Restricted triangular difference mean graphs

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Abstract:

Let $G = (V, E)$ be a graph with p vertices and q edges. Consider an injection $f : V(G) \rightarrow \{1, 2, 3, \dots, pq\}$. Define $f^* : E(G) \rightarrow \{T_1, T_2, T_3, \dots, T_q\}$, where T_q is the q^{th} triangular number such that $f^*(e) = \left\lceil \frac{|f(u) - f(v)|}{2} \right\rceil$ for all edges $e = uv$. If $f^*(E(G))$ is a sequence of consecutive triangular numbers $T_1, T_2, T_3, \dots, T_q$, then the function f is said to be restricted triangular difference mean. A graph that admits restricted triangular difference mean labeling is called restricted triangular difference mean graph. In this paper, we investigate restricted triangular difference mean behaviour of some standard graphs.

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1. Introduction

By a graph, we mean a finite, simple and undirected one. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Terms and notations not defined here are used in the sense of Harary[1]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a detailed survey is available in [2].

The concept of triangular difference mean labeling was introduced by Jeyanti et al.[3]. A triangular difference mean labeling of a graph $G = (p, q)$ is an injection $f : V(G) \rightarrow Z^+$, where Z^+ is a set of positive integers such that for each edge $e = uv$, the edge labels are defined as $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ such that the values of the edges are the first q triangular numbers $T_1, T_2, \dots, T_q$. A graph that admits restricted triangular difference mean labeling is called triangular difference mean graph.

In this paper we define a new labeling called restricted triangular difference mean labeling. Let $G = (V, E)$ be a graph with p vertices and q edges. Consider an injection $f : V(G) \rightarrow \{1, 2, 3, \dots, pq\}$. Define $f^* : E(G) \rightarrow \{T_1, T_2, T_3, \dots, T_q\}$, where T_q is the q^{th} triangular number such that $f^*(e) = \left\lceil \frac{|f(u)-f(v)|}{2} \right\rceil$ for all edges $e = uv$. If $f^*(E(G))$ is a sequence of consecutive triangular numbers $T_1, T_2, T_3, \dots, T_q$, then the function f is said to be restricted triangular difference mean. A graph that admits restricted triangular difference mean labeling is called restricted triangular difference mean graph.

We use the following definitions in the subsequent sequel.

Definition 1.1. A vertex of degree one is called a pendant vertex and a pendant edge is an edge incident with a pendant vertex. The corona $G_1 \odot G_2$ of the graphs G_1 and G_2 is obtained by taking one copy of G_1 (with p vertices) and p copies of G_2 and then join the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2 .

Definition 1.2. The bistar $B_{m,n}$ is a graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end

of K_2 .

Definition 1.3. A triangular number is obtained by adding all positive integers less than or equal to a given positive integer n . If the n^{th} triangular number is denoted by T_n , then $T_n = \frac{1}{2}n(n+1)$.

2. Restricted Triangular Difference Mean graphs

Theorem 2.1. Any path $P_n (n \geq 1)$ is a restricted triangular difference mean graph.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the path P_n .

Then $E(P_n) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}$.

Define $f : V(P_n) \rightarrow \{1, 2, 3, \dots, pq\}$ is as follows:

$$f(v_1) = 1,$$

$$f(v_2) = 2T_{n-1},$$

$$f(v_i) = f(v_{i-1}) + 2(-1)^i T_{n-i+1} \text{ for } 3 \leq i \leq n.$$

For the vertex label f , the induced edge label f^* is as follows:

$f^*(e_i) = T_{n-i}$ for $1 \leq i \leq n-1$. Hence P_n is a restricted difference mean graph. The restricted triangular difference mean labeling of P_5 is given in Figure 1.

□

Figure 1

Theorem 2.2. The star graph $K_{1,n} (n \geq 1)$ admits restricted triangular difference mean labeling.

