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Implicative filters in quasi-ordered residuated systems

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Abstract:

The concept of residuated relational systems ordered under a quasiorder relation was introduced in 2018 by S. Bonzio and I. Chajda as a structure $\mathcal{A} = \langle A, \cdot, \rightarrow, 1, R \rangle$*, where (A, ·) is a commutative monoid with the identity 1 as the top element in this ordered monoid under a quasi-order R. The author introduced and analyzed the concepts of filters in this type of algebraic structures. In this article, as a continuation of previous author's research, the author introduced and analyzed the concept of implicative filters in quasi-ordered residuated systems.*

Keywords: Quasi-ordered residuated system; Implicative filter in quasi-ordered residuated system.

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1. Introduction

Let $(A, \cdot, 1)$ be a commutative semigroup with the identity 1. Suppose that on the carrier A there exists another operation $\rightarrow$ and one relation R that with multiplication in A have a link $(x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R$ for each $x, y, z \in A$. A relational system designed in this way, when R is a quasi-ordered relation on A, is in the focus of this paper.

The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda [2] in 2018. Previously, this concept was discussed in [1]. R. D. Maddux suggests that text [6] written by A. Tarski in 1941 is probably one of the first articles which relates to 'The calculus of relations' ([4], page 438). The approach outlined in [6] is worked out in more detail in [7]. According to R. D. Madduox already mentioned, the first definition of relation algebras appears in [3] (cited by [4], page 441). The approach outlined in [6] is worked out in more detail in [7]. In addition, according to R. D. Madduox, the first definition of relation algebras appears in [3] (cited by [4], page 441).

This paper continues the investigations of quasi-ordered residuated systems and of their filters which were started in the author article [5]. In particular, the concept of implicative filters in a quasi-ordered residuated system is introduced. Also, some conditions for a filter of such system to be an implicative filter are listed.

2. Preliminaries

2.1. Concept of quasi-ordered residuated systems

In article [2], S. Bonzio and I. Chajda introduced and analyzed the concept of 'residual relational systems'.

Definition 2.1. *([2], Definition 2.1) A residuated relational system is a structure $\mathcal{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:*

(1) $(A, \cdot, 1)$ is a commutative monoid; (2) $(\forall x \in A)((x, 1) \in R)$; (3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties for residuated relational systems are subsumed in the following

Theorem 2.1 ([2], Proposition 2.1). *Let $\mathcal{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ be a residuated relational system. Then*

(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$,
(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$,
(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$,
(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$,
(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a quasi-order relation $''$ on a set A is a binary relation which is reflexive and transitive (Some authors use the term pre-order relation).

Definition 2.2. *([2], Definition 3.1) A quasi-ordered residuated system is a residuated relational system $\mathcal{A} = \langle A, \cdot, \rightarrow, 1, \rangle$, where is a quasi-order relation in the monoid $(A, \cdot)$*

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([2], Proposition 3.1). *Let A be a quasi-ordered residuated system. Then*

(9) $(\forall x, y, z \in A)(xy \Longrightarrow (x \cdot zy \cdot z \wedge z \cdot xz \cdot y))$;
(10) $(\forall x, y, z \in A)(xy \Longrightarrow (y \rightarrow zx \rightarrow z \wedge z \rightarrow xz \rightarrow y))$;
(11) $(\forall x, y \in A)(x \cdot yx \wedge x \cdot yy)$.

Estimating that this topic is interesting ([1, 2, 5]), it is certain that there is interest in the development of the concept of some substructures and processes in these systems.

Let $L(a) = \{y \in A : ay\}$ be the left class and $R(b) = \{x \in A : xb\}$ be the right class of the relation generated by the elements a and b respectively. Then $R(1) = A$. Some authors use the notation $U(a)$ instead of $L(a)$ (see, for example [2]).

