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New approach to Somos's Dedekind eta-function identities of level 6

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Abstract:

In the present work, we prove few new Dedekind eta-function identities of level 6 discovered by Somos in two different methods. Also during this process, we give an alternate method to Somos's Dedekind eta-function identities of level 6 proved by B. R. Srivatsa Kumar et al. As an application of this, we establish colored partition identities.

Keywords: Modular equations; Dedekind eta-functions; Colored partitions.

MSC (2020): 11F20, 33C05, 11P83.

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1. Introduction

Throughout the paper, we use the standard q-series notation and $f(-q^k)$ is defined as

$$f(-q^k) := (q^k; q^k)_\infty = \prod_{n=1}^{\infty}(1 - q^{nk}) \qquad k \in \mathbf{Z}, |q| < 1.$$

Ramanujan's modular equation involve quotients of the function $f(-q^k)$ at certain arguments. For example [7, p. 204, Entry 51] and [11], let

$$P := \frac{f^2(-q)}{q^{1/12} f^2(-q^3)} \qquad \text{and} \qquad Q := \frac{f^2(-q^2)}{q^{1/6} f^2(-q^6)}$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{1.1}$$

After the publication of [7] many authors including C. Adiga et. al.[1, 2, 3], N. D. Baruah[4, 5], M. S. M. Naika[10], K. R. Vasuki et. al.[19, 20], N. Saikia [13], J. Yi [22], have found additional modular equations of the type (1.1) to evaluate various continued fractions, Weber class invariants, two parametric evaluation of theta function and many more.

In the spirit of Ramanujan, M. Somos [14] used computer to discover around 6277 new elegant Dedekind eta-function identities of various levels without offering the proof. He has a large list of eta-product identities and he runs PARI/GP scripts and it works as a sophisticated programmable calculator. In 2010, Z. Cao [8] has given the proof of Somos's dissection identities. B. Yuttanan [23], K. R. Vasuki and R. G. Veeresha [21], B. R. Srivatsa Kumar and Veeresha [15], Srivatsa Kumar and D. Anu Radha [16], Srivatsa Kumar et. al. [17, 18] have obtained the proofs for the levels 4, 6, 8, 10, 12 and 14. The purpose of this paper is to prove Somos's Dedekind eta-function identities of level 6 by employing Ramanujan's modular equations of degree 3 in two different methods. Also, we provide an alternate method to Somos's Dedekind eta-function identities of level 6 proved by Srivatsa Kumar et. al. [17]. Some Somos's identities are simple consequences of others and we do record only few of them in Section 3. Moreover, our proofs are similar to each other and so in the interest of brevity and avoiding the repetition of the same ideas, we provide proofs of few identities.

Furthermore we establish combinatorial interpretations of some of our theorems.

2. Preliminaries

We use the standard q-series notation as

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq)^n$$

and

$$(a_1,a_2,\dots a_n;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\dots(a_n;q)_\infty, \qquad |q|<1.$$

The Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab|<1.$$

In Ramanujan's notation Jacobi's triple product identity [6, p. 36] is stated as

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab|<1.$$

The two important special cases of $f(a,b)$ are

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty} q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty$$

and

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty.$$

Also after Ramanujan, define

$$\chi(q)=(-q;q^2)_\infty.$$

Note that, if $q=e^{2i\pi\tau}$ then $f(-q)=e^{-i\pi\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the Dedekind eta-function is defined as

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad Im(\tau)>0,$$

here τ is any complex number. A theta function identity which associates $f(-q), f(-q^2), f(-q^n)$ and $f(-q^{2n})$ is called theta function identity of level $2n$. After expressing Dedekind eta-function identities, which we are proving in Section 3, interms of $f(-q^n)$, we obtain the arguments in $f(-q), f(-q^2), f(-q^3)$ and $f(-q^6)$, namely $-q, -q^2, -q^3$ and $-q^6$ all have exponents dividing 6, which is thus equal to the 'level' of the identity 6. For convenience all through the paper, we write $f(-q^n) = f_n$.

