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On degree of approximation of Fourier series of functions in Besov Space using Nörlund mean

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Abstract

In the present article, we have established a result on degree of approximation of function in the Besov space by (N, r_n)*- mean of Trigonometric Fourier series.*

Key Words: *Degree of approximation, Banach space, Hölder space, Besov space, Trigonometric Fourier series,* (N, r_n)*-summability mean.*

MSC (2020): *41A25, 42A24.*

1. Introduction

The concept of approximating a function is due to the great mathematician Weierstrass. To minimize the error in the degree of approximation, different summability methods of Fourier series were introduced. For study of degree of approximation of problems the natural way to proceed is to consider few restrictions on some modulus of smoothness in Hölder space(H_α and $H_{(\alpha,p)}$ spaces). However, for generalized Lipschitz class ($Lip(\alpha, p)$) there is no such restriction on 'α'. we required a finer scale of smoothness than is provided by Lipschitz class. In the mean time, for each $\alpha \geq 0$, Besov developped a remarkable technique for restricting modulus of smoothness by introducing one more parameter. The degree of approximation of functions belonging to Lipschitz class have been studied by the researchers (see [8-12] and [19]), Hölder space have been studied by the researchers (see [2-3],[7],[13],[16-17] and [20]) and Zygmund class have been studied by the researchers (see [5-6],[14-15],[18] and [22]). This motivated us to establish a result on degree of approximation of Fourier series of functions in Besov space using Nörlund mean.

2. Definitions and Notations

Let h be a function, which is periodic in $[0, 2\pi]$ such that $\int_0^{2\pi} |h(x)|^p dx < \infty$.

Let us denote

$$L_p[0, 2\pi] = \left\{h : [0, 2\pi] \to R : \int_0^{2\pi} |h(x)|^p dx < \infty\right\}, p \geq 1.$$

The Fourier series of $h(x)$ is given by

$$\sum_{n=0}^{\infty} u_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n cosnx + b_n sinnx\right) \tag{2.1}$$

When $0 < p < 1$, we can still regard (2.1) as the Fourier series of h. The kth order modulus of smoothness of a function $h : A \to R$ is defined by [23]

$$w_k(h, u) = \sup_{0<g\leq u} = \{sup.|\Delta_g^k(h, y)| : y, y + kg \in A\}, u \geq 0, \tag{2.2}$$

where

$$\Delta_g^k(h,y)=\sum_{i=0}^{k}(-1)^{k-1}C(k,i)\ h(y+ig),k\in N$$

and

$$A=R,R+[a,b]\subset R$$

The kth order modulus of smoothness of $h\in L_p(A),0<p\leq\infty$ is defined by

$$w_k(h,u)_p=\sup_{0<g\leq u}\|\Delta_g^k(h,.)\|_p,u\geq 0. \tag{2.3}$$

Let $\alpha>0$ be given and $k=[\alpha]+1$. Then for $0<p$ and $q\leq\infty$, the Besov space (see[21]) $B_q^\alpha(L_p)$ is defined as

$$B_q^\alpha(L_p)=\{h\in L_p:|h|_{B_q^\alpha(L_p)}=\|w_k(h,.)\|_{(\alpha,q)}\text{ is finite}\} \tag{2.4}$$

where

$$\|w_k(h,.)\|_{(\alpha,q)}=\left\{\int_0^\infty(u^{-\alpha}w_k(h,u)_p)^q\frac{du}{u}\right\}^{\frac{1}{q}},\text{for }0<q<\infty$$

and

$$\|w_k(h,.)\|_{(\alpha,q)}=\sup_{u>0}u^{-\alpha}w_k(h,u)_p,\text{for }q=\infty.$$

