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Some generalized ostrowski type fractional integral inequalities for $MT-$convex functions with applications on special means

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Abstract

Some generalized Ostrowski-type integral inequalities for $r-$times differentiable functions whose absolute values are $MT-$convex have been discussed. Moreover, some applications on special bivariate means are obtained.

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1. Introduction

In 1938, A. M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows:

Theorem 1. *[3] Let $f : K \to \mathbf{R}$, K is an interval in $\mathbf{R}$, be a differentiable function in K^o, the interior of K and $a, b \in K^o$, $a < b$. If $|f'(t)| \leq M$ for all $t \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)M, \tag{1.1}$$

for $x \in [a, b]$, where M is a constant.

Inequality (1.1) has great importance in numerical analysis and in the theory of some special means, it attracts the attention and interest of mathematicians and researchers. It was generalized, extended and different variants were given. These studies include, among others, the works in [1, 2, 12, 15]. The development of the theory of fractional calculus is prior to the turn of 20th century. It has a lot of applications in different fields of science and engineering including viscoelastic materials, fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory, probability. Fractional calculus is the generalization of classical calculus. Like ordinary calculus, fractional derivatives and integrals are not defined in a unique way. Different authors have their contributions [6, 7, 11, 14]. For $n = 1$, the sum $\sum_{k=1}^{n-1}$ is vacuously considered to be zero. $\mathbf{R}$ is the set of real numbers. In the whole presentation all the integrals are finite while, $I := [a^\rho, (a + \eta(b, a))^\rho]$ is an interval in $\mathbf{R}$, U depicts a convex set in $\mathbf{R}$ and $\eta : M \times M \to \mathbf{R}$ a continuous function where $M \subseteq \mathbf{R}$. This work is organized in the following way. After this Introduction, in Section 2 some basic concepts are discussed. In Section 3 some results relating to the topic are established and Section 4 deals with some applications on special means.

2. Preliminaries

Definition 1. *[10] The left- and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ of f, denoted by $J^\alpha_{a+} f$ and $J^\alpha_{b-} f$ respectively, are defined as:*

$$J^\alpha_{a+} f(x) \;:=\; \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a, \tag{2.1}$$

$$J_{b-}^{\alpha} f(x) \quad := \quad \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b, \tag{2.2}$$

where, Γ is the Euler gamma function; $\alpha \to 1$ gives the classical integral and it may be noted that:

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x).$$

Definition 2. *[17] The left- and right-sided Hadamard fractional integrals of order $\alpha > 0$ of f, denoted by $H_{a+}^{\alpha} f$ and $H_{b-}^{\alpha} f$ respectively, are defined as:*

$$H_{a+}^{\alpha} f(x) \quad := \quad \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \tag{2.3}$$

$$H_{b-}^{\alpha} f(x) \quad := \quad \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt. \tag{2.4}$$

Definition 3. *[9] Let $X_c^p(a,b)$, for $c \in \mathbf{R}$ and $p \in [1,\infty]$, be the space of all real-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where*

$$\|f\|_{X_c^p} \quad := \quad \sqrt[p]{\int_a^b |t^c f(t)|^p \frac{dt}{t}}, \quad \forall\, p \in [1,\infty); \quad \|f\|_{X_c^\infty} := ess \sup_{a \le t \le b} |t^c f(t)|.$$

Recently Katugampola defined the following integrals generalizing both Riemann-Liouville and Hadamard fractional integrals.

Definition 4. *[7] Let $[a,b] \subseteq \mathbf{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined as:*

$${}^{\rho}I_{a+}^{\alpha} f(x) \quad := \quad \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x t^{\rho-1}(x^\rho - t^\rho)^{\alpha-1} f(t)dt, \tag{2.5}$$

$${}^{\rho}I_{b-}^{\alpha} f(x) \quad := \quad \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b t^{\rho-1}(t^\rho - x^\rho)^{\alpha-1} f(t)dt, \tag{2.6}$$

with $a < x < b$ and $\rho > 0$, provided that the integrals exist. For $\rho \to 1$, integrals in (2.5) and (2.6) coincide with the integrals in (2.1) and (2.2), respectively, while for $\rho \to 0^+$ integrals in (2.5) and (2.6) coincide with the integrals in (2.3) and (2.4) respectively. Katugampola fractional integral operators are well-defined on $X_c^p(a,b)$ [8].

