Proyecciones (Antofagasta. On line) | vol. 40, n. 1 (2021) | pp. 199-215.

doi 10.22199/issn.0717-6279-2021-01-0013
PROYECCIONES
Journal of Mathematics
ISSN 0717-6279 (On line)

Some trapezoid and midpoint type inequalities for newly defined quantum integrals

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Received: April 2019 | Accepted: September 2020

Abstract:

In this paper, we first obtain prove two new identities for the quantum integrals. Then we establish Trapezoid and Midpoint type inequalities for quantum integrals defined by Bermudo et al. in [3]. The inequalities in this study generalize some results obtained in earlier works

Keywords: Hermite-Hadamard inequality; q-integral; Quantum calculus; Convex function; Trapezoid; Midpoint.

MSC (2020): 34A08, 26A51, 26D15.

Cite this article as (IEEE citation style):

H. Budak, "Some trapezoid and midpoint type inequalities for newly defined quantum integrals", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 1, pp. 199-215, 2021, doi: 10.22199/issn.0717-6279-2021-01-0013

1. Introduction

The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [4], [14, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if $f : I \to \mathbf{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied.

The general structure of this paper consist of five main sections including introduction. In Section 2, we give some necessary important notations for concept q-calculus and we also mention some related works in the literature. In section 3 and Section 4, we provide Trapezoid and Midpoint type inequalities for q^b integrals, respectively. We also examine the relation between our results and inequalities presented in the earlier works. Finally, in Section 5, some conclusions and further directions of research are discussed. We note that the opinion and technique of this work may inspire new research in this area.

2. Preliminaries of q-Calculus and Some Inequalities

Many integral inequalities well known in classical analysis such as Hölder inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss- Cebysev and other integral inequalities have been proved and applied for q-calculus using classical convexity.For the other results for q-calculus please refer to [1, 2, 7, 6, 12, 13, 15, 17].

In this section we present some required definitions and related inequalities about q-calculus. Also, here and further we use the following notation(see [9]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \quad q \in (0,1).$$

In [8], Jackson gave the q-Jackson integral from 0 to b for $0 < q < 1$ asfollows:

$$\int_0^b f(x)\ d_q x\ = (1-q)\, b \sum_{n=0}^{\infty} q^n f\left(bq^n\right) \tag{2.1}$$

provided the sum converge absolutely.
Jackson in [8] gave the q-Jackson integral in a generic interval $[a, b]$ as:

$$\int_a^b f(x)\ d_q x\ = \int_0^b f(x)\ d_q x\ - \int_0^a f(x)\ d_q x\ .$$

Definition 1. *[16] For a continuous function $f : [a, b] \to \mathbf{R}$, then q_a-derivative of f at $x \in [a, b]$ is characterized by the expression*

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)\, a)}{(1-q)(x-a)},\ x = a. \tag{2.2}$$

Since $f : [a, b] \to \mathbf{R}$ is a continuous function, thus we have

$${}_aD_q f(a) = x \to a\lim_a D_q f(x)\,.$$

The function f is said to be q- differentiable on $[a, b]$ if

$${}_aD_q f(t)$$

exists for all $x \in [a, b]$. If $a = 0$ in (2.2), then ${}_0D_q f(x) = D_q f(x)$ where

$$D_q f(x)$$

is familiar q-derivative of f at $x \in [a, b]$ defined by the expression (see [9])

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)\, x},\ x = 0.$$

Definition 2. *[3] For a continuous function $f : [a, b] \to \mathbf{R}$, then q^b-derivative of f at $x \in [a, b]$ is characterized by the expression*

$${}^bD_q f(x) = \frac{f(qx + (1-q)\, b) - f(x)}{(1-q)(b-x)},\ x = b.$$

Definition 3. *[16] Let $f : [a,b] \to \mathbf{R}$ be a continuous function. Then, the q_a-definite integral on $[a,b]$ is defined as*

$$\int_a^b f(x)_a \, d_q x = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f\left(q^n b + (1-q^n)a\right)$$

$$= (b-a)\int_0^1 f\left((1-t)a + tb\right) d_q t.$$

In [1], Alp et al. proved the following q_a-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 1. *Let $f : [a,b] \to \mathbf{R}$ be a convex differentiable function on $[a,b]$ and $0 < q < 1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$f\left(\frac{q\underline{a} + \underline{b}}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x) \; {}_a d_q x \; \leq \frac{qf(a) + f(b)}{1+q}. \tag{2.3}$$