Proof. Let v be the apex vertex and $v_1, v_2, \dots, v_n$ be the pendant vertices of the star $K_{1,n}$. Then $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$.

Define $f : V(K_{1,n}) \longrightarrow \{1, 2, 3, \dots, pq\}$ as follows:

$$f(v) = 1, f(v_i) = 2T_i + 1 \text{ for } 1 \leq i \leq n-1, f(v_n) = 2T_n.$$

For the vertex label f , the induced edge label f^* is as follows:

$f^*(vv_i) = T_i$ for $1 \leq i \leq n$. Hence, the induced edge labels are the first n triangular numbers. Therefore, $K_{1,n}$ is a restricted triangular difference mean graph. The restricted triangular difference mean labeling of $K_{1,5}$ is shown in Figure 2.

Figure 2

Theorem 2.3. The comb graph $P_n \odot K_1$ admits restricted triangular difference mean labeling.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of the path P_n and $u_1, u_2, \dots, u_n$ be the pendant vertices adjacent to $v_1, v_2, \dots, v_n$ respectively. Then $E(P_n \odot K_1) = \{e_i = v_i v_{i+1}, e'_j = u_j v_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. Define $f : V(P_n \odot K_1) \rightarrow \{1, 2, 3, \dots, pq\}$ is as follows:

$$f(v_1) = 1, f(v_i) = f(v_{i-1}) + 2(-1)^i T_{2(n-i+1)} \text{ for } 2 \leq i \leq n, f(u_1) = 2T_{2n-1},$$

$$f(u_i) = f(v_i) + 2(-1)^{i+1} T_{2(n-i)+1} \text{ for } 2 \leq i \leq n.$$

For the vertex label f , the induced edge label f^* is as follows:

$$f^*(e_i) = T_{2(n-i)} \text{ for } 1 \leq i \leq n-1, f^*(e'_j) = T_{2(n-j)+1} \text{ for } 1 \leq j \leq n.$$

We can see that the edge labels are the triangular numbers: $T_1, T_2, \dots, T_{2n-1}$. Hence $P_n \odot K_1$ is a restricted triangular difference mean graph. The restricted triangular difference mean labeling of $P_5 \odot K_1$ is shown in Figure 3.

Figure 3

Theorem 2.4. A necessary condition for a graph to be a restricted triangular difference mean graph is that $p > q$.

Proof. Suppose that the graph $G = (V, E)$ with p vertices and q edges is a restricted triangular difference mean graph.

Suppose $p \leq q$.

Let $f : V(G) \rightarrow \{1, 2, 3, \dots, pq\}$ be a restricted triangular difference mean labeling of G . Then the induced edge labels are $\{T_1, T_2, \dots, T_q\}$.

Let u and v be any two adjacent vertices of G such that $f^*(uv) = T_q$.

Without loss of generality, we assume $f(u) > f(v)$.

Then $1 \leq f(u), f(v) \leq pq$.

Case(i). Suppose $|f(u) - f(v)|$ is odd.

$$\begin{aligned} \frac{|f(u) - f(v)| + 1}{2} &= T_q = \frac{q(q+1)}{2} \\ f(u) - f(v) &= q^2 + q - 1 \\ f(u) &= f(v) + q^2 + q - 1. \end{aligned}$$

Now, $p \leq q$

$$\begin{aligned} pq &\leq q^2 \\ pq &\leq q^2 < q^2 + q \\ f(u) &\leq pq < q^2 + q \\ f(v) + q^2 + q - 1 &< q^2 + q \\ f(v) - 1 &< 0 \end{aligned}$$

$f(v) < 1$, which is a contradiction.

Case(ii). Suppose $|f(u) - f(v)|$ is even.

$$\begin{aligned}\frac{|f(u)-f(v)|}{2} &= T_q = \frac{q(q+1)}{2} \\ f(u) - f(v) &= q^2 + q \\ f(v) + q^2 + q &< q^2 + q.\end{aligned}$$

$f(v) < 0$, which is a contradiction.