Clearly, $\|w_k(h,.)\|_{(\alpha,q)}$ is a seminorm (see[21]) if $1\leq p,\ q\leq\infty$. The Besov norm for $B_q^\alpha(L_p)$ is

$$\|h\|_{B_q^\alpha(L_p)}=\|h\|_p+\|w_k(h,.)\|_{(\alpha,q)} \tag{2.5}$$

Clearly, for fixed α and p

$$B_q^\alpha(L_p)\subset B_{q_1}^\alpha(L_p),q<q_1.$$

For fixed p and q

$$B_q^\alpha(L_p)\subset B_q^\beta(L_p),\beta<\alpha.$$

And for fixed α and q

$$B_q^\alpha(L_p) \subset B_q^\alpha(L_{p_1}), p_1 < p.$$

Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$ and $\{r_n\}$ represents the sequence of non negative integers such that

$$R_n = \sum_{k=0}^{n} r_k \to \infty \text{ as } n \to \infty,$$

then the (N, r_n) mean of $\{s_n\}$ generated by the sequence $\{r_n\}$ is given by

$$\tau_n^N = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} s_k, n = 0, 1, 2, ...$$

It is known that, (N, r_n) method is regular. (see [4]).
Let $S_k\Big(h; x\Big)$ denotes the k-th partial sum of the Fourier series (2.1). It is known (see [24])that

$$S_k\Big(h; x\Big) - h(x) = \frac{1}{\pi} \int_0^\pi \phi(x, u)\ D_k(u) du \tag{2.6}$$

where

$$D_k(u) = \frac{sin\Big(k + \frac{1}{2}\Big)u}{2sin\frac{u}{2}}$$

is the Drichlet's kernel and

$$\phi(x, u) = h(x + u) + h(x - u) - h(x) \tag{2.7}$$

Let $\sigma_n(h; x)$ be the (N, r_n) mean of the Fourier series (2.1) then

$$\sigma_n(h; x) = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} S_k(h; x) \tag{2.8}$$

We know that (from [24])

$$l_n(x) = \sigma_n(h; x) - h(x) = \frac{1}{\pi} \int_0^\pi \phi(x, u) K_n(u)\ du \tag{2.9}$$

where

$$K_n(u) = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} D_k(u) \tag{2.10}$$

We have also used the following additional notations in the rest part of our present article.

$$\Phi(x, u, v) = \phi(x + u, v) - \phi(x, v), \text{ for } 0 < \alpha < 1 \tag{2.11}$$

and

$$\Phi(x, u, v) = \phi(x + u, v) + \phi(x - u, v) - 2\phi(x, v), \text{ for } 1 \leq \alpha < 2 \tag{2.12}$$

For $k = [\alpha] + 1, p \geq 1$; we have

$$w_k(h, u)_p = w_1(h, u)_p, \text{ for } 0 < \alpha < 1 \tag{2.13}$$

and

$$w_k(h, u)_p = w_2(h, u)_p, \text{ for } 1 \leq \alpha < 2 \tag{2.14}$$

We write

$$L_n(x, u) = l_n(x + u) - l_n(x), \text{ for } 0 < \alpha < 1 \tag{2.15}$$

and

$$L_n(x, u) = l_n(x + u) + l_n(x - u) - 2l_n(x), \text{ for } 1 \leq \alpha < 2 \tag{2.16}$$

By using (2.9),(2.11) and (2.12), we have

$$L_n(x, u) = \frac{1}{\pi} \int_0^{\pi} \Phi(x, u, v)\ K_n(v)\ dv \tag{2.17}$$

Using the definition of $w_k(h, u)_p$, (2.15) and (2.16), we have

$$w_k(l_n, u)_p = \|L_n(., u)\|_p \tag{2.18}$$

3. Known Results

Using Fejer mean of Fourier series Prossdorff [20] first obtained the following result on approximation of functions in H_α space.