In [11] and [1], authors established some bounds for left and right hand sides of the inequality (2.3).
On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 4. *[3] Let $f : [a,b] \to \mathbf{R}$ be a continuous function. Then, the q^b-definite integral on $[a,b]$ is defined as*

$$\int_a^b f(x)^b \, d_q x = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f\left(q^n a + (1-q^n)b\right)$$

$$= (b-a)\int_0^1 f\left(ta + (1-t)b\right) d_q t.$$

Theorem 2. *[3] Let $f : [a,b] \to \mathbf{R}$ be a convex function on $[a,b]$ and $0 < q < 1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$f\left(\frac{a+qb}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x) \; {}^b d_q x \; \leq \frac{f(a) + qf(b)}{1+q}. \tag{2.4}$$

From Theorem 1 and Theorem 2, one can the following inequalities:

Corollary 1. *[3] For any convex function $f:[a,b]\to\mathbf{R}$ and $0<q<1$, we have*

$$f\left(\frac{qa+b}{1+q}\right)+f\left(\frac{a+qb}{1+q}\right)\leq\frac{1}{b-a}\left\{\int_a^b f(x)\ {}_ad_qx\ +\int_a^b f(x)\ {}^bd_qx\ \right\}\leq f(a)+f(b)$$

(2.5)

and

$$(2.6)\ f\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)}\left\{\int_a^b f(x)_a\,d_qx+\int_a^b f(x)^b\,d_qx\right\}\leq\frac{f(a)+f(b)}{2}.$$

In this paper we will find some bounds for the left and right hand sides of the inequality (2.4).

3. New Trapezoid Type Inequalities for Quantum Integrals

In this section we will prove some new Trapezoid type inequalities for functions whose q^b-derivatives are convex.

Lemma 1. *Let $f:[a,b]\subset\mathbf{R}\to\mathbf{R}$ be a q-differentiable function on (a,b) with ${}^bD_qf$ be continuous and integrable on $[a,b]$, then*

$$\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)^b\,d_qx$$

$$(3.1)\qquad=\frac{q(b-a)}{1+q}\int_0^1(1-(1+q)t)^b\,D_qf(ta+(1-t)b)\,d_qt$$

where $0<q<1$

Proof. By the Definition 2, we have

$${}^bD_qf(ta+(1-t)b)=\frac{f(qta+(1-qt)b)-f(ta+(1-t)b)}{(1-q)(b-a)t}.$$

Then it follows that

$$\int_0^1 (1-(1+q)\,t)^b\, D_q f\,(ta+(1-t)\,b)\,d_q t$$

$$= \int_0^1 (1-(1+q)\,t)\,\frac{f\,(qta+(1-qt)\,b)-f\,(ta+(1-t)\,b)}{(1-q)\,(b-a)\,t} d_q t$$

$$= \frac{1}{b-a}\int_0^1 \frac{f\,(qta+(1-qt)\,b)-f\,(ta+(1-t)\,b)}{(1-q)\,t} d_q t$$

$$-\frac{1+q}{b-a}\int_0^1 \frac{f\,(qta+(1-qt)\,b)-f\,(ta+(1-t)\,b)}{(1-q)} d_q t.$$

(3.2)

By the equality (2.1), we have

$$\frac{1}{b-a}\int_0^1 \frac{f\,(qta+(1-qt)\,b)-f\,(ta+(1-t)\,b)}{(1-q)\,t} d_q t$$

$$= \frac{1}{b-a}\sum_{k=0}^{\infty} f\left(q^{k+1}a+\left(1-q^{k+1}\right)b\right) - \frac{1}{b-a}\sum_{k=0}^{\infty} f\left(q^k a+\left(1-q^k\right)b\right)$$

$$= \frac{1}{b-a}\left[f(b)-f(a)\right].$$

(3.3)

