The edge-to-edge geodetic domination number of a graph

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Abstract:

Let $G = (V, E)$ be a connected graph with at least three vertices. A set $S \subseteq E(G)$ is called an edge-to-edge geodetic dominating set of G if S is both an edge-to-edge geodetic set of G and an edge dominating set of G . The edge-to-edge geodetic domination number $\gamma_{gee}(G)$ of G is the minimum cardinality of its edge-to-edge geodetic dominating sets. Some general properties satisfied by this concept are studied. Connected graphs of size m with edge-to-edge geodetic domination number 2 or m or $m - 1$ are characterized. We proved that if G is a connected graph of size $m \geq 4$ and $\bar{G}$ is also connected, then $4 \leq \gamma_{gee}(G) + \gamma_{gee}(\bar{G}) \leq 2m - 2$. Moreover we characterized graphs for which the lower and the upper bounds are sharp. It is shown that, for every pair of positive integers a, b with $2 \leq a \leq b$, there exists a connected graph G with $\gamma_{ee}(G) = a$ and $\gamma_{gee}(G) = b$. Also it is shown that, for every pair of positive integers a and b with $2 < a \leq b$, there exists a connected graph G with $\gamma_e(G) = a$ and $\gamma_{gee}(G) = b$, where $\gamma_e(G)$ is the edge domination number of G and $\gamma_{ee}(G)$ is the edge-to-edge geodetic number of G .

Keywords: Edge-to-edge geodetic domination number; Edge-to-edge geodetic number; Edge domination number; Domination number; Geodetic number

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [9]. Two vertices u and v are said to be *adjacent* in G if $uv \in E(G)$. The *neighborhood* $N(v)$ of the vertex v in G is the set of vertices adjacent to v . The degree of the vertex v is $\deg(v) = |N(v)|$. If $e = \{u, v\}$ is an edge of a graph G with $\deg(u) = 1$ and $\deg(v) > 1$, then we call e an *end edge*, u a *leaf* and v a *support vertex*. For any connected graph G , a vertex $v \in V(G)$ is called a *cut vertex* of G if $V(G) - v$ is disconnected. The *subgraph induced* by set S of vertices of a graph G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{uv \in E(G) : u, v \in S\}$. A vertex v is called an *extreme vertex* of G if $\langle N(v) \rangle$ is complete. An edge e is called an *extreme edge* of G if one end of e is an extreme vertex of G . Let $c(G)$ be the length of the longest cycle in G . Let $\lceil x \rceil$ denotes the smallest integer greater than or equal to x .

A set of vertices D in a graph G is called a *dominating set* of G if each vertex of $V(G) - D$ is adjacent to some vertex of D . The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G . For references on domination parameters in graphs see [9,11,12,13,19,20,26]. Two edges are *adjacent* if they have a common vertex. A set of edges S of G is called an *edge dominating set* of G if every edge of $E(G) - S$ is adjacent to an element of S . The *edge domination number* $\gamma_e(G)$ of G is the minimum cardinality of an edge dominating set of G . An edge dominating set of cardinality $\gamma_e(G)$ is called a γ_e -*set* of G . An *independent edge set* of a graph G is a subset of the edges such that no two edges in the subset share a vertex of G . An edge dominating set S is said to be an *independent edge dominating set* of G if S is an independent edge set of G . The edge domination number was studied in [2,3,8,16,17,18,23,28].

A vertex x is an *internal vertex* of an u - v path P if x is a vertex of P and $x \neq u, v$. An edge e of G is an *internal edge* of an u - v path P if e is an edge of P with both of its ends are in P . The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u - v path in G . An u - v path of length $d(u, v)$ is called an u - v *geodesic*. A vertex x is said to lie on an u - v geodesic P if x is a vertex of P including the vertices u and v . For a vertex v of G , the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v . The minimum eccentricity among the vertices of G is the *radius*, $\text{rad}(G)$ and the maximum eccentricity is its *diameter*, $\text{diam}(G)$. We denote $\text{rad}(G)$ by r and $\text{diam}(G)$ by d . A *star* is a tree with

diameter 2. A *double star* is a tree with diameter 3. The *closed interval* $I[u, v]$ consists of u, v and all vertices lying on some u - v geodesic of G . For a non-empty set $S \subseteq V(G)$, the set $I[S] = \cup_{u,v \in S} I[u, v]$ is the *closure* of S . A set $S \subseteq V(G)$ is called a *geodetic set* if $I[S] = V(G)$. Thus every vertex of G is contained in a geodesic joining some pair of vertices in S . The minimum cardinality of a geodetic set of G is called the *geodetic number* of G and is denoted by $g(G)$. A geodetic set of minimum cardinality is called *g -set* of G . For references on geodetic parameters in graphs see [4,5,6,7,10,14,15,22,23].