Theorem 3.1

Let $f \in H_\alpha(0 < \alpha \leq 1)$ *and* $0 \leq \beta < \alpha \leq 1$. *Then*

$$\|\sigma_n(f) - f\|_\beta = O\left(\frac{1}{n^{\alpha-\beta}}\right), \text{for } 0 < \alpha < 1$$

and

$$\|\sigma_n(f) - f\|_\beta = O\left(\left\{\frac{log\ n}{n}\right\}^{1-\beta}\right), \text{for } \alpha = 1,$$

where $\sigma_n(f)$ is the Fejer mean of the Fourier series of f.
Alexists [1] obtained a result by taking $\beta = 0$ in theorem-3.1.
Later Das, Ghosh and Ray [3] further generalized the work by studying the problem for functions in $H(\alpha, p)$ space $(0 < \alpha \leq 1,\ p \geq 1)$ by the matrix mean of the Fourier series.
In the present paper, we propose to study a result on the degree of approximation of Fourier series of functions in Besov space using Nörlund mean.

4. Main Theorem

Theorem 4.1
Let $0 \leq \beta < \alpha < 2$. *If* $h \in B^\alpha_q(L_p), p \geq 1$, *then*

$$\|l_n(.)\|_{B^\beta_q(L_p)} = O\left(\frac{1}{n^\alpha}\right) + O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right),\ \textit{for } 1 < q < \infty$$

and

$$\|l_n(.)\|_{B^\beta_q(L_p)} = O\left(\frac{1}{n^\alpha}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right),\ \textit{for } q = \infty$$

We require the following lemmas to prove our main theorem:

5. Lemmas

Lemma 5.1
Let $1 \leq p \leq \infty$ and $0 < \alpha < 2$. If $h \in L_p[0, 2\pi]$, then for $0 < u,\ v \leq \pi$

$$(i) \|\Phi(., u, v)\|_p \leq 4w_k(h, u)_p$$
$$(ii) \|\Phi(., u, v)\|_p \leq 4w_k(h, v)_p$$
$$(iii) \|\Phi(u)\|_p \leq 2w_k(h, u)_p$$

Lemma 5.2
Let $0 < \alpha < 2$. Suppose that $0 \leq \beta < \alpha$. If $h \in B_q^\alpha(L_p), p \geq 1, 1 < q < \infty$, then

$$(i) \int_0^\pi |K_n(v)| \left(\int_0^\pi \frac{\|\Phi(.,u,v)\|_p^q}{u^{\beta q}} \frac{du}{u} \right)^{\frac{1}{q}} dv = O(1) \left\{ \int_0^\pi \left(v^{\alpha-\beta} |K_n(v)| \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}}$$

$$(ii) \int_0^\pi |K_n(v)| \left(\int_0^\pi \frac{\|\Phi(.,u,v)\|_p^q}{u^{\beta q}} \frac{du}{u} \right)^{\frac{1}{q}} dv = O(1) \left\{ \int_0^\pi \left(v^{\alpha-\beta+\frac{1}{q}} |K_n(v)| \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}}$$

Lemma 5.3
Let $0 < \alpha < 2$. Suppose that $0 \leq \beta < \alpha$. If $h \in B_q^\alpha(L_p), p \geq 1$ and $q = \infty$ then

$$\sup_{0 < u \leq v \leq \pi} u^{-\beta} \|\Phi(., u, v)\|_p = O\left(v^{\alpha-\beta}\right).$$

Lemma 5.4
Let (N, r_n) kernel of the Fourier series be as defined in (10). Then

$$|K_n(v)| = O(n) \text{ for } 0 \leq v \leq \frac{\pi}{n}$$
and
$$|K_n(v)| = O\left(\frac{1}{v}\right) \text{ for } \frac{\pi}{n} \leq v \leq \pi$$

6. Proof of the Lemmas

Proof of Lemma-5.1
For $0 < \alpha < 1$, $k = [\alpha] + 1 = 1$.
By virtue of (2.11),

$$\Phi(x, u, v) = \phi(x + u, v) - \phi(x, v)$$

can be written as

$$\begin{aligned}\Phi(x,u,v) &= \{h(x+u+v)-h(x+v)\}\\ &\quad+\{h(x+u-v)-h(x-v)\}\\ &\quad-2\{h(x+u)-h(x)\}\end{aligned} \tag{6.1}$$

$$\begin{aligned}\Phi(x,u,v) &= \{h(x+u+v)-h(x+u)\}\\ &\quad+\{h(x-v+u)-h(x+u)\}-\{h(x+v)-h(x)\}\\ &\quad-\{h(x-v)-h(x)\}\end{aligned} \tag{6.2}$$