By the equality (2.1) and Definition 4, we get

$$\frac{1+q}{b-a}\int_0^1 \frac{f\,(qta+(1-qt)\,b)-f\,(ta+(1-t)\,b)}{(1-q)} d_q t$$

$$= \frac{1+q}{b-a}\sum_{k=0}^{\infty} q^k f\left(q^{k+1}a+\left(1-q^{k+1}\right)b\right) - \frac{1+q}{b-a}\sum_{k=0}^{\infty} q^k f\left(q^k a+\left(1-q^k\right)b\right)$$

$$= \frac{1+q}{q\,(b-a)}\sum_{k=1}^{\infty} q^k f\left(q^k a+\left(1-q^k\right)b\right) - \frac{1+q}{b-a}\sum_{k=0}^{\infty} q^k f\left(q^k a+\left(1-q^k\right)b\right)$$

$$= \frac{(1+q)(1-q)}{q(b-a)} \sum_{k=0}^{\infty} q^k f\left(q^k a + \left(1-q^k\right) b\right) - \frac{1+q}{q(b-a)} f(a)$$

$$= \frac{(1+q)}{q(b-a)^2} \int_a^b f(x)^b \, d_q x - \frac{1+q}{q(b-a)} f(a).$$

(3.4)

By substitute the equalities (3) and (3) in (3), we have

$$\int_0^1 (1-(1+q)t)^b \, D_q f(ta + (1-t)b) \, d_q t$$

$$= -\frac{(1+q)}{q(b-a)^2} \int_a^b f(x)^b \, d_q x + \frac{1+q}{q(b-a)} f(a) + \frac{1}{b-a} [f(b) - f(a)]$$

$$= \frac{(1+q)}{q(b-a)} \left[\frac{f(a) + qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x)^b \, d_q x \right]$$

which completes the proof. □

Remark 1. *If we take the limit $q \to 1^-$ in Lemma 1, then Lemma 1 reduces to [5, Lemma 2.1].*

Theorem 1. *Let $f : [a,b] \subset \mathbf{R} \to \mathbf{R}$ be a q-differentiable function on (a,b) with ${}^bD_q f$ be continuous and integrable on $[a,b]$. If $\left|{}^bD_q f\right|$ is convex on $[a,b]$ then we have the inequality*

$$\left| \frac{f(a) + qf(b)}{1+q} - \frac{1}{b-a} \int_a^b f(x) \ {}^b d_q x \right|$$

$$\leq (b-a) \left[\left|{}^bD_q f(a)\right| \frac{q^2(1+4q+q^2)}{(1+q+q^2)(1+q)^4} + \left|{}^bD_q f(b)\right| \frac{q^2(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^4} \right]$$

where $0 < q < 1$.

Proof. Taking modulus in Lemma 1 and using convexity of $\left|{}^{b}D_{q}f\right|$, we obtain

$$\left|\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_{a}^{b}f(x)\,{}^{b}d_{q}x\right|$$

$$=\frac{q(b-a)}{1+q}\left|\int_{0}^{1}\left(1-(1+q)t\right)^{b}D_{q}f\left(ta+(1-t)b\right)d_{q}t\right|$$

$$\leq\frac{q(b-a)}{1+q}\int_{0}^{1}\left|\left(1-(1+q)t\right)\right|\left|{}^{b}D_{q}f\left(ta+(1-t)b\right)\right|d_{q}t$$

$$\leq\frac{q(b-a)}{1+q}\int_{0}^{1}\left|\left(1-(1+q)t\right)\right|\left[t\left|{}^{b}D_{q}f(a)\right|+(1-t)\left|{}^{b}D_{q}f(b)\right|\right]d_{q}t$$

$$=\frac{q(b-a)}{1+q}\left[\left|{}^{b}D_{q}f(a)\right|\int_{0}^{1}\left|\left(1-(1+q)t\right)\right|td_{q}t+\left|{}^{b}D_{q}f(b)\right|\int_{0}^{1}\left|\left(1-(1+q)t\right)\right|(1-t)\,d_{q}t\right]$$