Let $G = (V, E)$ be a connected graph with at least three vertices. For subsets A and B of $V(G)$, the *distance* $d(A, B)$ is defined as $d(A, B) = \min\{d(x, y) : x \in A, y \in B\}$. An u - v path of length $d(A, B)$ is called an *A - B geodesic* joining the sets A and B , where $u \in A$ and $v \in B$. A vertex x is said to lie on an A - B geodesic if x is a vertex of an A - B geodesic [21]. A set $S \subseteq E(G)$ is called an *edge-to-vertex geodetic set* if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S . The *edge-to-vertex geodetic number* $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is called an *g_{ev} -set* of G . The edge-to-vertex geodetic number of a graph was studied in [21,24,25,27]. A set $S \subseteq E(G)$ is called an *edge-to-edge geodetic set* of G if every edge of G is an element of S or lies on a geodesic joining a pair of edges of S . The *edge-to-edge geodetic number* $g_{ee}(G)$ of G is the minimum cardinality of its edge-to-edge geodetic sets and any edge-to-edge geodetic set of cardinality $g_{ee}(G)$ is said to be a *g_{ee} -set* of G . This concept was studied in [1].

For $G = C_5$ with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $S_1 = \{v_1v_2, v_1v_5, v_3v_4\}$ is a g_{ee} -set of G and $S_2 = \{v_1v_2, v_4v_5\}$ is a γ_e -set of G so that $g_{ee}(G) = 3$ and $\gamma_e(G) = 2$. For $G = C_8$ with vertex set $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, $S_3 = \{v_1v_2, v_5v_6\}$ is a g_{ee} -set of G and $S_4 = \{v_1v_2, v_4v_5, v_7v_8\}$ is a γ_e -set of G so that $g_{ee}(G) = 2$ and $\gamma_e(G) = 3$. From this two examples we observed that S_2 is a γ_e -set of G but not g_{ee} -set of G and S_3 is a g_{ee} -set of G but not γ_e -set of G .

Domination in graphs has applications to several fields [20]. Domination arises in facility location problems, where the number of facilities (e.g., hospitals, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. Concepts related to geodesics also appear in the social sciences [7]. When members of a social group are represented by vertices, and particular relationships between members by edges of a graph, then the geodetic number can be considered as the smallest number n such that every member of a group

is contained in a minimal chain of relationships between chosen n members. Edge domination in graphs has some very interesting applications in a telephone switching network [28]. A telephone switching network built to route phone calls from incoming lines to outgoing trunks (we assume that a trunk can pass only one phone call at a time). The problem is to find the worst-case behavior of the network, i.e., the minimum number of routed calls when the network is saturated and no calls can be added. For this we construct a bipartite graph G by connecting a line to a trunk if and only if the line can be switched to the trunk. Then the problem is equivalent to finding the size of a minimum independent edge dominating set of G . The edge-to-vertex geodetic concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design [21]. In the case of designing the route for a shuttle, although all the vertices are covered by the shuttle when considering edge-to-vertex geodetic sets, some of the edges may be left out. This drawback is rectified in the case of edge-to-edge geodetic sets and hence considering edge-to-edge geodetic sets is more advantageous to the real life application of routing problem. In particular, the edge-to-edge geodetic sets are more useful than edge-to-vertex geodetic sets in the case of regulating and routing the goods vehicles to transport the commodities to important places. This is the motivation behind the study of edge-to-edge geodetic concepts. An edge-to-edge geodetic set can be used to improve the effectiveness in networks. Throughout the following G denotes a connected graph with at least three vertices.

2. The edge-to-edge geodetic domination number of a graph

It is seen that an edge dominating set is not in general an edge-to-edge geodetic set in a graph G . Also the converse is not valid in general. This has motivated us to study the new edge domination conception of edge-to-edge geodetic domination. We investigate those subsets of edges of a graph that are both an edge-to-edge geodetic set and an edge dominating set. We call these sets as edge-to-edge geodetic dominating sets. In this section, we look closely at the concept of edge-to-edge geodetic domination in a graph G , and obtain the edge-to-edge geodetic domination number of some families of graphs. Although the edge-to-edge geodetic domination number is greater than or equal to the edge-to-edge geodetic number for an arbitrary graph, the properties of the edge-to-edge geodetic geodetic

dominating sets and results regarding edge-to-edge geodetic domination number are quite different from that of edge-to-edge geodetic concepts.

Definition 2.1. Let $G = (V, E)$ be a connected graph with at least three vertices. A set $S \subseteq E(G)$ is called an *edge-to-edge geodetic dominating set* of G if S is both an edge-to-edge geodetic set and an edge dominating set. The *edge-to-edge geodetic domination number* $\gamma_{gee}(G)$ of G is the minimum cardinality of its edge-to-edge geodetic dominating sets and any edge-to-edge geodetic dominating set of minimum cardinality is said to be a γ_{gee} -set of G .

Example 2.2. For the graph G given in Figure 2.1, $\gamma_e(G) = 2$, $g_{ee}(G) = 2$ and $S_1 = \{v_1v_2, v_3v_4, v_5v_6\}$, $S_2 = \{v_1v_2, v_1v_3, v_5v_6\}$, $S_3 = \{v_1v_2, v_2v_3, v_5v_6\}$ and $S_4 = \{v_1v_2, v_4v_5, v_5v_6\}$ are the only four γ_{gee} -sets of G so that $\gamma_{gee}(G) = 3$.

Figure 2.1

Observation 2.3. (i) Each end edge of a connected graph G belongs to every edge-to-edge geodetic dominating set of G .