Theorem 2.5. *The complete graph K_n ($n \geq 1$) is a restricted triangular difference mean graph if and only if $n < 3$.*

Proof. Let $V(K_n) = \{v_i : 1 \leq i \leq n\}$ and $E(K_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_2 v_{i+2} : 1 \leq i \leq n-2\} \cup \{v_3 v_{i+3} : 1 \leq i \leq n-3\} \cup \dots \cup \{v_{n-1} v_n\}$.

The graph K_n has n vertices and $\binom{n}{2}$ edges.

When $n < 3$, the result is obvious.

Conversely, suppose K_n is a restricted triangular difference mean graph.

Suppose $n \geq 3$.

Let $f : V(K_n) \rightarrow \{1, 2, 3, \dots, pq = n \binom{n}{2}\}$ be defined as follows:

Let u and v be any two adjacent vertices of G such that $f^*(uv) = T_q$.

Let $e = uv$ be labeled with $\binom{n}{2}$.

Now $1 \leq f(u), f(v) \leq n \binom{n}{2}$.

Then either $\frac{|f(u)-f(v)|}{2} = T \binom{n}{2}$ or $\frac{|f(u)-f(v)|+1}{2} = T \binom{n}{2}$

Without loss of generality we assume $f(u) > f(v)$.

$$\begin{aligned}
f(u) &= \binom{n}{2} \left[\left[\binom{n}{2} + 1 \right] + f(v) \right] \\
&\geq \binom{n}{2} \left[\left[\binom{n}{2} + 1 \right] + 1 \right] \\
&> \binom{n}{2} \left[\left[\binom{n}{2} + 1 \right] \right]
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\binom{n}{2} \left[\left[\binom{n}{2} + 1 \right] \right] < f(u) \leq n \binom{n}{2} \\
\left[\left[\binom{n}{2} + 1 \right] \right] < n,
\end{aligned}$$

This is a contradiction.

or

$$\begin{aligned}
f(u) &= \binom{n}{2} \text{ or } \left[\left[\binom{n}{2} + 1 \right] + f(v) - 1 \right] \\
&\geq \binom{n}{2} \left[\left[\binom{n}{2} + 1 \right] \right].
\end{aligned}$$

Then we have $\binom{n}{2} \left[\left[\binom{n}{2} + 1 \right] \right] \leq f(u) \leq n \binom{n}{2}$

$$\left[\left[\binom{n}{2} + 1 \right] \right] \leq n, \text{ which implies } n \leq 2.$$

This is also a contradiction.

Theorem 2.6. *The bistar $B_{m,n}$ ($m \geq 1, n \geq 1$) is a restricted triangular difference mean graph.*

Proof. Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$

and $E(B_{m,n}) = \{uv, uu_i, vv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Define $f : V(B_{m,n}) \rightarrow \{1, 2, 3, \dots, pq\}$ is as follows:

$$f(u) = 1, f(v) = 3, f(u_1) = 2T_{m+n+1},$$

$$f(u_i) = 2T_{m+n-i+2} + 1 \text{ for } 2 \leq i \leq m, f(v_j) = 2T_{j+1} + 3 \text{ for } 1 \leq j \leq n.$$

For the vertex label f , the induced edge label f^* is as follows:

$f^*(uv) = T_1$, $f^*(uu_i) = T_{n+m-i+2}$ for $1 \leq i \leq m$ and $f^*(vv_j) = T_{j+1}$ for $1 \leq j \leq n$.

Hence, the induced edge labels are the first $m + n + 1$ triangular numbers.

Therefore, $B_{m,n}$ is a restricted triangular difference mean graph.

The restricted triangular difference mean labeling of $B_{4,5}$ is shown in Figure 4.

Figure 4

Theorem 2.7. A graph obtained by joining the roots of different stars to a new vertex, is a restricted triangular difference mean graph.