Before concluding this section, we now define a modular equation as defined by Ramanujan. A modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)},$$

where

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \qquad |z| < 1,$$

denotes an ordinary hypergeometric function with

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Then, we say that β is of n^{th} degree over α. Also, we define the multiplier m by

$$m := \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}.$$

3. Main results: Somos's identities of level 6

Theorem 3.1. *We have*

$$\frac{\chi^9(-q^3)}{q\chi^3(-q)} - 7 - \frac{8q\chi^3(-q)}{\chi^9(-q^3)} = \frac{f_1^4 f_2^4}{q f_3^4 f_6^4}.$$

Proof. On page 230 of his notebook [6, pp. 230-238, Entry 5] and [12], the following modular equation of degree 3, recorded by Ramanujan is as follows. If

$$P := [16\alpha\beta(1-\alpha)(1-\beta)]^{1/8} \qquad \text{and} \qquad Q := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$$

then

$$Q+\frac{1}{Q}+2\sqrt{2}\left(P+\frac{1}{P}\right)=0, \tag{3.1}$$

here β has degree three over α. Suppose that if

$$y=\pi_2F_1(1-x)/_2F_1(x) \qquad \text{and} \qquad z=_2F_1(x)$$

then we have from [6, pp. 122-124, Entry 10(i) and Entry 12(v)],

$$\varphi(e^{-y}):=\sqrt{z} \tag{3.2}$$

and

$$\chi(e^{-y}):=2^{1/6}\left(x(1-x)e^{-y}\right)^{-1/24}. \tag{3.3}$$

On transforming (3.1) using (3.3), we obtain

$$\frac{x^6}{y^6}+\frac{y^6}{x^6}=x^3y^3-\frac{8}{x^3y^3}, \tag{3.4}$$

where $x:=x(q)=q^{-1/24}\chi(q)$ and $y:=y(q)=q^{-1/8}\chi(q^3)$. On multiplying (3.4) throughout by $x^{-12}y^{-9}(8x^{12}+x^9y^9+16x^3y^3-16y^{12})$, we obtain

$$\frac{8x^3}{y^9}-\frac{16y^3}{x^9}-\frac{16y^{15}}{x^{21}}+\frac{80}{x^6y^6}-\frac{112y^6}{x^{18}}+\frac{128}{x^{15}y^3}-7+\frac{17y^{12}}{x^{12}}-\frac{y^9}{x^3}=0,$$

which is equivalent to

$$\left(\frac{8x^3}{y^9}-\frac{y^9}{x^3}-7\right)\left(\frac{4y^3}{x^9}-1\right)^2+\frac{9x^{12}}{y^{12}}\left(1-\frac{4x^3}{y^9}\right)^2=0. \tag{3.5}$$

Also from [6, pp. 230-238, Entry 5], we have

$$m=\frac{1-2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1-2(\alpha\beta)^{1/4}} \qquad \text{and} \qquad \frac{3}{m}=\frac{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}-1}{1-2(\alpha\beta)^{1/4}}$$

which implies

$$\frac{m^2}{3}=\frac{1-2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}-1}.$$

On transforming the above identity into theta functions from (3.2) and (3.3), we deduce

$$\frac{\varphi^4(q)}{3\varphi^4(q^3)} = \frac{1 - 4\frac{x^3}{y^9}}{\frac{4y^3}{x^9} - 1}. \tag{3.6}$$

Employing (3.6) in (3.5), we find that

$$\frac{8x^3}{y^9} - \frac{y^9}{x^3} - 7 + \frac{y^{12}}{x^{12}} \frac{\varphi^8(q)}{\varphi^8(q^3)} = 0. \tag{3.7}$$

By using q-identities, one can easily deduce that

$$\varphi(q) = \frac{f_2^5}{f_2^2 f_4^2}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}. \tag{3.8}$$

From (3.8), we observe that

$$\frac{\varphi(q)}{\varphi(q^3)} = \frac{\chi^2(q)}{\chi^2(q^3)} \frac{f_2}{f_6}. \tag{3.9}$$

Using (3.9) in (3.7), we obtain

$$\frac{8x^3}{y^9} - \frac{y^9}{x^3} - 7 + \frac{q^{-4/3}x^4}{y^4} \left(\frac{f_2}{f_6}\right)^8 = 0.$$

Now, replacing $q \to -q$ in the above and simplifying using (3.8), we deduce the result. □

Proof. [Alternate Proof]
Using (3.8) in Theorem 3.1 and then multiplying throughout by $qf_3^4 f_6^4 / f_1^4 f_2^4$, we obtain

$$\frac{f_3^{13}}{f_1^7 f_2 f_6^5} - \frac{8q f_6^{13}}{f_1 f_2^7 f_3^5} - \frac{7q f_3^4 f_6^4}{f_1^4 f_2^4} - 1 = 0. \tag{3.10}$$

From [19, Theorem 3.4(i)], if

$$A := \frac{f_1}{q^{1/24} f_2} \qquad \text{and} \qquad B := \frac{f_3}{q^{1/8} f_6}$$

then we have

$$(AB)^3 + \frac{8}{(AB)^3} = \left(\frac{B}{A}\right)^6 - \left(\frac{A}{B}\right)^6. \tag{3.11}$$