Definition 5. *[10] Let $f : U \to [0, \infty)$ be a function, then f is said to be convex (or that $f \in Conv(U)$), provided that: f(tx+(1-t)y)$\leq tf(x) + (1-t)f(y)$, $\forall\ x, y \in U;\ t \in [0, 1]$.*

Definition 6. *[3] Let $f : U \to [0, \infty)$ and $h : J \subseteq \mathbf{R} \to (0, \infty)$ be two functions such that $[0, 1] \subseteq J$, then f is called $h-$convex (or that $f \in SX(h, U)$), provided that f(tx + (1 - t)y) $\leq h(t)f(x)+h(1-t)f(y)$, $\forall\ x, y \in U$.*

Definition 7. *[13] Let $f : U \to [0, \infty)$ be a function, then f is said to be $MT-$convex (or that $f \in MT(U)$), provided that f(tx+(1-t)y)$\leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y)$, $\forall\ x, y \in U;\ t \in (0, 1)$.*

The $\rho-$gamma and $\rho-$Beta functions for any two positive real numbers x, y are, denoted by ${}^{\rho}\Gamma(x)$ and ${}^{\rho}B(x, y)$ respectively, defined as [4]:

$$(2.7) \qquad {}^{\rho}\Gamma(x) = \int_0^{\infty} e^{-t^{\rho}}(t^{\rho})^{x-\frac{1}{\rho}}dt; \quad {}^{\rho}B(x, y) = \frac{{}^{\rho}\Gamma(x){}^{\rho}\Gamma(y)}{{}^{\rho}\Gamma(x + y)}.$$

3. Results

Lemma 1. *[5] Let $f : I = [a^{\rho}, (a + \eta(b, a))^{\rho}] \subseteq \mathbf{R} \to \mathbf{R}$ be $r-$times differentiable function on I^o, interior of I, and $f^{(r)} \in X_c^p(a, a + \eta(b, a))$ for $r \in \mathbf{N}_0$; let $\beta := \alpha - j + 1 + \frac{1-n}{\rho}$ for $r \geq j$; $\rho, \alpha, \beta > 0$ and $n \in \mathbf{N}$. If $x^{\rho} \in I$ and $\eta(b, a) \leq b$, then*

$$\Omega(x; a, b, \rho) := \rho^{j-1}\frac{(x^{\rho} - a^{\rho})^{j+\beta-1} + [(a + \eta(b, a))^{\rho} - x^{\rho}]^{j+\beta-1}}{\eta(b, a)}f^{(r-1)}(x^{\rho})$$
$$+\sum_{k=1}^{j-1}\prod_{1\leq i\leq k}\frac{\rho(\alpha + r + 1 - i) - n}{\rho^{k-j+1}\ \eta(b, a)}\{(-1)^k(x^{\rho} - a^{\rho})^{j+\beta-k-1} + [(a + \eta(b, a))^{\rho}$$
$$-x^{\rho}]^{j+\beta-k-1}\}f^{(r-1-k)}(x^{\rho}) + \prod_{1\leq i\leq j}(-1)^j\frac{[\rho(\alpha + r + 1 - i) - n]\Gamma(\alpha)}{\rho^{1-\alpha}\ \eta(b, a)}$$
$$\times\left\{{}^{\rho}I_{x-}^{\alpha}f^{(r-j)}(a^{\rho}) + (-1)^{j+1}\{{}^{\rho}I_{x+}^{\alpha}f^{(r-j)}(a + \eta(b, a))^{\rho}\right\} = \rho^j\frac{(x^{\rho} - a^{\rho})^{j+\beta}}{\eta(b, a)}$$
$$\times\int_0^1 t^{\alpha\rho+r\rho-n}f^{(r)}(t^{\rho}x^{\rho} + (1 - t^{\rho})a^{\rho})dt - \rho^j\frac{[(a + \eta(b, a))^{\rho} - x^{\rho}]^{j+\beta}}{\eta(b, a)}$$
$$\times\int_0^1 t^{\alpha\rho+r\rho-n}f^{(r)}(t^{\rho}x^{\rho} + (1 - t^{\rho})(a + \eta(b, a))^{\rho})dt.$$

Theorem 1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be $r-$times differentiable function on I^o, interior of I, and $f^{(r)} \in X_c^p(a, a+\eta(b,a))$ for $r \in \mathbf{N}_0$; let $\beta := \alpha - j + 1 + \frac{1-n}{\rho}$ for $r \geq j$; $\rho, \alpha, \beta > 0$ and $n \in \mathbf{N}$. Moreover, if $\left|f^{(r)}\right| \in MT(I)$ is such that $\left|f^{(r)}(x^\rho)\right| \leq M$ for $M > 0$. If $x^\rho \in I$ and $\eta(b,a) \leq b$, then*

$$|\Omega(x;a,b,\rho)| \leq M\rho^j \ {}^\rho B\left(\alpha + r - \frac{1}{2} + \frac{1-n}{\rho}, \frac{1}{2}\right)$$
$$\times \frac{(x^\rho - a^\rho)^{j+\beta} + [(a+\eta(b,a))^\rho - x^\rho]^{j+\beta}}{2\ \eta(b,a)}.$$