$$=\frac{q(b-a)}{1+q}\left[\left|{}^{b}D_{q}f(a)\right|\frac{q\left(1+4q+q^{2}\right)}{\left(1+q+q^{2}\right)(1+q)^{3}}+\left|{}^{b}D_{q}f(b)\right|\frac{q\left(1+3q^{2}+2q^{3}\right)}{\left(1+q+q^{2}\right)(1+q)^{3}}\right]$$

which completes the proof. □

Remark 2. *If we take the limit $q\to1^{-}$ in Theorem 1, then Theorem 1 reduces to [5, Theorem 2.2].*

Theorem 2. *Let $f:[a,b]\subset\mathbf{R}\to\mathbf{R}$ be a q-differentiable function on (a,b) with ${}^{b}D_{q}f$ be continuous and integrable on $[a,b]$. If $\left|{}^{b}D_{q}f\right|^{p_1}$, $p_1\geq1$, is convex on $[a,b]$ then we have the inequality*

$$\left|\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_{a}^{b}f(x)\ {}^{b}d_{q}x\ \right|$$

$$\leq\ \frac{q(b-a)}{1+q}\left(\frac{q\left(2+q+q^{3}\right)}{(1+q)^{3}}\right)^{1-\frac{1}{p_1}}$$

$$\times\left(\frac{q\left(1+4q+q^{2}\right)}{\left(1+q+q^{2}\right)(1+q)^{3}}\left|\ {}^{b}D_{q}\ f(a)\right|^{p_1}+\frac{q^{2}\left(1+3q^{2}+2q^{3}\right)}{\left(1+q+q^{2}\right)(1+q)^{3}}\left|\ {}^{b}D_{q}\ f(b)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

where $0<q<1$.

Proof. Taking modulus in Lemma 1 and using the power mean inequality, we have

$$\left|\frac{f(a)+qf(b)}{1+q}-\frac{1}{b-a}\int_a^b f(x)^b\,d_qx\right|$$

$$\leq \frac{q(b-a)}{1+q}\int_0^1 |(1-(1+q)t)|\left|{}^bD_qf(ta+(1-t)b)\right|d_qt$$

$$\leq \frac{q(b-a)}{1+q}\left(\int_0^1 |(1-(1+q)t)|\,d_qt\right)^{1-\frac{1}{p_1}}$$

$$\left(\int_0^1 |(1-(1+q)t)|\left|{}^bD_qf(ta+(1-t)b)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}.$$

Since $\left|{}^bD_qf\right|^{p_1}$ is convex, we have

$$\int_0^1 |(1-(1+q)t)|\left|\ {}^bD_q\ f(ta+(1-t)b)\right|^{p_1}d_qt$$

$$\leq \int_0^1 |(1-(1+q)t)|\left[t\left|\ {}^bD_q\ f(a)\right|^{p_1}+(1-t)\left|\ {}^bD_q\ f(b)\right|^{p_1}\right]d_qt$$

$$= \frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\left|\ {}^bD_q\ f(a)\right|^{p_1}+\frac{q^2(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}\left|\ {}^bD_q\ f(b)\right|^{p_1}.$$

We also have

$$\int_0^1 |(1-(1+q)t)|\,d_qt=\frac{q(2+q+q^3)}{(1+q)^3}.$$

This completes the proof. □

4. New Midpoint Type Inequalities for Quantum Integrals

In this section we will prove some new Midpoint inequalities for functions whose q^b-derivatives are convex.

Lemma 2. *Let $f : [a,b] \subset \mathbf{R} \to \mathbf{R}$ be a q-differentiable function on (a,b) with ${}^{b}D_q f$ be continuous and integrable on $[a,b]$, then*

$$q(b-a)\left[\int_0^{\frac{1}{1+q}} t^b D_q f(ta+(1-t)b)\,d_qt + \int_{\frac{1}{1+q}}^1 \left(t-\frac{1}{q}\right)^b D_q f(ta+(1-t)b)\,d_qt\right]$$

$$= \frac{1}{b-a}\int_a^b f(x)^b\,d_qx - f\left(\frac{a+qb}{1+q}\right)$$

where $0<q<1$.