Employing P, Q, A and B as defined as in (1.1) and (3.11), (3.10) reduces to

$$Q^6(AB)^6 - ((PQ)^2+7)(PQ)^3(AB)^3 - 8P^6 = 0$$

equivalently

$$(AB)^3 = \frac{(PQ)^2+7 \pm \sqrt{((PQ)^2+7)^2(PQ)^6+32(PQ)^6}}{2Q^6}.$$

Since $AB > 0$, we choose

$$(AB)^3 = \frac{(PQ)^2+7 + \sqrt{((PQ)^2+7)^2(PQ)^6+32(PQ)^6}}{2Q^6}. \tag{3.12}$$

Using (3.12) in (3.11) and then factorizing, we obtain

$$L(P,Q)M(P,Q) = 0,$$

where

$$L(P,Q) = P^6 - 9P^2Q^2 - P^4Q^4 + Q^6$$

and

$$M(P,Q) = P^8Q^8(P^6 + 9P^2Q^2 + P^4Q^4 + Q^6).$$

But $L(P,Q)$ is nothing but (1.1) and it verifies Theorem 3.1. $\square$

Theorem 3.2. *We have*

$$f_1^9 f_3 f_6^3 + 9q f_1^4 f_2 f_6^8 - f_2^9 f_3^4 = 0.$$

Proof. On dividing Theorem 3.2 throughout by $f_1^9 f_3 f_6^3$, we obtain

$$1 + 9q\frac{f_2 f_6^5}{f_3 f_1^5} - \frac{f_2^9 f_3^3}{f_1^9 f_6^3} = 0. \tag{3.13}$$

Employing P, Q, A and B as defined as in (1.1) and (3.11), (3.13) reduces to

$$(AB)^3 = \left(\frac{Q}{P}\right)^3 - \frac{9}{PQ}. \tag{3.14}$$

Using (3.14) in (3.11) then factorizing, we obtain (1.1) and it verifies Theorem 3.2. $\square$

Theorem 3.3. *We have* $f_1^5 f_2^5 f_3 f_6 + 8q f_2^6 f_6^6 - f_1^6 f_3^6 - 9q f_1 f_2 f_3^5 f_6^5 = 0.$

Proof. On dividing Theorem 3.3 throughout by $f_1^5 f_2^5 f_3 f_6$, we obtain

$$1+\frac{8qf_2f_6^5}{f_1^5f_3}-\frac{f_1f_3^5}{f_2^5f_6}-9q\frac{f_3^4f_6^4}{f_1^4f_2^4}=0. \tag{3.15}$$

Employing P, Q, A and B as defined as in (1.1) and (3.11), (3.15) reduces to

$$(AB)^6+\left(\frac{9}{PQ}-PQ\right)(AB)^3-8=0$$

equivalently

$$(AB)^3=\frac{(PQ)^2-9\pm\sqrt{(PQ)^4+14(PQ)^2+81}}{2PQ}.$$

Since $AB>0$, we choose

$$(AB)^3=\frac{(PQ)^2-9+\sqrt{(PQ)^4+14(PQ)^2+81}}{2PQ}. \tag{3.16}$$

Using (3.16) in (3.11) and then factorizing, we obtain

$$L(P,Q)M(P,Q)=0,$$

where

$$L(P,Q)=P^6-9P^2Q^2-P^4Q^4+Q^6$$

and

$$M(P,Q)=-32P^2Q^2(P^6+9P^2Q^2+P^4Q^4+Q^6).$$

But *L(P,Q)* is nothing but (1.1) and it verifies Theorem 3.3.

□

Remark: Similarly, we can prove the remaining Dedekind eta-function identities of level 6 due to Somos [14]. Some of them are listed below:

$f_1f_2^3f_3^9+qf_1^4f_6^9-f_2^8f_3^4f_6=0,$

$f_1^{17}f_3^2f_6^7+17f_1^{12}f_2f_3f_6^{12}+72f_1^7f_2^2f_6^{17}-f_2^{13}f_3^{13}=0,$

$f_1^{12}f_2^3f_3^{12}f_6^3+8qf_1^6f_2^9f_3^6f_6^9-f_1^3f_2^{12}f_3^{15}+qf_1^{15}f_3^3f_6^{12}=0,$

and many more identities listed in [14, 17].