Proof. Applications of Lemma 1, boundedness, $MT-$convexity of $\left|f^{(r)}\right|$ and relation (2.7) yield:

$$\begin{aligned}
&|\Omega(x;a,b,\rho)| \\
&\leq \rho^j \frac{(x^\rho - a^\rho)^{j+\beta}}{\eta(b,a)} \int_0^1 t^{\alpha\rho + r\rho - n} |f^{(r)}(t^\rho x^\rho + (1-t^\rho)a^\rho)| dt \\
&+\rho^j \frac{[(a+\eta(b,a))^\rho - x^\rho]^{j+\beta}}{\eta(b,a)} \\
&\times \int_0^1 t^{\alpha\rho + r\rho - n} |f^{(r)}(t^\rho x^\rho + (1-t^\rho)(a+\eta(b,a))^\rho)| dt \\
&\leq \rho^j \frac{(x^\rho - a^\rho)^{j+\beta}}{\eta(b,a)} \\
&\times \int_0^1 t^{\alpha\rho + r\rho - n} \left[\frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}} |f^{(r)}(x^\rho)| + \frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}} |f^{(r)}(a^\rho)|\right] dt \\
&+\rho^j \frac{[(a+\eta(b,a))^\rho - x^\rho]^{j+\beta}}{\eta(b,a)} \\
&\times \int_0^1 t^{\alpha\rho + r\rho - n} \left[\frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}} |f^{(r)}(x^\rho) + \frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}} |f^{(r)}(a+\eta(b,a))^\rho|\right] dt \\
&\leq M\rho^j \frac{(x^\rho - a^\rho)^{j+\beta}}{\eta(b,a)} \int_0^1 t^{\alpha\rho + r\rho - n} \left[\frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}} + \frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}}\right] dt \\
&+M\rho^j \frac{[(a+\eta(b,a))^\rho - x^\rho]^{j+\beta}}{\eta(b,a)} \int_0^1 t^{\alpha\rho + r\rho - n} \left[\frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}} + \frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}}\right] dt \\
&\leq M\rho^j \frac{(x^\rho - a^\rho)^{j+\beta} + [(a+\eta(b,a))^\rho - x^\rho]^{j+\beta}}{2\eta(b,a)}
\end{aligned}$$

$$\times \int_0^1 [t^{\alpha\rho+r\rho-n+\frac{\rho}{2}}(1-t^\rho)^{-\frac{1}{2}} + t^{\alpha\rho+r\rho-n-\frac{\rho}{2}}(1-t^\rho)^{\frac{1}{2}}]dt$$
$$\leq M\rho^{j-1}\ {}^\rho B\left(\alpha+r-\frac{1}{2}+\frac{1-n}{\rho},\frac{1}{2}\right)$$
$$\times\frac{(x^\rho-a^\rho)^{j+\beta}+[(a+\eta(b,a))^\rho-x^\rho]^{j+\beta}}{2\ \eta(b,a)}.$$

This completes the proof. □

Corollary 1. *Let the conditions of Theorem 1 be satisfied for* $x=\frac{2a+\eta(b,a)}{2}$, *then*

$$\Bigg|\rho^{j-1}\frac{[(2a+\eta(b,a))^\rho-(2a)^\rho]^{j+\beta-1}+[(2(a+\eta(b,a)))^\rho-(2a+\eta(b,a))^\rho]^{j+\beta-1}}{2^{j\rho+\beta\rho-\rho}\ \eta(b,a)}$$
$$f^{(r-1)}\left(\frac{2a+\eta(b,a)}{2}\right)^\rho+\sum_{k=1}^{j-1}\prod_{1\leq i\leq k}2^{\rho+k\rho-j\rho-\beta\rho}\ \frac{\rho(\alpha+r+1-i)-n}{\rho^{k-j+1}\ \eta(b,a)}$$
$$\times\{[(2(a+\eta(b,a)))^\rho-(2a+\eta(b,a))^\rho]^{j+\beta-k-1}+(-1)^k[(2a+\eta(b,a))^\rho$$
$$-(2a)^\rho]^{j+\beta-k-1}\}f^{(r-1-k)}\left(\frac{2a+\eta(b,a)}{2}\right)^\rho+\prod_{1\leq i\leq j}(-1)^j\frac{[\rho(\alpha+r+1-i)-n]}{\rho^{1-\alpha}\ \eta(b,a)}$$
$$\times\Gamma(\alpha)\{\ {}^\rho I^\alpha_{\left(\frac{2a+\eta(b,a)}{2}\right)^-}f^{(r-j)}(a^\rho)+(-1)^{j+1}\ {}^\rho I^\alpha_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}f^{(r-j)}(a+\eta(b,a))^\rho\}\Bigg|$$
$$\leq M\rho^j\frac{[(2a+\eta(b,a))^\rho-(2a)^\rho]^{j+\beta}+[(2(a+\eta(b,a)))^\rho-(2a+\eta(b,a))^\rho]^{j+\beta}}{2^{\rho j+\rho\beta+1}\ \eta(b,a)}$$
$$\times\ {}^\rho B\left(\alpha+r-\frac{1}{2}+\frac{1-n}{\rho},\frac{1}{2}\right). \tag{3.1}$$