Proof. We have

$$q(b-a)\left[\int_0^{\frac{1}{1+q}} t^b D_q f(ta+(1-t)b)\,d_qt + \int_{\frac{1}{1+q}}^1 \left(t-\frac{1}{q}\right)^b D_q f(ta+(1-t)b)\,d_qt\right]$$

$$= q(b-a)\left[\int_0^1 t^b D_q f(ta+(1-t)b)\,d_qt - \frac{1}{q}\int_{\frac{1}{1+q}}^1 {}^{b}D_q f(ta+(1-t)b)\,d_qt\right]$$

$$= q(b-a)\left[\int_0^1 t^b D_q f(ta+(1-t)b)\,d_qt - \frac{1}{q}\int_0^1 {}^{b}D_q f(ta+(1-t)b)\,d_qt\right.$$

$$\left. +\frac{1}{q}\int_0^{\frac{1}{1+q}} {}^{b}D_q f(ta+(1-t)b)\,d_qt\right].$$

(4.1)

By the equality (2.1), we have

$$\int_0^1 t^b D_q f(ta+(1-t)b)\,d_qt = \int_0^1 \frac{f(qta+(1-qt)b) - f(ta+(1-t)b)}{(1-q)(b-a)}d_qt$$

$$= \frac{1}{b-a}\sum_{k=0}^{\infty} q^k f\left(q^{k+1}a + \left(1-q^{k+1}\right)b\right) - \frac{1}{b-a}\sum_{k=0}^{\infty} q^k f\left(q^k a + \left(1-q^k\right)b\right)$$

$$= \frac{1}{q(b-a)} \sum_{k=1}^{\infty} q^k f\left(q^k a + \left(1-q^k\right) b\right) - \frac{1}{b-a} \sum_{k=0}^{\infty} q^k f\left(q^k a + \left(1-q^k\right) b\right)$$

$$= \frac{1}{b-a}\left(\frac{1}{q} - 1\right) \sum_{k=1}^{\infty} q^k f\left(q^k a + \left(1-q^k\right) b\right) - \frac{f(a)}{q(b-a)}$$

$$= \frac{1}{q(b-a)} \int_0^1 f\left(ta + (1-t)b\right) d_q t - \frac{f(a)}{q(b-a)}.$$

(4.2)

Similarly we get

$$\frac{1}{q} \int_0^1 {}^b D_q f\left(ta + (1-t)b\right) d_q t = \frac{1}{q} \int_0^1 \frac{f\left(qta + (1-qt)b\right) - f\left(ta + (1-t)b\right)}{(1-q)(b-a)t} d_q t$$

$$= \frac{1}{q(b-a)} \sum_{k=0}^{\infty} f\left(q^{k+1} a + \left(1-q^{k+1}\right) b\right) - \frac{1}{q(b-a)} \sum_{k=0}^{\infty} f\left(q^k a + \left(1-q^k\right) b\right)$$

$$= \frac{1}{q(b-a)} \sum_{k=0}^{\infty} \left[f\left(q^{k+1} a + \left(1-q^{k+1}\right) b\right) - f\left(q^k a + \left(1-q^k\right) b\right)\right]$$

$$= \frac{1}{q(b-a)} \left[f(b) - f(a)\right]$$

(4.3)

and

$$\frac{1}{q} \int_0^{\frac{1}{1+q}} {}^b D_q f\left(ta + (1-t)b\right) d_q t = \frac{1}{q} \int_0^{\frac{1}{1+q}} \frac{f\left(qta + (1-qt)b\right) - f\left(ta + (1-t)b\right)}{(1-q)(b-a)t} d_q t$$

$$= \frac{1}{q(b-a)} \sum_{k=0}^{\infty} f\left(\frac{q^{k+1}}{1+q} a + \left(1 - \frac{q^{k+1}}{1+q}\right) b\right) - \frac{1}{q(b-a)} \sum_{k=0}^{\infty} f\left(\frac{q^k}{1+q} a + \left(1 - \frac{q^k}{1+q}\right) b\right)$$