(ii) If G is a star $K_{1,m}$ with size $m \geq 2$, then $\gamma_{gee}(G) = m$.

(iii) If G is a double star with size $m \geq 3$, then $\gamma_{gee}(G) = m - 1$.

(iv) If G is a complete bipartite graph $K_{r,s}$ ($2 \leq r \leq s$), then $\gamma_{gee}(G) = s$.

(v) If G is a complete graph K_n ($n \geq 4$), then

$$\gamma_{gee}(G) = \begin{cases} \frac{n}{2}; & \text{if } n \text{ is even} \\ \frac{n+1}{2}; & \text{if } n \text{ is odd} \end{cases}$$

(vi) If G is a cycle C_n ($n \geq 4$), then $\gamma_{gee}(G) = \begin{cases} 3; & \text{if } n = 5 \\ \lceil \frac{n}{3} \rceil; & \text{if } n \neq 5 \end{cases}$

3. Some results on the edge-to-edge geodetic domination number of a graph

In this section, we look at some relationships between the edge-to-edge geodetic domination number and other parameters. Further some improved upper bounds for edge-to-edge geodetic domination number of a graph are given.

Observation 3.1. *If G is a connected graph with size $m \geq 2$, then $2 \leq \max(\gamma_e(G), g_{ee}(G)) \leq \gamma_{gee}(G) \leq m$.*

Theorem 3.2. *Let G be a connected graph with size $m \geq 2$. If $\gamma_{gee}(G) = 2$, then $d(e, f) \leq 2$ for every $e, f \in E(G)$.*

Proof. Let $S = \{e, f\}$ be a γ_{gee} -set of G . Let us assume that $e = uv$, $f = u'v'$ and $e - f$ geodesic P be $u = u_0, u_1, u_2, \dots, u_n = u'$, where u_0 is either u or v and u_n is either u' or v' . On the contrary assume that $d(e, f) \geq 3$ for some $e, f \in E(G)$. Which implies $n \geq 3$. Then the edge $u_1u_2 \in E(G) - S$ is not dominated by any edge of S . Hence S is not a geodetic dominating set of G , which is a contradiction. Therefore $d(e, f) \leq 2$ for every $e, f \in E(G)$. $\square$

Remark 3.3. *The converse of the Theorem 3.2 need not be true. For $G = C_5$, $d(e, f) \leq 2$ for every $e, f \in E(G)$ but $\gamma_{gee}(G) = 3$.*

Corollary 3.4. *Let G be a connected graph of size $m \geq 4$. If $d(e, f) \geq 3$ for some $e, f \in E(G)$. Then $\gamma_{gee}(G) \geq 3$.*

Theorem 3.5. *Let G be a connected graph with $\gamma_{gee}(G) = 2$. Then $c(G) \leq 6$, where $c(G)$ is the length of the longest cycle in G .*

Proof. On the contrary, suppose that $c(G) \geq 7$. If $G = C_n$ ($n \geq 7$), then by Observation 2.3 (vi), $\gamma_{gee}(G) \geq 3$, which is not so. Therefore $G \neq C_n$ ($n \geq 7$). Let C $v_1, v_2, v_3, \dots, v_n, v_1$ be a longest cycle in G , where $n \geq 7$. Let $S = \{e, f\}$ be a γ_{gee} -set of G . Since $G \neq C_n$ ($n \geq 7$), then exists a vertex v in G such that v is not on C and v is adjacent to v_1 , say. Then either S is not an edge-to-edge geodetic set of G or not an edge dominating set of G , which is a contradiction. Therefore $c(G) \leq 6$. $\square$

Theorem 3.6. *If $G \neq K_{1,m}$ is a connected graph with size $m \geq 4$, then $\gamma_{gee}(G) \leq m - 1$.*

Proof. If $G = K_n$ ($n \geq 4$), then by Observation 2.3(v), $\gamma_{gee}(G) \leq m - 1$. So we have done. Therefore let us assume that $G \neq K_n$ ($n \geq 4$). Since $G \neq K_{1,m}$ ($m \geq 4$), then G contains at least one non extreme edge, say e . Then $S = E(G) - \{e\}$ is an edge-to-edge geodetic dominating set of G so that $\gamma_{gee}(G) \leq m - 1$. $\square$

Remark 3.7. From Observation 2.3(ii), the bound in Theorem 3.6 is sharp for a double star with m edges.

Theorem 3.8. Let G be a connected graph of size $m \geq 4$ with diameter $d \leq 3$ which is neither a star nor a double star. Then $\gamma_{gee}(G) \leq m - 2$.

Proof. Let G be a tree. Then by our assumption G is not a star or a double star. Then G contains at least three internal edges. Let $e_1, e_2, \dots, e_n$ ($n \geq 3$) be the internal edges of G . Then $S = E(G) - \{e_1, e_n\}$ is an edge-to-edge geodetic dominating set so that $\gamma_{gee}(G) \leq m - 2$. Next we assume that G is not a tree. So we have the following cases.

Case 1: $d = 1$. Then $G = K_n$. By Observation 2.3 (v), $\gamma_{gee}(G) \leq m - 2$.