Proof. Let $K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_k}$ be a k stars. Let G be a graph obtained by joining the central vertices of the stars to the new vertex u .

Let $n = n_1 + n_2 + \dots + n_k$.

Assign 1 to u ; $2T_{n+k}, 2T_{n+k-1}, \dots, 2T_{n+1}$ to the central vertices of the stars.

$2T_{n+k} - 2T_1, 2T_{n+k} - 2T_2, \dots, +1, 2T_{n+k} - 2T_{n_1}$ to the pendant vertices of the

first star; $+1, 2T_{n+k-1} - 2T_{n_1+1}, +1, 2T_{n+k-1} - 2T_{n_1+2}, \dots, 2T_{n+k-1} - 2T_{n_1+n_2}$

to the pendant vertices of the second star; and so on finally assign the

numbers $+1, 2T_{n+1} - 2T_{n_1+n_2+\dots+n_{k-1}+1}, +1, 2T_{n+1} - 2T_{n_1+n_2+\dots+n_{k-1}+2}, \dots,$

$+1, 2T_{n+1} - 2T_{n_1+n_2+\dots+n_{k-1}+n_k}$ to the pendant vertices of the last star.

We observe that the edge labels are the triangular numbers $T_1, T_2, \dots,$

$T_{k+n_1+n_2+\dots+n_{k-1}+n_k}$, also the vertex labels are all different. The restricted

triangular difference mean labeling of the tree given in Theorem 2.7 with

$k = 3, n_1 = 4, n_2 = 5$ and $n_3 = 4$ is shown in Figure 5.

Figure 5

Theorem 2.8. *The caterpillar $S(n_1, n_2, \dots, n_m)$ is a restricted triangular difference mean graph.*

Proof. Let $v_1, v_2, \dots, v_m$ be the vertices of the path P_m and $v_i^j (1 \leq i \leq n_j, 1 \leq j \leq m)$ be the pendant vertices incident with $v_j (1 \leq j \leq m)$.

Let $n = n_1 + n_2 + \dots + n_m$.

Then $V(S(n_1, n_2, \dots, n_m)) = \{v_j, v_i^j : 1 \leq i \leq n_j, 1 \leq j \leq m\}$ and

$E(S(n_1, n_2, \dots, n_m)) = \{v_t v_{t+1}, v_j v_i^j : 1 \leq t \leq m-1, 1 \leq i \leq n_j, 1 \leq j \leq m\}$.

Define $f : V(S(n_1, n_2, \dots, n_m)) \rightarrow \{1, 2, 3, \dots, pq\}$ is as follows:

$f(v_1) = 1, f(v_j) = 2(T_1 + T_2 + \dots + T_{j-1}) + 1$ for $2 \leq j \leq m$,

$f(v_i^1) = 2T_{n+m-i}$ for $1 \leq i \leq n_1$,

$f(v_i^j) = 2(T_1 + T_2 + \dots + T_{j-1}) + 2T_{m+n-n_1-n_2-\dots-n_{j-1}-i} + 1$ for $2 \leq j \leq m$ and $1 \leq i \leq n_j$.

For each vertex label f , the induced edge label f^* is as follows:

$f^*(v_j v_{j+1}) = T_j$ for $1 \leq j \leq m-1$,

$f^*(v_1 v_i^1) = T_{m+n-i}$,

$f^*(v_j v_i^j) = T_{m+n-n_1-n_2-\dots-n_{j-1}-i}$ for $2 \leq j \leq m$ and $1 \leq i \leq n_j$.

Then the edge labels are the triangular numbers $T_1, T_2, \dots, T_{m-1},$

$T_m, \dots, T_{m-1+n_1+\dots+n_m}$ and also the vertex labels are different.

Hence $S(n_1, n_2, \dots, n_m)$ is a restricted triangular difference mean graph.

The restricted triangular difference mean labeling of $S(3, 5, 4, 6)$ is shown in Figure 6.