Theorem 2. *Let the conditions of Theorem 1 be satisfied; Moreover, if* $p,q>1$ *for which* $p=\frac{q}{q-1}$ *and* $\left|f^{(r)}\right|^q\in MT(I)$, *then*

$$|\Omega(x;a,b,\rho)|\leq M\frac{(x^\rho-a^\rho)^{j+\beta}+[(a+\eta(b,a))^\rho-x^\rho]^{j+\beta}}{\eta(b,a)\sqrt[p]{p\alpha\rho+pr\rho-np+1}}$$
$$\times\sqrt[q]{\frac{\rho^{jq-1}}{2}\left[{}^\rho B\left(\frac{\rho}{2}+1,\frac{1}{2}\right)+{}^\rho B\left(1-\frac{\rho}{2},\frac{3}{2}\right)\right]}. \tag{3.2}$$

Proof. Applications of Lemma 1, Holder's inequality [15] and relation (2.7) yield:

$$|\Omega(x;a,b,\rho)|$$

$$\leq \rho^j \frac{(x^\rho - a^\rho)^{j+\beta} \sqrt[q]{\int_0^1 |f^{(r)}(t^\rho x^\rho + (1-t^\rho)a^\rho)|^q dt}}{\eta(b,a)\sqrt[p]{p\alpha\rho + rp\rho - np + 1}}$$

$$(3.3)+\rho^j \frac{[(a+\eta(b,a))^\rho - x^\rho]^{j+\beta} \sqrt[q]{\int_0^1 |f^{(r)}(t^\rho x^\rho + (1-t^\rho)(a+\eta(b,a))^\rho)|^q dt}}{\eta(b,a)\sqrt[p]{p\alpha\rho + rp\rho - np + 1}}$$

By boundedness of $\left|f^{(r)}(x^\rho)\right|$ and $\left|f^{(r)}\right|^q \in MT(I)$, the following inequalities hold.

$$\begin{aligned}
\int_0^1 |f^{(r)}(t^\rho x^\rho + (1-t^\rho)a^\rho)|^q dt &\leq \int_0^1 \left[\frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}}|f^{(r)}(x^\rho)|^q + \frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}}|f^{(r)}(a^\rho)|^q\right] dt \\
&\leq M^q \int_0^1 \left[\frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}} + \frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}}\right] dt \\
&\leq \frac{M^q}{2} \int_0^1 \left[t^{\frac{\rho}{2}}(1-t^\rho)^{-\frac{1}{2}} + t^{-\frac{\rho}{2}}(1-t^\rho)^{\frac{1}{2}}\right] dt \\
&\leq \frac{M^q}{2\rho}\left[{}^\rho B\left(\frac{\rho}{2}+1, \frac{1}{2}\right) + {}^\rho B\left(1-\frac{\rho}{2}, \frac{3}{2}\right)\right]. \qquad (3.4)
\end{aligned}$$

$\int_0^1 |f^{(r)}(t^\rho x^\rho + (1-t^\rho)(a+\eta(b,a))^\rho)|^q dt$
$\leq \frac{M^q}{2\rho}\left[{}^\rho B\left(\frac{\rho}{2}+1, \frac{1}{2}\right) + {}^\rho B\left(1-\frac{\rho}{2}, \frac{3}{2}\right)\right]$. A combination of (3.3)-(3), yields the desired result. □

Corollary 2. *Let the conditions of Theorem 2 be satisfied for* $x = \frac{2a+\eta(b,a)}{2}$, *then*

$$\begin{aligned}
&\Bigg|\rho^{j-1}\frac{[(2a+\eta(b,a))^\rho - (2a)^\rho]^{j+\beta-1} + [(2(a+\eta(b,a)))^\rho - (2a+\eta(b,a))^\rho]^{j+\beta-1}}{2^{j\rho+\beta\rho-\rho}\ \eta(b,a)} \\
&f^{(r-1)}\left(\frac{2a+\eta(b,a)}{2}\right)^\rho + \sum_{k=1}^{j-1}\prod_{1\leq i\leq k} 2^{\rho+k\rho-j\rho-\beta\rho}\ \frac{\rho(\alpha+r+1-i)-n}{\rho^{k-j+1}\ \eta(b,a)} \\
&\times\{[(2(a+\eta(b,a)))^\rho - (2a+\eta(b,a))^\rho]^{j+\beta-k-1} + (-1)^k[(2a+\eta(b,a))^\rho \\
&-(2a)^\rho]^{j+\beta-k-1}\} f^{(r-1-k)}\left(\frac{2a+\eta(b,a)}{2}\right)^\rho + \prod_{1\leq i\leq j}(-1)^j\frac{[\rho(\alpha+r+1-i)-n]}{\rho^{1-\alpha}\ \eta(b,a)} \\
&\times\Gamma(\alpha)\{\ {}^\rho I^\alpha_{\left(\frac{2a+\eta(b,a)}{2}\right)^-} f^{(r-j)}(a^\rho) + (-1)^{j+1}\ {}^\rho I^\alpha_{\left(\frac{2a+\eta(b,a)}{2}\right)^+} f^{(r-j)}(a+\eta(b,a))^\rho\}\Bigg| \\
&\leq M\frac{[(2a+\eta(b,a))^\rho - (2a)^\rho]^{j+\beta} + [(2(a+\eta(b,a)))^\rho - (2a+\eta(b,a))^\rho]^{j+\beta}}{\sqrt[p]{(p\alpha\rho + pr\rho - pn + 1)}\ \eta(b,a)}
\end{aligned}$$

$$\times\sqrt[q]{\frac{\rho^{jq-1}}{2^{\rho qj+\rho q\beta+1}}\left[{}^{\rho}B\left(\frac{\rho}{2}+1,\frac{1}{2}\right)+{}^{\rho}B\left(1-\frac{\rho}{2},\frac{3}{2}\right)\right]}. \tag{3.5}$$