Case 2: $d = 2$. If $G = C_n$ ($n \geq 4$), then G is either C_4 or C_5 . By Observation 2.3 (vi), $\gamma_{gee}(G) \leq m - 2$. If $G \neq C_n$ ($n \geq 4$), then G contains a cycle. Since G is connected and $d = 2$, it follows that $c(G) \leq 4$. So we have the following two subcases.

Subcase 2a: $c(G) = 4$. Let $C : v_1, v_2, v_3, v_4, v_1$ be a cycle in G . Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1 , say. Then $S = E(G) - \{v_1v_2, v_1v_4\}$ is an edge-to-edge geodetic dominating set of G so that $\gamma_{gee}(G) \leq m - 2$.

Subcase 2b: $c(G) = 3$. Let $C : v_1, v_2, v_3, v_1$ be a cycle in G . Since $m \geq 4$, $G \neq C_3$. Then the graph is like G given in Figure 3.1, which has edge-to-edge geodetic domination number at most $m - 2$.

Figure 3.1

Case 3: $d = 3$. If $G = C_n$ ($n \geq 6$), then $G = C_6$ or C_7 . By Observation 2.3 (vi), $\gamma_{gee}(G) \leq m - 2$. If $G \neq C_n$ ($n \geq 6$), since G is connected and $d = 3$, then $c(G) \leq 6$. We have the following four subcases.

Subcase 3a: $c(G) = 6$. Let $C : v_1, v_2, v_3, v_4, v_5, v_6, v_1$ be a cycle in G . Since G is not a cycle, there exists a vertex v such that v is not on C and v is adjacent to v_1 , say. Without loss of generality, let us assume that v, v_1, v_2, v_3 be a diametral path in G . Then $S = E(G) - \{v_1v_2, v_2v_3\}$ is an edge-to-edge geodetic dominating set of G and so $\gamma_{gee}(G) \leq m - 2$.

Subcase 3b: $c(G) = 5$. Let $C : v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle in G . Since G is not a cycle, there exists a vertex v not in C and v is adjacent to v_1 , say. Since $d = 3$, either v is an end vertex of G or v is adjacent to a vertex, say v_2 in G . If v is an end vertex, then $S = E(G) - \{v_1v_2, v_1v_5\}$ is an edge-to-edge geodetic dominating set of G so that $\gamma_{gee}(G) \leq m - 2$. If v is adjacent to v_2 . Then $S_1 = E(G) - \{v_4v_5, v_1v_5\}$ is an edge-to-edge geodetic set of G so that $\gamma_{gee}(G) \leq m - 2$.

Subcase 3c: $c(G) = 4$. Let $C : v_1, v_2, v_3, v_4, v_1$ be a cycle in G . Since $G \neq C_4$, there exists a vertex v such that v is not on C and v is adjacent to v_1 , say. Without loss of generality, let v_4, v_2, v_1, v be a diametral path in G . Then $S = E(G) - \{v_1v_2, v_3v_4\}$ is an edge-to-edge geodetic dominating set of G so that $\gamma_{gee}(G) \leq m - 2$.

Subcase 3d: $c(G) = 3$. Let $C : v_1, v_2, v_3, v_1$ be a cycle in G . Since $m \geq 4$, $G \neq C_3$. Then the graph is like G given in Figure 3.2, which has an edge-to-edge geodetic domination number at most $m - 2$. $\square$

Figure 3.2

Remark 3.9. The bound in Theorem 3.8 is sharp. For the graph G given in Figure 3.3, $S = E(G) - \{v_1v_2, v_1v_3\}$ is the unique γ_{gee} -set of G so that $\gamma_{gee}(G) = m - 2$.

Figure 3.3

In the following theorem we give an improved upper bound for the edge-to-edge geodetic domination number of a graph in terms of its size m and diameter d .

Theorem 3.10. If $G \neq K_{1,m}$ is a connected graph with $m \geq 4$, then $\gamma_{gee}(G) \leq m - \lceil \frac{2d-3}{3} \rceil$, where d is the diameter of G .

Proof. **Case 1.** $d \leq 3$. Since $G \neq K_{1,m}$, by Theorem 3.6, $\gamma_{gee}(G) \leq m - 1 \leq m - \lceil \frac{2d-3}{3} \rceil$.

Case 2. $d \geq 4$. Let u and v be vertices of G for which $d(u, v) = d$, where d is the diameter of G . Let $P : u = u_0, u_1, u_2, \dots, u_d = v$ be a $u - v$ path of length d . Let $e_i = u_{i-1}u_i$ ($1 \leq i \leq d$).

Subcase 2a. $d \equiv 0 \pmod{3}$. Let $S = E(G) - \{u_1u_2, u_2u_3, u_4u_5, u_5u_6, \dots, u_{d-2}u_{d-1}\}$. Then S is an edge dominating set of G . Let e be any edge of G . If $e = u_iu_{i+1}$ ($1 \leq i \leq d-1$), then e lies on the $e_1 - e_d$ geodesic $P_1 : u_1, u_2, \dots, u_{d-1}$. If $e \neq u_iu_{i+1}$ ($1 \leq i \leq d-1$), then e is an edge of S . Therefore, S is an edge-to-edge geodetic set of G . Thus S is an edge-to-edge geodetic dominating set of G . Consequently, $\gamma_{gee}(G) \leq |S| = m - (\frac{2d-3}{3}) \leq m - \lceil \frac{2d-3}{3} \rceil$.

