.10. *If G is a k -bal graph on $k \geq 8$, then:*

- $c_c(v_2) > c_c(v_1) = c_c(v_3) > c_c(c_1) = c_c(c_2) = \dots = c_c(c_k) > c_c(s_1) = c_c(s_2) = \dots = c_c(s_k)$
- $c_{sc}(v_3) > c_{sc}(v_2) > c_{sc}(v_1) > c_{sc}(c_1) = c_{sc}(c_2) = \dots = c_{sc}(c_k) > c_{sc}(s_1) = c_{sc}(s_2) = \dots = c_{sc}(s_k)$

Proof. For the closeness centrality, by direct calculation, it follows that $c_c(v_1) = (4k + 3)^{-1}$, $c_c(v_2) = (4k + 2)^{-1}$, $c_c(v_3) = (4k + 3)^{-1}$, $c_c(c_i) = (5k + 5)^{-1}$, $c_c(s_i) = (6k + 4)^{-1}$, $1 \leq i \leq k$ and the first part of the result is proven.

For the second part, from Proposition 3.3 it follows that $c_{sc}(c_i) = c_{sc}(c_j)$ and $c_{sc}(s_i) = c_{sc}(s_j)$ for $i, j \in \{1, \dots, k\}$. So, to completely order the vertices of G by the spectral closeness centrality, if the principal eigenvector for $D(G)$ is $\mathbf{x} = (x_{s_1}, \dots, x_{s_k}, x_{c_1}, \dots, x_{c_k}, x_{v_1}, x_{v_2}, x_{v_3})^T$, it is enough to prove that $x_{s_k} > x_{c_k} > x_{v_1} > x_{v_2} > x_{v_3}$ or, analogously, that all entries of the vector $\mathbf{y}_k = (x_{s_k} - x_{c_k}, x_{c_k} - x_{v_1}, x_{v_1} - x_{v_2}, x_{v_2} - x_{v_3}, x_{v_3})^T$ are positive.

Let M_k be the natural quotient matrix associated with $D(G)$, that is,

$$M_k = \begin{pmatrix} 2(k-1) & 4k & 3 & 2 & 1 \\ 4k & (k-1) & 1 & 2 & 3 \\ 3k & k & 0 & 1 & 2 \\ 2k & 2k & 1 & 0 & 1 \\ k & 3k & 2 & 1 & 0 \end{pmatrix}.$$

Then, $\mathbf{z}_k = (x_{s_k}, x_{c_k}, x_{v_1}, x_{v_2}, x_{v_3})^T$ is an eigenvector of M_k associated with its largest eigenvalue ∂ (see, for example, Brouwer and Haemers [3], Section 2.3). Let P be a matrix, with inverse P^{-1} , as below:

$$P = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As PM_kP^{-1} and M_k are similar, these matrices have the same eigenvalues. Moreover, \mathbf{y}_k is an eigenvector for PM_kP^{-1} associated with the eigenvalue ∂ , since $PM_kP^{-1}\mathbf{y}_k = PM_k\mathbf{z}_k = \partial P\mathbf{z}_k = \partial\mathbf{y}_k$.

Thus, \mathbf{y}_k is the eigenvector associated with $(\partial)^r, r \in \mathbf{N}$, the largest eigenvalue for $(PM_k P^{-1})^r$.

We claim that for any $k \geq 8$, all the entries of the matrix $(PM_k P^{-1})^7$ are non negatives. In this case, by Perron-Frobenius Theorem, all entries of the eigenvector \mathbf{y}_k are different to zero and must have the same sign. The result follows from the fact that $x_{v_3} > 0$, since \mathbf{x} is the principal eigenvector of $D(G)$.

Indeed, let $f: \mathbf{R} \rightarrow \mathbf{R}^{5 \times 5}$ such that

$$f(x) = \begin{pmatrix} -2x-2 & x-1 & x+1 & x+1 & x-1 \\ x & x-1 & x & x+1 & x+2 \\ x & 0 & -1 & 0 & 1 \\ x & 0 & -1 & -2 & -1 \\ x & 4x & 4x+2 & 4x+3 & 4x+3 \end{pmatrix}^7.$$

So, $f(k) = (PM_k P^{-1})^7$, for all $k \in \mathbf{N}$. Let $f_{m,n}(x), 1 \leq m, n \leq 5$, be the function in position (m, n) of $f(x)$. It is easy get computationally the Taylor Series, centered in $x_0 = 8$, for each $f_{m,n}(x)$. As $f_{m,n}(8), f'_{m,n}(8), \dots, f_{m,n}^{(7)}(8)$ are all non negatives, and $f_{m,n}^{(i)}(x) = 0, \forall i > 7$, since $f_{m,n}(x)$ are polynomials with degree no more than 7, it follows that

$$f_{m,n}(x) = \sum_{i=0}^{\infty} \frac{f_{m,n}^{(i)}(8)}{i!} (x-8)^i \geq 0, \forall x \geq 8 \text{ and } 1 \leq m, n \leq 5.$$

Thus, the vector y_k has strictly positive entries, for $k \geq 8$, and the result is proven.

□

For a k -bal graph, $2 \leq k \leq 7$, the more central vertices according closeness and spectral closeness centralities are the same. In these cases, it can be computationally determined that:

- $c_c(v_2) > c_c(v_1) = c_c(v_3) > c_c(c_1) = c_c(c_2) = \dots = c_c(c_k) > c_c(s_1) = c_c(s_2) = \dots = c_c(s_k)$
- $c_{sc}(v_2) > c_{sc}(v_3) > c_{sc}(v_1) > c_{sc}(c_1) = c_{sc}(c_2) = \dots = c_{sc}(c_k) > c_{sc}(s_1) = c_{sc}(s_2) = \dots = c_{sc}(s_k)$.

The more central vertex by the spectral closeness centrality, depending on the value of k , can be discussed from the point of view of the interpretation of this measure. For any k -bal graph, $k \geq 2$, the vertex v_3 has a small transmission value, with $Tr(v_3) = Tr(v_2) + 1$, where v_2 has the

smallest transmission. Furthermore, as k increases, more pendant vertices are connected with v_3 and more vertices are added in the complete block. Vertices in the first group have the smallest spectral closeness centrality in the graph and they are close to v_3 . The second group is formed by more central vertices, distant from v_3 .

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