4. Applications to Colored Partitions

The identities proved in Section 3 have applications in colored partitions. In this Section, we present partition interpretation for Theorem 3.1. This concept was first introduced by S. S. Huang [9]. Now we define colored partition as given in the literature.
"A positive integer n has l colors if there are l copies of n available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are colored partitions".
For instance if 3 colors are assigned to 1, then the possible colored partitions of 2 are $1_b + 1_b$, $1_y + 1_y$, $1_i + 1_i$, $1_b + 1_y$, $1_b + 1_i$, $1_i + 1_y$ and 2, where we represent the indices b (blue), y (yellow) and i (indigo) to recognize three colors of 1. For our convenience, we use the notation,

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty, \qquad (r < s); r, s \in \mathbf{N}. \tag{4.1}$$

As an example of this, $(q^{3\pm}; q^8)_\infty$ means $(q^3, q^5; q^8)_\infty$, which is $(q^3; q^8)_\infty$ $(q^5; q^8)_\infty$. Also the number of partitions of n where all the parts are congruent to l modulo m with n colors is given by the generating function as

$$\frac{1}{(q^l; q^m)^n_\infty}.$$

Theorem 4.1. *Let $\alpha_1(n)$ represent the partitions of n being divided into parts congruent to ± 1 modulo 6 with 7 colors and ± 2 modulo 6 with 8 colors. Let $\alpha_2(n)$ is chosen to represent the partitions of n into parts congruent to ± 1 modulo 6 with 4 colors, ± 2 modulo 6 with 8 colors and $+3$ modulo 6 with 6 colors. Let $\alpha_3(n)$ indicates the partitions of n being split into parts congruent to ± 1 modulo 6 with 1 colors, ± 2 modulo 6 with 8 colors and $+3$ modulo 6 with 12 colors. Let $\alpha_4(n)$ is taken to represent the partitions of n into several parts congruent to $+3$ modulo 6 with 6 colors. Then we have*

$$\alpha_1(n) - 7\alpha_2(n-1) - 8\alpha_3(n-2) - \alpha_4(n) = 0, \qquad n \geq 2.$$

Proof. On using (3.8) in Theorem 3.1, dividing throughout by $f_1^4 f_2^4 f_3^9 f_6^9$ and then simplifying subject to the common base q^6, we deduce

$$\frac{1}{(q^1_7, q^2_8, q^4_8, q^5_7; q^6)_\infty} - \frac{7q}{(q^1_4, q^2_8, q^3_6, q^4_8, q^5_4; q^6)_\infty} - \frac{8q^2}{(q^1_1, q^2_8, q^{12}_3, q^4_8, q^5_1; q^6)_\infty} - \frac{1}{(q^3_6; q^6)_\infty} = 0.$$

On using (4.1) in the above, we have

$$\frac{1}{(q_7^{1\pm}, q_8^{2\pm}; q^6)_\infty} - \frac{7q}{(q_4^{1\pm}, q_8^{2\pm}, q_6^{3+}; q^6)_\infty} - \frac{8q^2}{(q_1^{1\pm}, q_8^{2\pm}, q_{12}^{3+}; q^6)_\infty} - \frac{1}{(q_6^{3+}; q^6)_\infty} = 0. \tag{4.2}$$

We observe that (4.2) gives the generating functions for $\alpha_1(n), \alpha_2(n), \alpha_3(n)$ and $\alpha_4(n)$ respectively. Hence the above identity is equivalent to

$$\sum_{n=0}^{\infty} \alpha_1(n) q^n - 7 \sum_{n=0}^{\infty} \alpha_2(n) q^{n+1} - 8 \sum_{n=0}^{\infty} \alpha_3(n) q^{n+2} - \sum_{n=0}^{\infty} \alpha_4(n) q^n = 0$$

and we set $\alpha_1(0) = \alpha_2(0) = \alpha_3(0) = \alpha_4(0) = 1$. Now on extracting terms of q^n in the above, we obtain the required result. □

Example: For $n = 2$, the partition representations are given by the following table.

$\alpha_1(2) = 36:$	$1_r + 1_r, 1_w + 1_w, 1_b + 1_b, 1_y + 1_y, 1_o + 1_0, 1_g + 1_g, 1_m + 1_m, 1_r + 1_w, 1_r + 1_b, 1_r + 1_y, 1_r + 1_o, 1_r + 1_m, 1_r + 1_g, 1_w + 1_b, 1_w + 1_y, 1_w + 1_0, 1_w + 1_m, 1_w + 1_g, 1_b + 1_y, 1_b + 1_o, 1_b + 1_m, 1_b + 1_g, 1_y + 1_o, 1_y + 1_m, 1_y + 1_g, 1_o + 1_m, 1_o + 1_g, 1_m + 1_g, 2_r, 2_w, 2_b, 2_y, 2_o, 2_m, 2_g, 2_v.$
$\alpha_2(1) = 4:$	$1_r, 1_w, 1_b, 1_g.$
$\alpha_3(0) = 1:$	
$\alpha_4(2) = 0:$	

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