Applying Minkowski's inequality to (6.1) and (6.2), we get for $p \geq 1$

$$\|\Phi(.,u,v)\|_p \leq 4w_k(h,u)_p$$

Which completes the proof of (i).
Again, For $1 < \alpha < 2$, $k = [\alpha] + 1 = 2$.
By virtue of (2.12),

$$\Phi(x,u,v) = \phi(x+u,v) + \phi(x-u,v) - 2\phi(x,v)$$

can be written as

$$\begin{aligned}\Phi(x,u,v) &= \{h(x+u+v)+h(x+u-v)-2h(x+u)\}\\ &\quad+\{h(x-u+v)+h(x-u-v)-2h(x-u)\}\\ &\quad-2\{h(x+v)+h(x-v)-2h(x)\}\end{aligned} \tag{6.3}$$

$$\begin{aligned}\Phi(x,u,v) &= \{h(x+u+v)+h(x-u+v)-2h(x+v)\}\\ &\quad+\{h(x+u-v)+h(x-u-v)-2h(x-v)\}\\ &\quad-2\{h(x+u)+h(x-u)-2h(x)\}\end{aligned} \tag{6.4}$$

If we apply Minkowski's inequality to (6.3) and (6.4), we get

$$\|\Phi(.,u,v)\|_p \leq 4w_k(h,v)_p$$

Which completes the proof of (ii).
We have omitted the proof of (iii) as it is trivial.

Proof of Lemma-5.2
For the proof of (i), applying lemma-5.1(i), we have

$$\int_0^{\pi} |K_n(v)| \Big(\int_0^{\pi} \frac{\|\Phi(.,u,v)\|_p^q}{u^{\beta q}} \frac{du}{u} \Big)^{\frac{1}{q}} dv$$

$$= O(1) \int_0^{\pi} |K_n(v)| \Big\{ \int_0^{\pi} \Big(\frac{w_k(h,u)_p}{u^{\alpha}} \Big)^q u^{(\alpha-\beta)q} \frac{du}{u} \Big\}^{\frac{1}{q}} dv$$

$$= O(1) \int_0^{\pi} |K_n(v)| v^{\alpha-\beta} \Big\{ \int_0^{\pi} \frac{w_k(h,u)_p}{u^{\alpha}} \frac{du}{u} \Big\}^{\frac{1}{q}} dv$$

$$= O(1) \int_0^{\pi} |K_n(v)| v^{\alpha-\beta} \, dv$$

(By definition of Besov space and 2nd mean value theorem)

$$= O(1) \Big\{ \int_0^{\pi} \Big(|K_n(v)| v^{\alpha-\beta} \Big)^{\frac{q}{q-1}} dv \Big\}^{1-\frac{1}{q}} \Big\{ \int_0^{\pi} dv \Big\}^{\frac{1}{q}}$$

(By applying Hölder's inequality)

$$= O(1) \Big\{ \int_0^{\pi} \Big(|K_n(v)| v^{\alpha-\beta} \Big)^{\frac{q}{q-1}} dv \Big\}^{1-\frac{1}{q}}$$