$$= \frac{1}{q(b-a)} \sum_{k=0}^{\infty} \left[f\left(\frac{q^{k+1}}{1+q} a + \left(1 - \frac{q^{k+1}}{1+q}\right) b\right) - f\left(\frac{q^k}{1+q} a + \left(1 - \frac{q^k}{1+q}\right) b\right)\right]$$

$$= \frac{1}{q(b-a)} \left[-f\left(\frac{a+qb}{1+q}\right) - f(b)\right].$$

(4.4)

If we substitute the equalities (4)-(**??**) in (4), we get

$$q(b-a)\left[\int_0^{\frac{1}{1+q}} t^b D_q f(ta+(1-t)b)\,d_qt+\int_{\frac{1}{1+q}}^1 \left(t-\frac{1}{q}\right)^b D_q f(ta+(1-t)b)\,d_qt\right]$$

$$=\frac{1}{q(b-a)}\int_0^1 f(ta+(1-t)b)\,d_qt-f\left(\frac{a+qb}{1+q}\right)$$

$$=\frac{1}{b-a}\int_a^b f(x)^b\,d_qx-f\left(\frac{a+qb}{1+q}\right)$$

which completes the proof. □

Remark 3. *If we take the limit $q\to 1^-$ in Lemma 2, then Lemma 2 reduces to [10, Lemma 2.1].*

Theorem 1. *Let $f:[a,b]\subset\mathbf{R}\to\mathbf{R}$ be a q-differentiable function on (a,b) with ${}^bD_qf$ be continuous and integrable on $[a,b]$. If $\left|{}^bD_qf\right|$ is convex on $[a,b]$ then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)^b\,d_qx-f\left(\frac{a+qb}{1+q}\right)\right|$$

$$\leq q(b-a)\left[\left|{}^bD_qf(a)\right|\frac{3}{(1+q)^3(1+q+q^2)}+\left|{}^bD_qf(b)\right|\frac{-1+2q+2q^2}{(1+q)^3(1+q+q^2)}\right]$$

where $0<q<1$.

Proof. Taking modulus in Lemma 2 and using convexity of $\left|{}^bD_qf\right|$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)^b\,d_qx-f\left(\frac{a+qb}{1+q}\right)\right|$$

$$\leq q(b-a)\left[\left|\int_0^{\frac{1}{1+q}} t^bD_qf(ta+(1-t)b)\,d_qt\right|+\left|\int_{\frac{1}{1+q}}^1\left(t-\frac{1}{q}\right)^bD_qf(ta+(1-t)b)\,d_qt\right|\right]$$

$$\leq q(b-a)\left[\int_0^{\frac{1}{1+q}} t\left|{}_b^{}D_q f(ta+(1-t)b)\right| d_qt + \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)\left|{}_b^{}D_q f(ta+(1-t)b)\right| d_qt\right]$$

$$\leq q(b-a)\left[\int_0^{\frac{1}{1+q}} t\left[t\left|{}_b^{}D_q f(a)\right|+(1-t)\left|{}_b^{}D_q f(b)\right|\right] d_qt\right.$$

$$\left. + \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)\left[t\left|{}_b^{}D_q f(a)\right|+(1-t)\left|{}_b^{}D_q f(b)\right|\right] d_qt\right]$$

$$= q(b-a)\left[\left|{}_b^{}D_q f(a)\right|\left(\int_0^{\frac{1}{1+q}} t^2 d_qt + \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right) t\, d_qt\right)\right.$$

$$\left. + \left|{}_b^{}D_q f(b)\right|\left(\int_0^{\frac{1}{1+q}} t(1-t)\, d_qt + \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)(1-t)\, d_qt\right)\right].$$