Corollary 3. *Let the conditions of Theorem 2 be satisfied for $x=a$ and $x=a+\eta(b,a)$, then*

$$\begin{aligned}
&\Bigg|\frac{\rho^{j-1}}{\eta(b,a)}[(a+\eta(b,a))^{\rho}-a^{\rho}]^{j+\beta-1}\{f^{(r-1)}(a^{\rho})+f^{(r-1)}(a+\eta(b,a))^{\rho}\}\\
&+\sum_{k=1}^{j-1}\prod_{1\le i\le k}\frac{\rho(\alpha+r+1-i)-n}{\rho^{k-j+1}\ \eta(b,a)}\Big\{[(a+\eta(b,a))^{\rho}-a^{\rho}]^{j+\beta-k-1}f^{(r-k-1)}(a^{\rho})\\
&+(-1)^{k}[(a+\eta(b,a))^{\rho}-a^{\rho}]^{j+\beta-k-1}f^{(r-k-1)}(a+\eta(b,a))^{\rho}\Big\}+\prod_{1\le i\le j}(-1)^{j}\\
&\times\frac{[\rho(\alpha+r+1-i)-n]}{\rho^{1-\alpha}\ \eta(b,a)}\Gamma(\alpha)\{\ {}^{\rho}I^{\alpha}_{a^{-}}f^{(r-j)}(a^{\rho})+{}^{\rho}I^{\alpha}_{(a+\eta(b,a))^{-}}f^{(r-j)}(a^{\rho})\\
&+(-1)^{j+1}\ {}^{\rho}I^{\alpha}_{a^{+}}f^{(r-j)}(a+\eta(b,a))^{\rho}+(-1)^{j+1}{}^{\rho}I^{\alpha}_{(a+\eta(b,a))^{+}}f^{(r-j)}(a+\eta(b,a))^{\rho}\Big\}\Bigg|\\
&\le 2M\frac{[(a+\eta(b,a))^{\rho}-a^{\rho}]^{j+\beta}}{\eta(b,a)\sqrt[p]{p\alpha\rho+pr\rho-np+1}}\sqrt[q]{\frac{\rho^{jq-1}}{2^{1-q}}\left[{}^{\rho}B\left(\frac{\rho}{2}+1,\frac{1}{2}\right)+{}^{\rho}B\left(1-\frac{\rho}{2},\frac{3}{2}\right)\right]}.
\end{aligned}$$

Theorem 3. *Let the conditions of Theorem 1 be satisfied; Moreover, if $q\ge 1$ for which $\left|f^{(r)}\right|^{q}\in MT(I)$, then*

$$\begin{aligned}
|\Omega(x;a,b,\rho)|\le\ & M\frac{(x^{\rho}-a^{\rho})^{j+\beta}+[(a+\eta(b,a))^{\rho}-x^{\rho}]^{j+\beta}}{\sqrt[q]{(\alpha\rho+r\rho-n+1)^{q-1}}\ \eta(b,a)}\\
&\times\sqrt[q]{\frac{\rho^{jq-1}}{2}\ {}^{\rho}B\left(\alpha+r-\frac{1}{2}+\frac{1-n}{\rho},\frac{1}{2}\right)}.
\end{aligned} \tag{3.6}$$

Proof. Applications of Lemma 1, power mean inequality [16] and relation (2.7) yield:

$$\begin{aligned}
&|\Omega(x;a,b,\rho)|\\
&\le\rho^{j}\frac{(x^{\rho}-a^{\rho})^{j+\beta}}{\eta(b,a)}\int_{0}^{1}t^{\alpha\rho+r\rho-n}|f^{(r)}(t^{\rho}x^{\rho}+(1-t^{\rho})a^{\rho})|dt\\
&+\rho^{j}\frac{[(a+\eta(b,a))^{\rho}-x^{\rho}]^{j+\beta}}{\eta(b,a)}\int_{0}^{1}t^{\alpha\rho+r\rho-n}|f^{(r)}(t^{\rho}x^{\rho}+(1-t^{\rho})(a+\eta(b,a))^{\rho})|dt
\end{aligned}$$

$$\leq \rho^j \frac{(x^\rho - a^\rho)^{j+\beta}(\sqrt[q]{\int_0^1 t^{\alpha\rho+r\rho-n}|f^{(r)}(t^\rho x^\rho + (1-t^\rho)a^\rho)|^q dt})}{\eta(b,a)\sqrt[q]{(p\alpha\rho + rp\rho - np + 1)^{q-1}}}$$
$$+\rho^j \frac{[(a+\eta(b,a))^\rho - x^\rho]^{j+\beta}}{\eta(b,a)\sqrt[q]{(p\alpha\rho + rp\rho - np + 1)^{q-1}}}$$
$$\times \sqrt[q]{\int_0^1 t^{\alpha\rho+r\rho-n}|f^{(r)}(t^\rho x^\rho + (1-t^\rho)(a+\eta(b,a))^\rho)|^q dt} \tag{3.7}$$