Subcase 2b. $d+1 \equiv 0 \pmod{3}$. Let $S = E(G) - \{u_1u_2, u_2u_3, u_4u_5, u_5u_6, \dots, u_{d-4}u_{d-3}, u_{d-3}u_{d-2}\}$. Then S is an edge dominating set of G . By the similar argument made in Subcase 2a, we get $\gamma_{gee}(G) \leq m - (\frac{2d-3}{3}) \leq m - \lceil \frac{2d-3}{3} \rceil$.

Subcase 2c. $d-1 \equiv 0 \pmod{3}$. Let $S = E(G) - \{u_1u_2, u_2u_3, u_4u_5, u_5u_6, \dots, u_{d-3}u_{d-2}, u_{d-2}u_{d-1}\}$. Then S is an edge dominating set of G . By the similar argu-

ment made in Subcase 2a, we get $\gamma_{gee}(G) \leq m - \lceil \frac{2d-3}{3} \rceil$.
 $\square$

Remark 3.11. The bound in Theorem 3.10 is sharp. For the graph $G = P_5$, $\gamma_{gee}(G) = 2$, $m = 4$, $d = 4$ and $m - \lceil \frac{2d-3}{3} \rceil = 2$. Therefore $\gamma_{gee}(G) = m - \lceil \frac{2d-3}{3} \rceil$. Also the bound in Theorem 3.10 can be strict. For the graph G given in Figure 3.4, $\gamma_{gee}(G) = 3$, $m = 9$, $d = 5$ and $m - \lceil \frac{2d-3}{3} \rceil = 6$. Thus $\gamma_{gee}(G) < m - \lceil \frac{2d-3}{3} \rceil$.

4. Graph with small and large edge-to-edge geodetic domination number

In this section, we characterized connected graphs G of size m with $\gamma_{gee}(G) = 2$ or m or $m - 1$. For this purpose we introduce the collection $\mathfrak{S}$ of graphs given in Figure 4.1.

Figure 4.1.

Theorem 4.1. Let G be a connected graph with size $m \geq 2$. Then $\gamma_{gee}(G) = 2$ if and only if G is either P_3 or P_4 or P_5 or C_4 or $K_4 - \{e\}$ or K_4 or C_6 or the class of graphs $\mathfrak{S}$ given in Figure 4.1.

Proof. Let $\gamma_{gee}(G) = 2$. Then by Theorem 3.2, $d(e, f) \leq 2$ for every $e, f \in E(G)$. If G is a tree, then $G = P_3$ or P_4 or P_5 . Which satisfies the requirements of the theorem. Let us assume that G is not a tree. Let $c(G)$ be the length of a longest cycle in G . By Theorem 3.5, $c(G) \leq 6$. Let C be a longest cycle in G . We consider four cases.

Case 1. $c(G) = 3$. Let C be v_1, v_2, v_3, v_1 . First assume that G contains only one cycle C with $|C| = 3$. If $G = C_3$, then $\gamma_{gee}(G) = 3 > 2$, which is a contradiction. Therefore $G \neq C_3$. Then there exists a vertex u_1 not in C and u_1 is adjacent to v_1 , say. Since G contains only one cycle, v_1 is a cut vertex of G . Since $\gamma_{gee}(G) = 2$, there exist only two components, say G_1 and G_2 of $G - v_1$ such that $|V(G_1)| \leq 2$ and $|V(G_2)| \leq 2$. Since G is not a tree, $|V(G_1)| = |V(G_2)| \neq 1$. If $|V(G_1)| = 1$ and $|V(G_2)| = 2$, then the graph G is given in class of graphs $\mathfrak{S}$ of Figure 4.1 (a), which satisfies the requirements of the theorem. If $|V(G_1)| = |V(G_2)| = 2$, then the graph G is given in class of graphs $\mathfrak{S}$ of Figure 4.1 (b), which satisfies the requirements of the theorem. Suppose that G contains another one cycle C_1 with $|C| = |C_1| = 3$. If C and C_1 have a common vertex, then the graph G is given in the class of graphs $\mathfrak{S}$ of Figure 4.1(c,d), which satisfies the requirements of the theorem. If C and C_1 have a common edge, then the graph G is either $K_4 - \{e\}$ or K_4 , which satisfies the requirements of the theorem. If C and C_1 is connected by a path $P: x, x_1, x_2, \dots, x_n = y$, where $x \in V(C)$ and $y \in V(C_1)$ and $n \geq 1$. Then $d(e, f) \geq 3$ for some $e, f \in E(G)$ and so $\gamma_{gee}(G) \geq 3$, which is a contradiction.

