Proyecciones Journal of Mathematics
Vol. 41, N^o 1, pp. 83-99, February 2022.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-4050

(Completely) weak simple semigroups and (Completely) weak 0-simple semigroups

Leila Nouri
University of Sistan and Baluchestan, Iran
and
Hossein Mohammadzadeh Saany
University of Sistan and Baluchestan, Iran
Received : March 2020. Accepted : February 2021

Abstract

The structure theorems for (Completely) simple semigroups and (Completely) 0-simple semigroups have proved a powerful tool in the investigation of such semigroups. In this paper, first of all, we define weak simple semigroups and weak 0-simple semigroups and compare them with simple semigroups and 0-simple semigroups respectively. Then we give examples of these semigroups and describe the structure of them. Also, we define completely weak simple semigroup and completely weak 0-simple semigroup. Finally, by using Green's equivalences, we prove some results and give equivalences, for these semigroups.

MSC(2010): *Primary 20M10; Secondary 20M12.*

Keywords: *Ideal, (0-) simple semigroups, Completely (0-) simple semigroups, Regular semigroup, Green's equivalences.*

1. Preliminaries

This paper is concerned with the study of a certain basic type of semigroup, known as a (completely) weak (0-) simple semigroup. First, we recall from [2] that a semigroup without zero is called *simple* if it has no proper ideals. A semigroup S with zero (S should have at least two elements) is called 0-*simple* if

(i) $\{0\}$ and S are its only ideals;

(ii) $S^2 \neq \{0\}$.

For background material on algebraic structures and some other terminologies of these concepts, we refer the reader to [1, 2, 3].

2. Weak (0-) Simple Semigroups

In this section, first of all, we define weak simple semigroups and weak 0-simple semigroups and describe the structure of 0-simple semigroup. Then compare weak simple semigroups and weak 0-simple semigroups with simple semigroups and 0-simple semigroups, respectively. Then we give examples of weak simple semigroups and weak 0-simple semigroups. Also, we prove some results about these semigroups.

Definition 2.1. *We say that the semigroup S without zero is weak simple if the powers of S are the only ideals of S.*

Indeed, if S is a weak simple semigroup and $S^2 = S$, then S is a simple semigroup.

Definition 2.2. *We say that the semigroup S with zero is weak 0-simple, if*

(i) $\{0\}$ *and the powers of S are the only ideals of S;*

(ii) $S^n \neq \{0\}$, *for every* $n \geq 2(n \in \mathbf{N})$.

Indeed, if S is a weak 0-simple semigroup and $S^2 = S$, then S is a 0-simple semigroup.

Description of the structure of weak 0-simple semigroup:

Let S is a weak 0-simple semigroup such that $S^2 \neq S$. Let $\alpha \in S \setminus S^2$. It is clear that $S^2 \cup \{\alpha\}$ is an ideal of S and S^2 is a proper subset of $S^2 \cup \{\alpha\}$. Then $S = S^2 \cup \{\alpha\}$ because S is a weak 0-simple semigroup. Now if $S^2 = S^3$, then the powers of S will be static, for every $n \geq 2$. Otherwise, if $S^2 \neq S^3$, then we let $u \in S^2 \setminus S^3$. We can write $u = xy$, for $x, y \in S$. According to $S = S^2 \cup \{\alpha\}$, there are three cases that can arise:

(i) x and y belong to S^2;

(ii) only one of x or y belongs to S^2 and the other one equals α;

(iii) $x = \alpha = y$.

But (i) and (ii) lead to $u \in S^3$, which is a contradiction. Thus only case (iii) can occur. So $u = \alpha^2$. Now the subset $S^3 \cup \{\alpha^2\}$ is an ideal of S such that
$$S^3 \subseteq S^3 \cup \{\alpha^2\} \subseteq S^2.$$
Thus $S^2 = S^3 \cup \{\alpha^2\}$, and so $S = S^3 \cup \{\alpha, \alpha^2\}$, such that $\alpha^i \in S^i \setminus S^{i+1}$, for $i = 1, 2$. Again if $S^3 = S^4$, then the powers of S will be static, for every $n \geq 3$. Otherwise, if $S^3 \neq S^4$, then in a similar way, we can show $S^3 = S^4 \cup \{\alpha^3\}$, and so
$$S = S^4 \cup \{\alpha, \alpha^2, \alpha^3\},$$
such that $\alpha^i \in S^i \setminus S^{i+1}$, for $i = 1, 2, 3$. Therefore by using induction, we have the following theorem.

Theorem 2.3. *Let S is a weak 0-simple semigroup. Then one of the following statements holds:*

(i) *$S^2 = S$, and so S is a 0-simple semigroup.*

(ii) *There exists $n \geq 2$, such that $S^{n-1} \neq S^n$ and $S^n = S^{n+1}$. In this case, there exists $\alpha \in S$, such that $S^i = S^{i+1} \cup \{\alpha^i\}$, and $\alpha^i \in S^i \setminus S^{i+1}$, for $1 \leq i \leq n-1$. Also $S = S^{i+1} \cup \{\alpha, \alpha^2, ..., \alpha^i\}$, $1 \leq i \leq n-1$.*

(iii) *$S^n \neq S^{n+1}$, for every $n \geq 1$. In this case, there exists $\alpha \in S$, such that for every $i \geq 1$,*
$$S^i = S^{i+1} \cup \{\alpha^i\}.$$
Thus $S = S^{i+1} \cup \{\alpha, \alpha^2, ..., \alpha^i\}$, such that $\alpha^j \in S^j \setminus S^{j+1}$, $j \in \mathbf{N}$.