Which completes the proof of (i).
For the proof of (ii), applying lemma-5.1(ii), we have

$$\int_0^{\pi} |K_n(v)| \Big(\int_0^{\pi} \frac{\|\Phi(.,u,v)\|_p^q}{u^{\beta q}} \frac{du}{u} \Big)^{\frac{1}{q}} dv$$

$$= O(1) \int_0^{\pi} |K_n(v)| w_k(h,v)_p \Big\{ \int_0^{\pi} \frac{du}{u^{\beta q+1}} \Big\}^{\frac{1}{q}} dv$$

$$= O(1) \int_0^{\pi} |K_n(v)| \; w_k(h,v)_p \; v^{-\beta} \, dv$$

$$= O(1) \int_0^{\pi} |K_n(v)| v^{\alpha-\beta-\frac{1}{q}} \Big\{ \frac{w_k(h,v)_p}{v^{\alpha+\frac{1}{q}}} \Big\} dv$$

$$= O(1) \Big\{ \int_0^{\pi} \Big(\frac{w_k(h,v)_p}{v^{\alpha}} \Big)^q \frac{dv}{v} \Big\}^{\frac{1}{q}} \Big\{ \int_0^{\pi} \Big(|K_n(v)| v^{\alpha-\beta-\frac{1}{q}} \, dv \Big) \Big\}^{1-\frac{1}{q}}$$

(By using Hölder's inequality)

$$= O(1) \Big\{ \int_0^{\pi} \Big(|K_n(v)| v^{\alpha-\beta-\frac{1}{q}} \, dv \Big) \Big\}^{1-\frac{1}{q}}$$

(By using the definition of Besov space)

This completes the proof of (ii).

Proof of Lemma-5.3
For $0 < u \leq v \leq \pi$, using lemma-5.1(i), we have

$$\begin{aligned}\sup_{0<u\leq v\leq\pi} u^{-\beta}\|\Phi(.,u,v)\|_p &= \sup_{0<u\leq v\leq\pi} u^{\alpha-\beta}\left\{u^{-\alpha}\|\Phi(.,u,v)\|_p\right\}\\ &\leq 4v^{\alpha-\beta}\sup_u\left(u^{-\alpha}w_k(h,u)_p\right)\\ &= 4O\left(v^{\alpha-\beta}\right) \text{ (by the hypothesis)}\end{aligned}$$

Again, for $0 < v \leq u \leq \pi$, using lemma-5.1(ii), we have

$$\begin{aligned}\sup_{0<v\leq u\leq\pi} u^{-\beta}\|\Phi(.,u,v)\|_p &\leq 4w_k(h,v)_p \sup_{0<v\leq u\leq\pi} u^{-\beta}\\ &\leq 4v^{\alpha-\beta}\sup_v\left(v^{-\alpha}w_k(h,v)_p\right)\\ &= 4O\left(v^{\alpha-\beta}\right) \text{ (by the hypothesis)}\end{aligned}$$

This completes the proof of the Lemma 5.3.

Proof of Lemma-5.4
For $0 \leq v \leq \frac{\pi}{n}$ and $sin\ nv = n\ sin\ v$, then

$$\begin{aligned}|K_n(v)| &= \left|\frac{1}{R_n}\sum_{k=0}^{n} r_{n-k}\frac{sin\left(k+\frac{1}{2}\right)v}{2sin\ \frac{v}{2}}\right|\\ &= \left|\frac{1}{R_n}\sum_{k=0}^{n} r_{n-k}\frac{(2k+1)sin\ \frac{v}{2}}{2sin\ \frac{v}{2}}\right|\\ &= \left|\frac{(2n+1)}{2R_n}\sum_{k=0}^{n} r_{n-k}\right|\\ &= O(n)\end{aligned}$$

Again, for $\frac{\pi}{n} \leq v \leq \pi$, $sin\frac{v}{2} \geq \frac{v}{\pi}$ and $sin\ nv \leq 1$, we have

$$\begin{aligned} |K_n(v)| &= \left| \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} \frac{sin\left(k+\frac{1}{2}\right)v}{2sin\ \frac{v}{2}} \right| \\ &\leq \left| \frac{1}{2R_n} \sum_{k=0}^{n} r_{n-k} \frac{\pi}{v} \right| \\ &= \left| \frac{\pi}{2vR_n} \sum_{k=0}^{n} r_{n-k} \right| \\ &= O\left(\frac{1}{v}\right) \end{aligned}$$