2.2. Concepts of filters

In the article [5], in order to determine the concept of filters in quasi-ordered residuated systems, the relationships between the following conditions are analyzed: (F0) $1 \in F$; (F1) $(\forall u, v \in A)((u \cdot v \in F \Longrightarrow (u \in F \wedge v \in F))$; (F2) $(\forall u, v \in A)((u \in F \wedge uv) \Longrightarrow v \in F)$; and (F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \Longrightarrow v \in F)$.

It is shown ([5], Proposition 3.2) that $(F2) \Longrightarrow (F1)$. In addition, it is shown ([5], Proposition 3.4) that for every nonempty subset of F of system $\mathcal{A}$ is valid $(F2) \Longrightarrow (F0)$. Additionally, it has been shown to be valid

Proposition 2.2 ([5], Proposition 3.6). *Let F be a submonoid of the monoid $(A, \cdot)$ in a quasi-ordered residuated system $\mathcal{A} = \langle(A, \cdot, \rightarrow, 1, \rangle$. Then $F(2) \implies F(3)$.*

Based on our previous analysis of the interrelationship between conditions (F1), F(2) and (F3) in a quasi-ordered residual system, we introduced the concept of filters by the following definition.

Definition 2.3. *([5], Definition 3.1) For a non-empty subset F of a quasi-ordered residuated system $\mathcal{A}$ we say that it is a filter of $\mathcal{A}$ if it satisfies conditions (F2) and (F3).*

3. The concept of implicative filters

In this section, we introduce the concept of implicative filters in quasi-ordered residuared systems and analyze it.

Definition 3.1. *For a non-empty subset F of a quasi-ordered residuated system $\mathcal{A}$ we say that the implicative filter in $\mathcal{A}$ if (F2) and the following condition (IF) $(\forall u, v, z \in A)((u \rightarrow (v \rightarrow z) \in F \wedge u \rightarrow v \in F) \implies u \rightarrow z \in F)$*
are valid.

It is immediately seen that $1 \in F$ and F satisfies condition (F1) because F satisfies the condition (F2) and F is a non-empty subset.

Proposition 3.1. *Let F be an implicative filter of a quasi-ordered residuated system $\mathcal{A}$. Then the following holds (F4) $(\forall u, v \in A)(u \rightarrow (u \rightarrow v) \in F \implies u \rightarrow v \in F)$.*

Proof. If we put $v = u$ in (IF), we immediately obtain the claim of this proposition, since for every $u \in A$ always $u \rightarrow u \in F$ holds for every non-empty set F satisfying condition (F2). Indeed, $u \rightarrow u \in F$ follows from uu; whence $1(u \rightarrow u)$ by $1 \in F$ and (F2). □

In what follows, we need the following lemma

Lemma 3.1. *Let a subset F of a quasi-ordered residuated system $\mathcal{A}$ satisfies the condition (F2). Then the following holds (12) $(\forall u \in A)(u \in F \iff 1 \rightarrow u \in F)$.*

Proof. Since $(\forall x \in A)(1 \to xx)$ and $(\forall x \in A)(x1 \to x)$, by Proposition 2.3 (d) in [2], the proof of this lemma follows from (F2). $\square$

Let us show that every implicative filter in $\mathcal{A}$ is a filter of $\mathcal{A}$.

Theorem 3.1. *Every implicative filter in a quasi-ordered residuated system $\mathcal{A}$ is a filter of $\mathcal{A}$.*

Proof. Let F be an implicative filter in a quasi-ordered residuated system $\mathcal{A}$. To prove that F is a filter of $\mathcal{A}$, it suffices to prove that F satisfies condition (F3). Let $u, v \in A$ be arbitrary elements such that $u \to v \in F$ and $u \in F$. Then $1 \to (u \to v) \in F$ and $1 \to u \in F$ by (12). Thus $1 \to v \in F$ by (IF). Hence $v \in F$ by (12). So, the set F is a filter of $\mathcal{A}$. $\square$

We intend to more accurately describe this class of filters in quasi-ordered residuated systems. In what follows, we need the following two lemmas