It can be easily that

$$\int_0^{\frac{1}{1+q}} t^2 d_qt = \frac{1}{(1+q)^3(1+q+q^2)},$$

$$\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right) t = \frac{2}{(1+q)^3(1+q+q^2)},$$

$$\int_0^{\frac{1}{1+q}} t(1-t)\, d_qt = \frac{q}{(1+q)^2(1+q+q^2)}$$

and

$$\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)(1-t)\, d_qt = \frac{-1+q+q^2}{(1+q)^3(1+q+q^2)}.$$

By these equalities the proof is completed. □

Remark 4. *If we take the limit $q \to 1^-$ in Theorem 1, then Theorem 1 reduces to [10, Theorem 2.2].*

Theorem 2. *Let $f : [a,b] \subset \mathbf{R} \to \mathbf{R}$ be a q-differentiable function on (a,b) with ${}_bD_q f$ be continuous and integrable on $[a,b]$. If $\left|{}_bD_q f\right|^{p_1}$, $p_1 \geq 1$, is convex on $[a,b]$ then we have the inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\ {}_bd_qx\ - f\left(\frac{a+qb}{1+q}\right)\right| \\
\leq\ & q(b-a)\left(\frac{1}{(1+q)^3}\right)^{1-\frac{1}{p_1}} \\
&\times\left[\left(\frac{1}{(1+q)^3(1+q+q^2)}\left|{}_bD_q f(a)\right|^{p_1}+\frac{q}{(1+q)^2(1+q+q^2)}\left|{}_bD_q f(b)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&\left.\left(\frac{2}{(1+q)^3(1+q+q^2)}\left|{}_bD_q f(a)\right|^{p_1}+\frac{-1+q+q^2}{(1+q)^3(1+q+q^2)}\left|{}_bD_q f(b)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where $0<q<1$.

Proof. Taking modulus in Lemma 2 and using the power mean inequality, we obtain

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\ {}_bd_qx\ - f\left(\frac{a+qb}{1+q}\right)\right| \\
\leq\ & q(b-a)\left[\left(\int_0^{\frac{1}{1+q}} t d_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^{\frac{1}{1+q}} t\left|\ {}_bD_q\ f(ta+(1-t)b)\right|^{p_1} d_qt\right)^{\frac{1}{p_1}}\right. \\
&\left.\left(\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right) d_qt\right)^{1-\frac{1}{p_1}}\left(\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)\left|\ {}_bD_q\ f(ta+(1-t)b)\right|^{p_1} d_qt\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Since $\left|{}_bD_q f\right|^{p_1}$ is convex, we have

$$\int_0^{\frac{1}{1+q}} t \left| {}^bD_q \ f\left(ta+(1-t)\,b\right)\right|^{p_1} d_qt$$

$$\leq \int_0^{\frac{1}{1+q}} t\left[t\left|{}^bD_qf\left(a\right)\right|^{p_1}+\left(1-t\right)\left|{}^bD_qf\left(b\right)\right|^{p_1}\right] d_qt$$

$$= \frac{1}{\left(1+q\right)^3\left(1+q+q^2\right)}\left|{}^bD_qf\left(a\right)\right|^{p_1}+\frac{q}{\left(1+q\right)^2\left(1+q+q^2\right)}\left|{}^bD_qf\left(b\right)\right|^{p_1}$$

and similarly

$$\int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right)\left| {}^bD_q \ f\left(ta+(1-t)\,b\right)\right|^{p_1} d_qt$$

$$\leq \frac{2}{\left(1+q\right)^3\left(1+q+q^2\right)}\left|{}^bD_qf\left(a\right)\right|^{p_1}+\frac{-1+q+q^2}{\left(1+q\right)^3\left(1+q+q^2\right)}\left|{}^bD_qf\left(b\right)\right|^{p_1}.$$

On the other hand we can easily see that $\int_0^{\frac{1}{1+q}} td_qt = \frac{1}{(1+q)^3} = \int_{\frac{1}{1+q}}^1 \left(\frac{1}{q}-t\right) d_qt$. This completes the proof. □

5. Conclusions

In this paper, we establish some Trapezoid and Midpoint type inequalities for q^b-integrals. In order to validate that their generalized behavior, we show the relation of our results with previously published ones. In the future works, authors can obtain similar inequalities by using the different kind convexities.

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