7. Proof of Main Theorem

Proof of Theorem-4.1

We first consider the case $1 < q < \infty$, we have for $p \geq 1$, $0 \leq \beta < \alpha < 2$

$$\|l_n(.)\|_{B_q^\beta L(P)} = \|l_n(.)\|_p + \|w_k(l_n, .)\|_{(\beta,q)} \tag{7.1}$$

Applying lemma-5.1(iii) in (2.9), we get

$$\|l_n(.)\|_p \leq \frac{1}{\pi} \int_0^\pi \|\phi.(v)\|_p |k_n(v)| dv$$

$$\Rightarrow \|l_n(.)\|_p \leq \frac{2}{\pi} \int_0^\pi |k_n(v)| w_k(h, v)_p\ dv \tag{7.2}$$

Applying Hölder's inequality, we have

$$\begin{aligned} \|l_n(.)\|_p \ &\leq\ \frac{2}{\pi} \left\{ \int_0^\pi \left(|k_n(v)| v^{\alpha+\frac{1}{q}} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}} \left\{ \int_0^\pi \left(\frac{w_k(h,v)_p}{v^{\alpha+\frac{1}{q}}} \right)^q dv \right\}^{\frac{1}{q}} \\ &= O(1) \left\{ \int_0^\pi \left(|k_n(v)| v^{\alpha+\frac{1}{q}} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}} \text{(by defn. of Besov space)} \\ &= O(1) \left[\left\{ \int_0^{\frac{\pi}{n}} \left(|K_n(v)| v^{\alpha+\frac{1}{q}} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}} \right. \\ &\quad \left. + \left\{ \int_{\frac{\pi}{n}}^\pi \left(|K_n(v)| v^{\alpha+\frac{1}{q}} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}} \right] \\ &= O(1)[I + J] \ \ \text{(say)} \end{aligned} \tag{7.3}$$

Now,

$$I = \left\{ \int_0^{\frac{\pi}{n}} \left(|K_n(v)| v^{\alpha+\frac{1}{q}} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}}$$

$$= O(n) \left\{ \int_0^{\frac{\pi}{n}} \left(v^{\alpha+\frac{1}{q}} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}}$$

$$= O\left(\frac{1}{n^\alpha}\right) \tag{7.4}$$

Again,

$$J = \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(|K_n(v)| v^{\alpha+\frac{1}{q}} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}}$$

$$= \left\{ \int_{\frac{\pi}{n}}^{\pi} \left(v^{\alpha+\frac{1}{q}-1} \right)^{\frac{q}{q-1}} dv \right\}^{1-\frac{1}{q}}$$

$$= \left\{ \int_{\frac{\pi}{n}}^{\pi} v^{\frac{\alpha\ q}{q-1}-1} dv \right\}^{\frac{q-1}{q}}$$

$$= O\left(\frac{1}{n^\alpha}\right) \tag{7.5}$$

Now,

$$\begin{aligned}
\|w_k(l_n, .)\|_{(\beta,q)} &= \left\{ \int_0^{\pi} \left(\frac{w_k(l_n, u)_p}{u^\beta} \right)^q \frac{du}{u} \right\}^{\frac{1}{q}} \\
&= \left\{ \int_0^{\pi} \left(\frac{\|L_n(., u)\|_p}{u^\beta} \right)^q \frac{du}{u} \right\}^{\frac{1}{q}} \\
&= \left\{ \int_0^{\pi} \left(\int_0^{\pi} |L_n(x, u)|^p dx \right)^{\frac{q}{p}} \frac{du}{u^{u^{\beta q+1}}} \right\}^{\frac{1}{q}} \\
&= \left\{ \int_0^{\pi} \left(\int_0^{\pi} \left| \frac{1}{\pi} \int_0^{\pi} \Phi(x, u, v)\ K_n(v)\ dv \right|^p dx \right)^{\frac{q}{p}} \frac{du}{u^{u^{\beta q+1}}} \right\}^{\frac{1}{q}}
\end{aligned}$$