By boundedness of $\left|f^{(r)}(x^\rho)\right|$ and $\left|f^{(r)}\right|^q \in MT(I)$, following inequalities hold.

$$\int_0^1 t^{\alpha\rho+r\rho-n}|f^{(r)}(t^\rho x^\rho + (1-t^\rho)a^\rho)|^q dt$$
$$\leq \int_0^1 t^{\alpha\rho+r\rho-n}\left[\frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}}|f^{(r)}(x^\rho)|^q + \frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}}|f^{(r)}(a^\rho)|^q\right] dt$$
$$\leq M^q \int_0^1 t^{\alpha\rho+r\rho-n}\left[\frac{\sqrt{t^\rho}}{2\sqrt{1-t^\rho}} + \frac{\sqrt{1-t^\rho}}{2\sqrt{t^\rho}}\right] dt$$
$$\leq \frac{M^q}{2}\int_0^1 t^{\alpha\rho+r\rho-n+\frac{\rho}{2}}(1-t^\rho)^{-\frac{1}{2}} + t^{\alpha\rho+r\rho-n-\frac{\rho}{2}}(1-t^\rho)^{\frac{1}{2}} dt$$
$$\leq \frac{M^q}{2\rho}\,{}^\rho B\left(\alpha + r - \frac{1}{2} + \frac{1-n}{\rho}, \frac{1}{2}\right). \tag{3.8}$$

$\int_0^1 t^{\alpha\rho+r\rho-n}\left|f^{(r)}(t^\rho x^\rho + (1-t^\rho)(a+\eta(b,a))^\rho)\right|^q dt$
$\leq \frac{M^q}{2\rho}\,{}^\rho B\left(\alpha + r - \frac{1}{2} + \frac{1-n}{\rho}, \frac{1}{2}\right)$. A combination of (3.7)-(3), yields the desired result. □

Corollary 4. *Let the conditions of Theorem 3 be satisfied for* $x = \frac{2a+\eta(b,a)}{2}$, *then*

$$\left|\rho^{j-1}\frac{[(2a+\eta(b,a))^\rho - (2a)^\rho]^{j+\beta-1} + [(2(a+\eta(b,a)))^\rho - (2a+\eta(b,a))^\rho]^{j+\beta-1}}{2^{j\rho+\beta\rho-\rho}\ \eta(b,a)}\right.$$
$$f^{(r-1)}\left(\frac{2a+\eta(b,a)}{2}\right)^\rho + \sum_{k=1}^{j-1}\prod_{1\leq i\leq k} 2^{\rho+k\rho-j\rho-\beta\rho}\ \frac{\rho(\alpha+r+1-i)-n}{\rho^{k-j+1}\ \eta(b,a)}$$
$$\times\{[(2(a+\eta(b,a)))^\rho - (2a+\eta(b,a))^\rho]^{j+\beta-k-1} + (-1)^k[(2a+\eta(b,a))^\rho$$

$$-(2a)^{\rho}]^{j+\beta-k-1}\}f^{(r-1-k)}\left(\frac{2a+\eta(b,a)}{2}\right)^{\rho}+\prod_{1\leq i\leq j}(-1)^{j}\frac{[\rho(\alpha+r+1-i)-n]}{\rho^{1-\alpha}\ \eta(b,a)}$$

$$\times\Gamma(\alpha)\{\ {}^{\rho}I^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^{-}}f^{(r-j)}(a^{\rho})+(-1)^{j+1}\ {}^{\rho}I^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^{+}}f^{(r-j)}(a+\eta(b,a))^{\rho}\}\Bigg|$$

$$\leq M\frac{[(2a+\eta(b,a))^{\rho}-(2a)^{\rho}]^{j+\beta}+[(2(a+\eta(b,a)))^{\rho}-(2a+\eta(b,a))^{\rho}]^{j+\beta}}{\sqrt[q]{(\alpha\rho+r\rho-n+1)^{q-1}}\ \eta(b,a)}$$

$$\times\sqrt[q]{\frac{\rho^{jq-1}}{2^{\rho qj+\rho q\beta+1}}\ {}^{\rho}B\left(\alpha+r-\frac{1}{2}+\frac{1-n}{\rho},\frac{1}{2}\right)}.$$

(3.9)

Remark 1. *On letting $\rho, n, r, j \to 1$ and $\eta(b,a) = b - a$,*

- *Theorem 1 reduces to [13, Theorem 5]. Moreover, for $\alpha \to 1$ Theorem 1 reduces to [18, Theorem 2] and Corollary 1 reduces to [18, Remark 1].*
- *Theorem 2 reduces to [13, Theorem 6]. Moreover, for $\alpha \to 1$ Theorem 2 reduces to [18, Theorem 3] and Corollary 2 reduces to [18, Remark 2].*
- *for $\alpha \to 1$ Corollary 3 reduces to [18, Corollary 1].*
- *Theorem 3 reduces to [13, Theorem 7]. Moreover, for $\alpha \to 1$ Theorem 3 reduces to [18, Theorem 4] and Corollary 4 reduces to [18, Remark 4].*