Lemma 3.2. *Let F be a subset of a quasi-ordered residuated system $\mathcal{A}$. Then the condition $(F2)$ is equivalent to the condition (F5) $(\forall u, v, z \in A)((u \cdot v \in F \wedge uv \to z) \Longrightarrow z \in F)$.*

Proof. $(F2) \Longrightarrow (F5)$. Let $u, v, z \in A$ such that $x \cdot v \in F$ and $uv \to z$. Then $u \cdot v \in F \wedge u \cdot vz$ by (3). Thus $z \in F$ by (F2). $(F5) \Longrightarrow (F2)$. Conversely, let us assume that (F4) holds. Let $u, v \in A$ such that $u \in F \wedge uv$. Then $u \cdot 1 \in F \wedge u1 \to v$. Thus $v \in F$ according (F5). So, the formula (F2) is proven. $\square$

Lemma 3.3. *Let $a \in A$ be an element of a quasi-ordered residuated system $\mathcal{A}$. Then $L(a) = \{y \in A : ay\}$ is a filter of $\mathcal{A}$ if and only if the following holds (13) $(\forall u, v \in A)((au \wedge auv) \Longrightarrow av)$.*

Proof. Each set $L(a)$ satisfies condition (F2) according to Proposition 3.1 (13) in [5]. Let $u, v \in A$ be such $u \in L(a)$ and $u \to v \in L(a)$. Then au and $au \to v$. Thus au and auv by (3). Hence av by (13), i.e. $v \in L(a)$. We have shown that $L(a)$ satisfies condition (F3). Therefore, $L(a)$ is a filter of $\mathcal{A}$. Conversely, it is obvious that condition (F3) is transformed into condition (13) in the case $F = L(a)$. $\square$

Another important result in this report is the following theorem

Theorem 3.2. *In a quasi-ordered residuated system $\mathcal{A}$, the set $L(1)$ is an implicative filter if and only if $L(a)$ is a filter of $\mathcal{A}$ for any $a \in A$.*

Proof. Each set $L(a)$ satisfies condition (F2) according to Proposition 3.1 (13) in [5].

Assume that $L(a)$ is a filter of $\mathcal{A}$ for all $a \in A$. Let $u, v, z \in A$ be such that $u \to (v \to z) \in L(1)$ and $u \to v \in L(1)$. Then $1u \to (v \to z)$ and $1u \to v$. Thus $u \cdot vz$ and uv by (3). Hence uz by (13) because $L(u)$ is a filter of $\mathcal{A}$. So, $u \to z \in L(1)$. Therefore, the set $L(1)$ is an implicative filter.

Conversely, suppose that $L(1)$ is an implicative filter of $\mathcal{A}$. Let $u \in L(a)$ and $u \to v \in L(a)$. Then $au \wedge au \to v$ and $1a \to u \wedge 1a \to (u \to v)$. This means $a \to u \in L(1) \wedge a \to (u \to v) \in L(1)$. Since $L(1)$ is an implicative filter of $\mathcal{A}$, it follows $a \to v \in L(1)$. hence $1a \to v$ and av. So, $v \in L(a)$. Hence, $L(a)$ is a filter of $\mathcal{A}$. $\square$

The following theorem gives another condition for a filter of a quasi-ordered residuates system $\mathcal{A}$ to be an implicative filter in $\mathcal{A}$.

Theorem 3.3. *Let F be a filter of a quasi-ordered sesiduated system $\mathcal{A}$. Then F is an implicative filter in $\mathcal{A}$ if and only if the set $F_a = \{x \in A : a \to x \in F\}$ is a filter of $\mathcal{A}$ for any $a \in A$.*