By repeated application of generalized Minkowski's inequality, we have

$$\begin{aligned}
\|w_k(l_n, .)\|_{(\beta,q)} &\leq \frac{1}{\pi} \left[\int_0^{\pi} \left\{ \int_0^{\pi} \left(\int_0^{\pi} |\Phi(x, u, v)|^p\ |K_n(v)|^p\ dx \right)^{\frac{1}{p}} dv \right\}^q \frac{du}{u^{u^{\beta q+1}}} \right]^{\frac{1}{q}} \\
&= \frac{1}{\pi} \left[\int_0^{\pi} \left\{ \int_0^{\pi} |K_n(v)|\ \|\Phi(., u, v)\|_p\ dv \right\}^q \frac{du}{u^{u^{\beta q+1}}} \right]^{\frac{1}{q}}
\end{aligned}$$

$$\leq \frac{1}{\pi}\int_0^{\pi}\left(\int_0^{\pi}|K_n(v)|^q\ \|\Phi(.,u,v)\|_p^q\frac{du}{u^{\beta q+1}}\right)^{\frac{1}{q}}dv$$
$$= \frac{1}{\pi}\int_0^{\pi}\left\{\left(\int_0^{v}+\int_v^{\pi}\right)\frac{\|\Phi(.,u,v)\|_p^q\ du}{u^{\beta q+1}}\right\}|K_n(v)|\ dv$$
$$\leq \frac{1}{\pi}\int_0^{\pi}\left\{\int_0^{v}\frac{\|\Phi(.,u,v)\|_p^q\ du}{u^{\beta q+1}}\right\}|K_n(v)|\ dv$$
$$+\frac{1}{\pi}\int_0^{\pi}\left\{\int_v^{\pi}\frac{\|\Phi(.,u,v)\|_p^q\ du}{u^{\beta q+1}}\right\}|K_n(v)|\ dv$$

Applying lemma-5.2, we have

$$\|w_k(l_n,.)\|_{(\beta,q)} = O(1)\Bigg[\left\{\int_0^{\pi}\left(|K_n(v)|\ v^{\alpha-\beta}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$
$$+\left\{\int_0^{\pi}\left(|K_n(v)|\ v^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}\Bigg]$$
$$= O(1)[I^{'}+J'],\ \text{(say)} \tag{7.6}$$

Now,

$$I^{'} = \left\{\int_0^{\pi}\left(|K_n(v)|\ v^{\alpha-\beta}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$
$$\leq \left\{\int_0^{\frac{\pi}{n}}\left(|K_n(v)|\ v^{\alpha-\beta}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$
$$+\left\{\int_{\frac{\pi}{n}}^{\pi}\left(|K_n(v)|\ v^{\alpha-\beta}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$
$$= \left\{O(n)\int_0^{\frac{\pi}{n}}v^{(\alpha-\beta)\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}+\left\{O(1)\int_{\frac{\pi}{n}}^{\pi}\left(v^{\alpha-\beta-1}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$
$$= O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) \tag{7.7}$$

Similarly,

$$J^{'} = \left\{\int_0^{\pi}\left(|K_n(v)|\ v^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$
$$\leq \left\{\int_0^{\frac{\pi}{n}}\left(|K_n(v)|\ v^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$

$$+\left\{\int_{\frac{\pi}{n}}^{\pi}\left(|K_n(v)|\ v^{\alpha-\beta+\frac{1}{q}}\right)^{\frac{q}{q-1}}dv\right\}^{1-\frac{1}{q}}$$

$$= O(n)\left(\int_0^{\frac{\pi}{n}} v^{\frac{q}{q-1}(\alpha-\beta+1)-1}dv\right)^{\frac{q-1}{q}} + o(1)\left(\int_0^{\frac{\pi}{n}} v^{\frac{q}{q-1}(\alpha-\beta)-1}dv\right)^{\frac{q-1}{q}}$$

$$= O\left(\frac{1}{n^{\alpha-\beta}}\right) \tag{7.8}$$

Using (7.6),(7.7) and (7.8), we get

$$\|w_k(l_n,.)\|_{(\beta,q)} = O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right) \tag{7.9}$$