4. Applications

Let $a, b > 0$ with $a \neq b$, then the arithmetic mean and $p-$logarithmic mean on a, b is defined as:

$$A(a,b) := \tfrac{a+b}{2}$$

$$L_p(a,b) := \begin{cases} \left[\frac{(b^{p+1}-a^{p+1})}{(b-a)(p+1)}\right]^{\frac{1}{p}}, & p \neq 0, p \neq -1; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0; \\ \frac{b-a}{\log b - \log a}, & p = -1. \end{cases}$$

Proposition 1. *Let $f:(0,1]\to\mathbf{R}$, be defined by $f(x)=-\ln x^s$ where $s>0$, obviously f is $MT-$convex and hence from inequality (3.1), the following inequality holds:*

$$\Bigg|s(-1)^{r-1}\frac{\rho^{2j+\beta-2}\ \Gamma(r-1)}{2^{j\rho+\beta\rho-\rho}[\eta(b,a)]^{2-j-\beta}}\Big\{[L_{\rho-1}^{\rho-1}(2a+\eta(b,a),2a)]^{j+\beta-1}$$
$$+[L_{\rho-1}^{\rho-1}(2(a+\eta(b,a),2a+\eta(b,a))]^{j+\beta-1}\Big\}\,[A(a,a+\eta(b,a))]^{\rho(1-r)}$$
$$+\sum_{k=1}^{j-1}\prod_{1\le i\le k}s(-1)^{r-k-1}\Gamma(r-k-1)2^{\rho+k\rho-j\rho-\beta\rho}\ \frac{\rho(\alpha+r+1-i)-n}{\rho^{2k-2j+2-\beta}[\eta(b,a)]^{k+2-j-\beta}}$$
$$\times[A(a,a+\eta(b,a))]^{\rho(1+k-r)}\{[L_{\rho-1}^{\rho-1}(2(a+\eta(b,a)),2a+\eta(b,a))]^{j+\beta-k-1}$$
$$+(-1)^k[L_{\rho-1}^{\rho-1}(2a+\eta(b,a),2a)]^{j+\beta-1}\}+\prod_{1\le i\le j}s(-1)^r\frac{[\rho(\alpha+r+1-i)-n]}{\rho^{2-\alpha}\ \eta(b,a)}$$
$$\times\Gamma(\alpha)\Gamma(r-j)\Bigg\{(-1)^{\alpha-1}a^{\rho\alpha+j-r}\Bigg[\mathbf{B}\left(\left(\frac{2a+\eta(b,a)}{2a}\right)^{\rho};\frac{j-r}{\rho}+1,\alpha\right)$$
$$-\mathbf{B}\left(a^{\rho(\rho-1)};\frac{j-r}{\rho}+1,\alpha\right)\Bigg]+\Bigg[\mathbf{B}\left((a+\eta(b,a))^{\rho(\rho-1)};\frac{j-r}{\rho}+1,\alpha\right)$$
$$-\mathbf{B}\left(\left(\frac{2a+\eta(b,a)}{2(a+\eta(b,a))}\right)^{\rho};\frac{j-r}{\rho}+1,\alpha\right)\Bigg](-1)^{j+1}(a+\eta(b,a))^{\rho\alpha+j-r}\Bigg\}\Bigg|$$
$$\le M\rho^j\ {}^{\rho}B\left(\alpha+r-\frac{1}{2}+\frac{1-n}{\rho},\frac{1}{2}\right)$$
$$\times\frac{[(2a+\eta(b,a))^{\rho}-(2a)^{\rho}]^{j+\beta}+[(2(a+\eta(b,a)))^{\rho}-(2a+\eta(b,a))^{\rho}]^{j+\beta}}{2^{\rho j+\rho\beta+1}\ \eta(b,a)}.$$

where, $\mathbf{B}$ is the Incomplete Beta function.

Proposition 2. *Let $f:\mathbf{R}\to\mathbf{R}^+$ be defined by $f(x)=\frac{x^{-m}}{m}$ where $0\neq m\in\mathbf{R}$, obviously f is $MT-$convex and hence from inequality (3.5) the following inequality holds:*