Figure 6

Theorem 2.9. *The complete bipartite graph $K_{m,n}$ is a restricted triangular difference mean graph if and only if $n < 2$ or $m < 2$.*

Proof. If $m = n = 1$, then the graph $K_{1,1} = K_2$ which is a restricted triangular difference mean graph.

If $m = 1$ or $r = 1$, $K_{m,n}$ is a star graph and hence it is a restricted triangular difference mean graph.

Conversely suppose that $m, n \geq 2$.

Let G be a restricted triangular difference mean graph $K_{m,n}$.

Let $V(G) = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Then G has $m + n$ vertices and mn edges.

Let $f : V(G) \rightarrow \{1, 2, 3, \dots, pq = mn(m + n)\}$ be defined as follows:

Now $1 \leq f(u_i), f(v_j) \leq mn(m + n)$ for all u_i and v_j .

Let $e = u_i v_j$ be labeled with T_{mn} .

Without loss of generality, let $f(u_i) > f(v_j)$.

Case(i).

Suppose $\frac{|f(u_i) - f(v_j)|}{2} = T_{mn} = \frac{mn(mn+1)}{2}$

Then

$$\begin{aligned}
f(u_i) &= f(v_j) + mn(mn + 1) \\
&\geq 1 + mn(mn + 1) \\
&> mn(mn + 1).
\end{aligned}$$

That is, we have

$$\begin{aligned}
mn(mn + 1) &< f(u_i) \leq mn(m + n) \\
mn + 1 &< m + n \text{ for all } m, n \geq 2,
\end{aligned}$$

which is a contradiction.

Case(ii).

Suppose $\frac{|f(u_i) - f(v_j)| + 1}{2} = T_{mn} = \frac{mn(mn+1)}{2}$

Then

$$\begin{aligned}
f(u_i) &= f(v_j) + mn(mn + 1) - 1 \\
&\geq mn(mn + 1).
\end{aligned}$$

If $mn(mn + 1) \leq f(u_i) \leq mn(m + n)$, then $mn + 1 \leq m + n$ for all $m, n \geq 2$, which is a contradiction.

Theorem 2.10. *The wheel graph $W_n, n > 2$ is not a restricted triangular difference mean graph.*

Proof. Let G be a wheel graph $W_n, n > 2$.

Let $V(G) = \{v, v_i : 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_jv_{j+1} : 1 \leq i \leq n, 1 \leq j \leq n - 1\} \cup \{v_nv_1\}$.

Then the graph G has $n + 1$ vertices and $2n$ edges.

Let $f : V(G) \rightarrow \{1, 2, 3, \dots, pq = 2n(n + 1)\}$ be defined as follows:

Suppose G is a restricted triangular difference mean graph.

Let $f(v) = a$ and $f(v_i) = b$ for some i .

Suppose $a > b$. Then $1 \leq b < a \leq 2n(n + 1)$.

Now we consider two cases.

Case(i).

Suppose $\frac{|f(v) - f(v_i)|}{2} = T_{2n} = \frac{2n(2n+1)}{2}$

$$\begin{aligned}
\frac{a-b}{2} &= \frac{2n(2n+1)}{2} \\
a &= 2n(2n+1) + b \\
&\geq 2n(2n+1) + 1 \\
&> 2n(2n+1) \\
&> 2n(n+1),
\end{aligned}$$

which is a contradiction.

Case(ii).

Suppose $\frac{|f(v)-f(v_i)|+1}{2} = T_{2n} = \frac{2n(2n+1)}{2}$

$$\begin{aligned}
a - b + 1 &= 2n(2n+1) \\
a &= 2n(2n+1) + b - 1 \\
&\geq 2n(2n+1) \\
&> 2n(n+1),
\end{aligned}$$

which is also a contradiction.

Hence the wheel graph W_n , $n > 2$ is not a restricted triangular difference mean graph.

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