By (7.4),(7.5) and (7.9), we get

$$\|l_n(.)\|_{B_q^\beta L(P)} = O\left(\frac{1}{n^{\alpha}}\right) + O\left(\frac{1}{n^{\alpha-\beta-\frac{1}{q}}}\right) + O\left(\frac{1}{n^{\alpha-\beta}}\right) \tag{7.10}$$

Now, we consider the case when $q = \infty$.

$$\|w_k(l_n,.)\|_{(\beta,\infty)} = \sup_u \frac{\|L_n(.,u)\|_p}{u^\beta}$$

$$= \sup_{u>0} \frac{u^{-\beta}}{\pi}\left[\int_0^\pi \left|\int_0^\pi \Phi(x,u,v)K_n(v)\ dv\right|^p dx\right]^{\frac{1}{p}}$$

Applying generalized Minkowski's inequality, we get

$$\|w_k(l_n,.)\|_{(\beta,\infty)} \le \sup_{u>0} \frac{u^{-\beta}}{\pi}\int_0^\pi \left\{\int_0^\pi |\Phi(x,u,v)|^p |K_n(v)|^p\ dx\right\}^{\frac{1}{p}} dv$$

$$= \sup_{u>0} \frac{u^{-\beta}}{\pi}\int_0^\pi |K_n(v)|\ \|\Phi(.,u,v)\|_p\ dv$$

$$\le \frac{1}{\pi}\int_0^\pi |K_n(v)| \left\{\sup_{u>0} u^{-\beta}\|\Phi(.,u,v)\|_p\right\} dv$$

$$= O(1)\int_0^\pi v^{\alpha-\beta}|K_n(v)|\ dv \text{ (by lemma-5.3)}$$

$$= O(1)\left[\int_0^{\frac{\pi}{n}} v^{\alpha-\beta}|K_n(v)|\ dv + \int_{\frac{\pi}{n}}^\pi v^{\alpha-\beta}|K_n(v)|\ dv\right]$$

$$= O(n)\int_0^{\frac{\pi}{n}} v^{\alpha-\beta}\ dv + O(1)\int_{\frac{\pi}{n}}^\pi v^{\alpha-\beta-1}\ dv \text{ (by lemma-5.4)}$$

$$= O\left(\frac{1}{n^{\alpha-\beta}}\right) \tag{7.11}$$

Also, for $q = \infty$,

$$\begin{aligned}
\|l_n(.)\|_p &\leq \frac{2}{\pi}\int_0^{\pi} |K_n(v)| w_k(h,v)_p \, dv \\
&= O(1)\int_0^{\pi} |K_n(v)| \; v^{\alpha} \, dv \text{ (by the hypothesis)} \\
&= O(1)\Big[\int_0^{\frac{\pi}{n}} |K_n(v)| \; v^{\alpha} \, dv + \int_{\frac{\pi}{n}}^{\pi} |K_n(v)| \; v^{\alpha} \, dv\Big] \\
&= O(n)\int_0^{\frac{\pi}{n}} \; v^{\alpha} \, dv + O(1)\int_{\frac{\pi}{n}}^{\pi} \; v^{\alpha-1} \, dv \\
&= O\Big(\frac{1}{n^{\alpha}}\Big) \qquad (7.12)
\end{aligned}$$

From (7.11) and (7.12), we get

$$\begin{aligned}
\|l_n(.)\|_{B^{\beta}_{\infty}L(P)} &= \|l_n(.)\|_p + \|w_k(l_n,.)\|_{(\beta,\infty)} \\
&= O\Big(\frac{1}{n^{\alpha}}\Big) + O\Big(\frac{1}{n^{\alpha-\beta}}\Big) \qquad (7.13)
\end{aligned}$$

This completes the proof of Theorem 4.1.

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