$$\Bigg|\frac{\rho^{2j+\beta-2}(-1)^{r-1}\Gamma(m+r-1)}{2^{j\rho+\beta\rho-\rho}[\eta(b,a)]^{2-j-\beta}}\Big\{[L_{\rho-1}^{\rho-1}(2a+\eta(b,a),2a)]^{j+\beta-1}+$$
$$[L_{\rho-1}^{\rho-1}(2(a+\eta(b,a)),2a+\eta(b,a))]^{j+\beta-1}\}[A(a,a+\eta(b,a))]^{\rho(1-m-r)}$$
$$+\sum_{k=1}^{j-1}\prod_{1\le i\le k}(-1)^{r-k-1}\Gamma(m+r-k-1)2^{\rho+k\rho-j\rho-\beta\rho}\ \frac{\rho(\alpha+r+1-i)-n}{\rho^{2+\beta-2k-2j}[\eta(b,a)]^{k+2-j-\beta}}$$
$$\times[A(a,a+\eta(b,a))]^{\rho(1+k-m-r)}\{[L_{\rho-1}^{\rho-1}(2(a+\eta(b,a)),2a+\eta(b,a))]^{j+\beta-k-1}+(-1)^k$$

$$\times[L_{\rho-1}^{\rho-1}(2a+\eta(b,a),2a)]^{j+\beta-k-1}\}+\prod_{1\le i\le j}(-1)^r\frac{[\rho(\alpha+r+1-i)-n]}{\rho^{2-\alpha}\ \eta(b,a)}\Gamma(\alpha)$$
$$\times\Gamma(m+r-j)\left\{(-1)^{\alpha-1}a^{\rho\alpha+j-m-r}\left[\mathbf{B}\left(\left(\frac{2a+\eta(b,a)}{2a}\right)^{\rho};\frac{j-m-r}{\rho}+1,\alpha\right)\right.\right.$$
$$-\mathbf{B}\left(a^{\rho(\rho-1)};\frac{j-m-r}{\rho}+1,\alpha\right)\Big]+\Big[\mathbf{B}\left((a+\eta(b,a))^{\rho(\rho-1)};\frac{j-m-r}{\rho}+1,\alpha\right)$$
$$\left.\left.-\mathbf{B}\left(\left(\frac{2a+\eta(b,a)}{2(a+\eta(b,a))}\right)^{\rho};\frac{j-m-r}{\rho}+1,\alpha\right)\right](-1)^{j+1}(a+\eta(b,a))^{\rho\alpha+j-m-r}\right\}\Bigg|$$
$$\le M\frac{[(2a+\eta(b,a))^{\rho}-(2a)^{\rho}]^{j+\beta}+[(2(a+\eta(b,a)))^{\rho}-(2a+\eta(b,a))^{\rho}]^{j+\beta}}{\sqrt[p]{(p\alpha\rho+pr\rho-pn+1)}\ \eta(b,a)}$$
$$\times\sqrt[q]{\frac{\rho^{jq-1}}{2^{\rho qj+\rho q\beta+1}}\ \left[{}^{\rho}B\left(\frac{\rho}{2}+1,\frac{1}{2}\right)+{}^{\rho}B\left(1-\frac{\rho}{2},\frac{3}{2}\right)\right]}.$$

Proposition 3. *Let $f:\mathbf{R}\to\mathbf{R}^+$ be defined by $f(x)=x^n$ for $n\in\mathbf{R}$, obviously f is $MT-$convex and hence from inequality* (3.6), *the following inequality holds:*

$$\Bigg|\frac{2\ \rho^{2j+\beta-2}}{[\eta(b,a)]^{2-j-\beta}}\Gamma(r)[L_{\rho-1}^{\rho-1}(a+\eta(b,a),a)]^{j+\beta-1}+\sum_{k=1}^{j-1}\prod_{1\le i\le k}\frac{\rho(\alpha+r+1-i)-n}{[\eta(b,a)]^{k+2-j-\beta}}$$
$$\times\frac{\Gamma(r-k)}{\rho^{2+\beta-2k-2j}}(1+(-1)^k)[L_{\rho-1}^{\rho-1}(a+\eta(b,a),a)]^{j+\beta-k-1}+\prod_{1\le i\le j}(-1)^j$$
$$\times\frac{\rho(\alpha+r+1-i)-n}{\alpha\ \rho^{2-\alpha}\ \eta(b,a)}\Gamma(\alpha)\Gamma(r-j+1)\left\{(1+(-1)^{j+1})((a+\eta(b,a))^{\rho}-a^{\rho})^{\alpha}\right.$$
$$\left.-2(a^{\rho^2}-a^{\rho})^{\alpha}-2(-1)^{j+1}((a+\eta(b,a))^{\rho}-(a+\eta(b,a))^{\rho^2})^{\alpha}\right\}\Bigg|$$
$$\le 2M\frac{[(a+\eta(b,a))^{\rho}-a^{\rho}]^{j+\beta}}{\eta(b,a)\sqrt[p]{p\alpha\rho+pr\rho-np+1}}\sqrt[q]{\frac{\rho^{jq-1}}{2^{1-q}}\left[{}^{\rho}B\left(\frac{\rho}{2}+1,\frac{1}{2}\right)+{}^{\rho}B\left(1-\frac{\rho}{2},\frac{3}{2}\right)\right]}.$$

Remark 2. *On letting $\rho,n,r,j,\alpha\to1$ and $\eta(b,a)=b-a$,*

- *for $s\to1$ Proposition* 1 *reduces to [18, Proposition 2].*
- *Proposition* 3 *reduces to [18, Proposition 3].*

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