Case 2. $c(G) = 4$. Let C be v_1, v_2, v_3, v_4, v_1 . If $G = C_4$, then G satisfies the requirements of the theorem. If $G \neq C_4$, then there exists a vertex u_1 not in C and u_1 is adjacent to v_1 , say. Since $\gamma_{gee}(G) = 2$, there exists a vertex u_2 not in C and u_2 is adjacent to v_3 . If $\deg(u_1) = 1$ and $\deg(u_2) = 1$, then the graph G is given in class of graphs $\mathfrak{S}$ of Figure 4.1(e). If $\deg(u_1) = 2$ and $\deg(u_2) = 2$, then the graph G is given in class of graphs $\mathfrak{S}$ of Figure 4.1(f). If $\deg(u_1) = 1$ and $\deg(u_2) = 2$ or if $\deg(u_1) = 2$ and $\deg(u_2) = 1$, then $\gamma_{gee}(G) \geq 3$, which is not so. If $\deg(u_1) \geq 2$ and $\deg(u_2) \geq 3$ or if $\deg(u_1) \geq 3$ and $\deg(u_2) \geq 2$, and also since $\gamma_{gee}(G) = 2$, then there must be at least one u_1v_1 - u_2v_3 geodesic. Then the graph G given in class of graphs $\mathfrak{S}$ of Figure 4.1(g,h,i,j,k). Note that $S = \{u_1v_1, u_2v_3\}$ is a γ_{gee} -set of G so that $\gamma_{gee}(G) = 2$, G satisfies the requirements of the theorem.

Case 3. $c(G) = 5$. Let C be $v_1, v_2, v_3, v_4, v_5, v_1$. If $G = C_5$, then $\gamma_{gee}(G) =$

3, which is a contradiction. Therefore $G \neq C_5$. Then there exists a vertex u_1 not in C and u_1 is adjacent to v_1 , say. If $\deg(u_1) = 1$, then since $\gamma_{gee}(G) = 2$, there must be at least one u_1v_1 - u_3v_4 geodesic. Then the graph G given in class of graphs $\mathfrak{S}$ of Figure 4.1(l). Note that $S = \{u_1v_1, u_3v_4\}$ is a γ_{gee} -set of G so that $\gamma_{gee}(G) = 2$. Therefore G satisfies the requirements of the theorem. If $\deg(u_1) \geq 2$, then there must be at least one u_1 - v_3 geodesic or u_1 - v_4 geodesic or v_1 - v_4 geodesic or v_1 - v_3 geodesic or u_1 - v_3 . Then the graph G given in class of graphs $\mathfrak{S}$ of Figure 4.1(m,n). Note that $S = \{u_1v_1, v_3v_4\}$ is a γ_{gee} -set of G so that $\gamma_{gee}(G) = 2$, which G satisfies the requirements of the theorem.

$\mathfrak{S}$ of Figure 4.1(g), which satisfies the requirements of the theorem. If $\deg(v) \geq 2$, since $d(e, f) \leq 2$ for every $e, f \in E(G)$, then there must be a path P of length three which joins two non adjacent vertices of C . Let the path P be v_1, u_2, u_3, v_4 , where $u_2 \neq v_2$ and $u_3 \neq v_3$. Then $\gamma_{gee}(G) = 2$. Therefore the graph G is given in class of graphs $\mathfrak{S}$ of Figure 4.1 (h), which satisfies the requirements of the theorem. If $v_2v_5 \in E(G)$, then $\gamma_{gee}(G) = 2$. Therefore the graph G is given in class of graphs $\mathfrak{S}$ of Figure 4.1(i), which satisfies the requirements of the theorem.

Case 4. $c(G) = 6$. Let C be $v_1, v_2, v_3, v_4, v_5, v_6, v_1$. If $G = C_6$, then G satisfies the requirement of the theorem. If $G \neq C_6$, there exist at least two v_1v_2 - v_4v_5 geodesics. Then the graph G given in class of graphs $\mathfrak{S}$ of Figure 4.1(p). Note that $S = \{v_1v_2, v_4v_5\}$ is a γ_{gee} -set of G so that $\gamma_{gee}(G) = 2$, which G satisfies the requirements of the theorem.

The converse is clear. $\square$

Theorem 4.2. Let G be a connected graph of size $m \geq 2$. Then $\gamma_{gee}(G) = m$ if and only if G is the star $K_{1,m}$ or K_3 .

Proof. Let $G = K_{1,m}$ ($m \geq 2$). Then by Observation 2.3 (ii), $\gamma_{gee}(G) = m$. If $G = K_3$, then $S = E(G)$ is the unique γ_{gee} -set of G so that $\gamma_{gee}(G) = 3 = m$. For the converse, let G be a connected graph of size $m \geq 2$ and $\gamma_{gee}(G) = m$. Then by Theorem 3.10, $d \leq 2$. If $d = 1$, then $G = K_n$ ($n \geq 3$). If $n \geq 4$, then by Observation 2.3 (v), $\gamma_{gee}(G) \leq m - 1$, which is not so. Therefore $m = 3$. Which implies $G = K_3$, which satisfies the requirements of the theorem. Let $d = 2$. Let $m = 3$. If G is a tree, then $G = K_{1,3}$, which satisfies the requirement of this theorem. If G is not a tree, then $G = K_3$, which satisfies the requirements of this theorem. Let $m = 2$. Then $G = K_{1,2}$, which satisfies the requirements of this theorem.