Proof. Note that $F_1 = F$ by Lemma 3.1. (1) Assume that F is an implicative filter in $\mathcal{A}$. Since $a1$ by (2) for every $a \in A$, we have $1a \to 1$ by (3). Then $1 \in F$ and $1a \to 1$ implies $a \to 1 \in F$ by (F2). Thus $1 \in F_a$. Let us prove that F_a satisfies condition (F2). Let $u, v \in A$ be such that $u \in F_a$ and uv. Then $a \to u \in F$ and $a \to ua \to v$ according to (10). Thus $a \to v \in F$ by (F2). Hence $v \in F_a$. Let $u, v \in A$ be such $u \in F_a$ and $u \to v \in F_a$. Then from $a \to u \in F$ and $a \to (u \to v) \in F$ it follows $a \to v \in F$ by (IF). Hence $v \in F_a$. This shows that F_a satisfies condition (F3). So, since F_a satisfies conditions (F2) and (F3), it is a filter of $\mathcal{A}$. (2) Conversely, suppose that F_a is a filter of $\mathcal{A}$ for any $a \in A$. Let $u, v, z \in A$ be elements such that $u \to (v \to z) \in F$ and $u \to v \in F$. Then $v \to z \in F_u$ and $v \in F_u$. Thus $z \in F_u$ because F_u is a filter of $\mathcal{A}$. Hence $u \to z \in F$. So, F is an implicative filter in $\mathcal{A}$. $\square$

To design another condition equivalent to a condition (IF), we need the following lemmas

Lemma 3.4. *Let a subset F of a quasi-ordered residuated system $\mathcal{A}$ satisfy the condition (F2). Then the following holds (14) $(\forall u, v, z \in A)(u \to (v \to z) \in F \iff v \to (u \to z) \in F)$.*

Proof. Let $u, v, z \in A$ be elements such that $u \to (v \to z) \in F$. From here and from the condition $(\forall u, v, z \in A)(u \to (v \to z)v \to (u \to z)$ ([2], Proposition 3.1(f)) follows $v \to (u \to z) \in F$. according to (F2). Reverse implication is obtained from previously proven if variables u and v change places. □

Lemma 3.5. *Let A be a quasi-ordered residuated system. Then (15) $(\forall x, y, z \in A)(y \to z(x \to y) \to (x \to z))$.*

Proof. In [2] ((Proposition 3.1(g)) is shown that the following

$$(\forall x, y, z \in A)((x \to y) \cdot (y \to z)(x \to z))$$

holds. Since multiplication in $\mathcal{A}$ is commutative, we have

$$(\forall x, y, z \in A)((y \to z) \cdot (x \to y)(x \to z)).$$

From here we get (15) according to (3). □

Theorem 3.4. *Let F be a non-empty subset of a quasi-ordered residuated system $\mathcal{A}$ satisfying (F2). If F satisfies the additional condition (F6) $(\forall u, v, z \in A)((u \to (v \to (v \to z)) \in F \wedge u \in F) \Longrightarrow v \to z \in F)$ then F is an implicative filter in $\mathcal{A}$.*

Proof. Let us show that the set F satisfies condition (FI). Let $u, v, z \in A$ be arbitrary elements such that $u \to (v \to z) \in F$ and $u \to v \in F$. Using (14) and (15) we have

$$v \to (u \to z) \in F \text{ and } v \to (u \to z)(u \to v) \to (u \to (u \to z)).$$

Thus

$$(u \to v) \to (u \to (u \to z)) \in F \wedge u \to v \in F.$$

Hence $u \to z \in F$ by (F6). So, F is an implicative filter. □

We conclude this report with the following theorem

Theorem 3.5. *If a non-empty subset F of a quasi-ordered residuates system $\mathcal{A}$ satisfies conditions (F2), (F3) and (F4), then F is an implicative filter in $\mathcal{A}$.*

Proof. To prove that F is an implicative filter in $\mathcal{A}$, it suffices to show that (F6) is a valid formula. Let $u, v, z \in A$ be such that $u \to (v \to (v \to z)) \in F$ and $u \in F$. Then $v \to (v \to z) \in F$ and $u \in F$ by (F3). Thus $v \to z \in F$ by (F4). □

Competing Interests.

The author declares that there is none conflict of interest regarding the publication of this manuscript.

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