Let $m \geq 4$. If $G \neq K_{1,m}$, then by Theorem 3.6, $\gamma_{gee}(G) \leq m - 1$, which is a contradiction. Therefore $G = K_{1,m}$. First assume that $m \geq 4$. Therefore $G = K_3$ or $G = K_{1,m}$. $\square$

Theorem 4.3. Let G be a connected graph of size $m \geq 3$. Then $\gamma_{gee}(G) = m - 1$ if and only if G is a double star.

Proof. If G is a double star, then by Observation 2.3(iii), $\gamma_{gee}(G) = m - 1$. For the converse, let G be connected graph of size $m \geq 3$ and $\gamma_{gee}(G) = m - 1$. Let $m = 3$. If G is a tree, then $G = P_4$ or $K_{1,3}$. If $G = P_4$, then $\gamma_{gee}(G) = 2 = m - 1$, which satisfies the requirement of the theorem. If $G = K_{1,3}$, then G is a star. By Theorem 4.2, $\gamma_{gee}(G) = 3 = m$, which is a contradiction. If G is not a tree, then $G = K_3$. By Theorem 4.2, $\gamma_{gee}(G) = 3 = m$, which is a contradiction. Let $m \geq 4$. Then by Theorem 3.10, $m - 1 \leq \lceil \frac{2d-3}{3} \rceil$
 $\Rightarrow d \leq 3$

If G is neither a star nor a double star, then by Theorem 3.8, $\gamma_{gee}(G) \leq m - 2$, which is a contradiction. Therefore G is either a star or a double star. If G is a star, then by Theorem 4.2, $\gamma_{gee}(G) = m$, which is a contradiction. If G is a double star, then by Observation 2.3 (iii), $\gamma_{gee}(G) = m - 1$, which satisfies the requirements of the theorem. $\square$

In the following, we present the Nordhaus-Gaddum type relations for edge-to-edge geodetic dominating number of a graph.

Theorem 4.4. If G and $\bar{G}$ are connected graphs of size $m \geq 4$, then $4 \leq \gamma_{gee}(G) + \gamma_{gee}(\bar{G}) \leq 2m - 2$. Moreover the lower and bound are sharp if and only if $G = P_4$.

Proof. Since G and $\bar{G}$ are connected graphs, $\gamma_{gee}(G) \geq 2$ and $\gamma_{gee}(\bar{G}) \geq 2$ and so $\gamma_{gee}(G) + \gamma_{gee}(\bar{G}) \geq 4$. Since $\bar{G}$ is connected, by Theorem 4.2, $\gamma_{ee}(\bar{G}) \leq m - 1$. Which implies $\gamma_{ee}(\bar{G}) \leq m - 1$. Hence $\gamma_{gee}(G) + \gamma_{gee}(\bar{G}) \leq 2m - 2$. Therefore $4 \leq \gamma_{gee}(G) + \gamma_{gee}(\bar{G}) \leq 2m - 2$. Next we prove that $\gamma_{gee}(G) + \gamma_{gee}(\bar{G}) = 4$ if and only if $G = P_4$. From Theorem 4.1, we can easily verified that $\gamma_{gee}(G) = 2$ and $\gamma_{gee}(\bar{G}) = 2$ if and only if $G = P_4$. Therefore $\gamma_{gee}(G) + \gamma_{gee}(\bar{G}) = 4$ if and only if $G = P_4$. Next we prove that $\gamma_{gee}(G) + \gamma_{gee}(\bar{G}) = 2m - 2$ if and only if $G = P_4$. By Theorem 4.3, $\gamma_{gee}(G) = m - 1$ and $\gamma_{ee}(\bar{G}) = m - 1$ if and only if G is a double star. For a double star $G = \bar{G}$ if and only if $m = 3$. Therefore $\gamma_{gee}(G) = m - 1$ and $\gamma_{gee}(\bar{G}) = m - 1$ if and only if $G = P_4$. Hence $\gamma_{gee}(G) + \gamma_{gee}(\bar{G}) = 2m - 2$ if and only if $G = P_4$. $\square$

5. Realization Results

In this section we give some realization results concerning the edge-to-edge geodetic domination number. Further, we will also make use of the following Theorem 5.1, which is proved in [1].

Theorem 5.1. [1] *Each end edge of G belongs to every edge-to-edge geodetic set of G .*

Theorem 5.2. *For any integer $a \geq 2$, there exists a connected graph G with $\gamma_e(G) = g_{ee}(G) = \gamma_{gee}(G) = a$.*

Proof. Let $P_i : u_i, v_i (1 \leq i \leq a)$ be a copy of path on two vertices. Let G be a graph obtained from $P_i (1 \leq i \leq a)$ by adding a new vertex x and join x with each $u_i (1 \leq i \leq a)$ and with each $v_i (1 \leq i \leq a)$. The graph G is shown in Figure 5.1.

Figure 5.1

First we claim that $\gamma_e(G) = a$. Let $H_i = \{xu_i, xv_i, u_i v_i\} (1 \leq i \leq a)$. It is easily observed that every edge dominating set of G contains at least one edge from each $H_i (1 \leq i \leq a)$ and so $\gamma_e(G) \geq a$. Now $S = \{u_1 v_1, u_2 v_2, \dots, u_a v_a\}$ is an edge dominating set of G so that $\gamma_e(G) = a$. Next we show that $g_{ee}(G) = a$. It is easily observed that every g_{ee} -set of G contains only the edge $u_i v_i$ of $H_i (1 \leq i \leq a)$ and so $g_{ee}(G) \geq a$. Now S is an edge-to-edge geodetic set of G so that $g_{ee}(G) = a$. Next we show that $\gamma_{gee}(G) = a$. Since S is both a γ_e -set of G and a g_{ee} -set of G , S is a γ_{gee} -set of G so that $\gamma_{gee}(G) = a$. $\square$

Theorem 5.3. *For every pair of positive integers a, b with $2 \leq a \leq b$, there exists a connected graph G with $g_{ee}(G) = a$ and $\gamma_{gee}(G) = b$.*

Proof.

For $2 \leq a = b$, the result from Theorem 5.2. Therefore let $2 \leq a < b$. Let $P_i : u_i, v_i$ ($1 \leq i \leq b - a$) be a copy path of order 2. Let G be a graph obtained from P_i ($1 \leq i \leq b - a$) by adding new vertices x, y, z and $z_1, z_2, \dots, z_{a-1}$ and joining x with y , y with each u_i ($1 \leq i \leq b - a$), z with each v_i ($1 \leq i \leq b - a$) and z with each z_i ($1 \leq i \leq a - 1$). The graph G is shown in Figure 5.2.

Figure 5.2

First we claim that $g_{ee}(G) = a$. Let $X = \{xy, zz_1, zz_2, \dots, zz_{a-1}\}$ be the set of all end edges of G . By Theorem 5.1, X is a subset of every edge-to-edge geodetic set of G and so $g_{ee}(G) \geq a$. It is clear that X is an edge-to-edge geodetic set of G so that $g_{ee}(G) = a$.

Next we show that $\gamma_{gee}(G) = b$. By Observation 2.3(i), X is a subset of every edge-to-edge geodetic dominating set of G . For $1 \leq i \leq b - a$, let $H_i = \{yu_i, u_i v_i, v_i z\}$. Then it is observed that every edge-to-edge geodetic dominating set of G contains at least one edge from each H_i ($1 \leq i \leq b - a$) and so $\gamma_{gee}(G) \geq b$. Let $Y = \{u_1 v_1, u_2 v_2, \dots, u_{b-a} v_{b-a}\}$. Then $S = X \cup Y$ is an edge-to-edge geodetic dominating set of G so that $\gamma_{gee}(G) = b$. $\square$

Theorem 5.4. For every pair of positive integers a, b with $2 < a \leq b$, there exists a connected graph G with $\gamma_e(G) = a$ and $\gamma_{gee}(G) = b$.

Proof. For $2 \leq a = b$, the result from Theorem 5.2. Therefore let $2 \leq a < b$. Let $Q_i : u_i, v_i, w_i, x_i$ ($1 \leq i \leq a - 2$) be a copy of path on four vertices. Let G be a graph obtained from Q_i ($1 \leq i \leq a - 2$) by adding new vertices $p, q, r, s, z_1, z_2, \dots, z_{b-a}$ and join p with q , q with each u_i ($1 \leq i \leq a - 2$), r with each x_i ($1 \leq i \leq a - 2$), r with s and s with each z_i ($1 \leq i \leq b - a$). The graph G is shown in Figure 5.3.

G
Figure 5.3

First we claim that $\gamma_e(G) = a$. It is easily observed that every minimum edge dominating set of G contains the edges $v_i w_i$ ($1 \leq i \leq a-2$) and so $\gamma_e(G) \geq a-2$. Let $S = \{v_1 w_1, v_2 w_2, \dots, v_{a-2} w_{a-2}\}$. Then S is not an edge dominating set of G and so $\gamma_e(G) \geq a-1$. It is easily verified that $S \cup \{f\}$, where $f \notin S$ is not an edge dominating set of G and $\gamma_e(G) \geq a$. Let $S_1 = S \cup \{pq, rs\}$. Then S_1 is the edge dominating set of G so that $\gamma_e(G) = a$. Next we show that $\gamma_{gee}(G) = b$. Let $X = \{pq, sz_1, sz_2, \dots, sz_{b-a}\}$ be the set of all end edges of G . By Observation 2.3(i), X is a subset of every edge-to-edge geodetic dominating set of G and so $\gamma_{gee}(G) \geq b-a+1$. It is easily observed that every minimum edge-to-edge geodetic dominating set of G contains the edges $v_i w_i$ ($1 \leq i \leq a-2$) and so $\gamma_{gee}(G) \geq b-1$. Let $Z = \{v_1 w_1, v_2 w_2, \dots, v_{a-2} w_{a-2}\}$ and $M = S \cup Z$. Then M is not an edge-to-edge geodetic dominating set of G and so $\gamma_{gee}(G) \geq b$. Now $M \cup \{rs\}$ is an edge-to-edge geodetic dominating set of G so that $\gamma_{gee}(G) = b$. $\square$

Conclusion

This paper introduces a new parameter known as edge-to-edge geodetic domination number of a graph. Further, this concept can be extended to conditional parameters.

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