Note that if S is a weak 0-simple semigroup such that $S^2 \neq S$, then $|S \setminus S^2| = 1$. Since if $\alpha, \beta \in S \setminus S^2$, then $S^2 \cup \{\alpha\}$ and $S^2 \cup \{\beta\}$ are ideals of S such that $S^2 \subset S^2 \cup \{\alpha\}$ and $S^2 \subset S^2 \cup \{\beta\}$. Therefore $S^2 \cup \{\alpha\} = S = S^2 \cup \{\beta\}$, and so $\alpha = \beta$. By a similar argument, if $S^n \neq S^{n+1}$, then $|S^n \setminus S^{n+1}| = 1$.

Theorem 2.4. *Let S is a weak 0-simple semigroup such that $S^2 \neq S$. Then S is a monogenic semigroup with zero.*

Proof. Since $S^2 \neq S$, by Theorem 2.3, one of the following cases is held:

Case 1. There exists $n \geq 2$, such that $S^{n-1} \neq S^n$ and $S^n = S^{n+1}$, that is,

$$S^{n+1} = S^n \subset S^{n-1} \subset \ldots \subset S^2 \subset S.$$

Let $S \setminus S^2 = \{\alpha\}$. Then $S = S^n \cup \{\alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. Let $0 \neq x \in S^m$, $m \geq n$. Then $x = x_1 x_2 \ldots x_m$, such that $x_i \in S \setminus S^2 = \{\alpha\}$, $1 \leq i \leq m$, that is $x = \alpha^m$. Hence

$$S^n = \{\alpha^n, \alpha^{n+1}, \alpha^{n+2}, \ldots\} \cup \{0\}$$

and

$$S^{n+1} = \{\alpha^{n+1}, \alpha^{n+2}, \alpha^{n+3}, \ldots\} \cup \{0\}.$$

Since $S^n = S^{n+1}$, there exits $j \geq 1$ such that $\alpha^n = \alpha^{n+j}$. Let r is the smallest natural number such that $\alpha^n = \alpha^{n+r}$. Then $S^n = \{\alpha^n, \alpha^{n+1}, \ldots, \alpha^{n+r-1}\} \cup \{0\}$, and so

$$S = \{\alpha, \alpha^2, \ldots, \alpha^n, \alpha^{n+1}, \ldots, \alpha^{n+r-1}\} \cup \{0\},$$

which is a finite monogenic semigroup with zero of index n and period $r = |S^n|$.

Case 2. $S^n \neq S^{n+1}$, for every $n \geq 1$. That is,

$$\ldots \subset S^n \subset S^{n-1} \subset \ldots \subset S^2 \subset S.$$

In this case, by Theorem 2.3(iii), there exists $\alpha \in S$ such that $S = S^{i+1} \cup \{\alpha, \alpha^2, ..., \alpha^i\}$, for every $i \geq 1$, which implies S is an infinite monogenic semigroup with zero. □

From Theorems 2.3 and 2.4, we have the following corollary.

Corollary 2.5. *For a semigroup S the following statements are equivalent:*

(i) *S is a weak 0-simple semigroup such that is not 0-simple;*

(ii) *S is an infinite monogenic semigroup with zero or is a finite monogenic semigroup with zero of index n $(n > 1)$ and period r.*

Now we give examples of weak simple semigroups and weak 0-simple semigroups.

Example 2.6. *Let $S =< a >= \{a^i | i \geq 1\}$, is an infinite monogenic semigroup. Then only ideals of S, are powers of S, and so S is a weak simple semigroup. Also*

$$... \underset{\neq}{\subset} S^4 \underset{\neq}{\subset} S^3 \underset{\neq}{\subset} S^2 \underset{\neq}{\subset} S.$$

That is, the chain of subsemigroups of S is not static.

If we add zero to the above infinite monogenic semigroup, then the made semigroup is weak 0-simple, such that the chain of subsemigroups of S is not static.

Example 2.7. *Let $S =< b >= \{b, b^2, ..., b^m, b^{m+1}, ..., b^{m+r-2}, b^{m+r-1}\}$, is a finite monogenic semigroup, of index m, and period r. Obviously,*

$$S^2 =< b >^2 = \{b^2, b^3, ..., b^m, b^{m+1}, ..., b^{m+r-2}, b^{m+r-1}\},$$

$$S^3 =< b >^3 = \{b^3, ..., b^m, b^{m+1}, ..., b^{m+r-2}, b^{m+r-1}\},$$

$$\vdots$$

$$S^m =< b >^m = \{b^m, b^{m+1}, ..., b^{m+r-2}, b^{m+r-1}\},$$

$$S^{m+1} = S^m.$$

Thus $S^{m+1} = S^m \underset{\neq}{\subset} S^{m-1} \underset{\neq}{\subset} ... \underset{\neq}{\subset} S^2 \underset{\neq}{\subset} S$. Also every ideal of S is a power of S. Therefore S is a weak simple semigroup, such that the powers of S will be static, for every $k \geq m$. In this case, S^m is a group.

Notice that the converse of Theorem 2.3 is not true in general, that is, if there exists a semigroup S with zero, such that satisfies in condition (ii) or condition (iii) of Theorem 2.3, then S necessity is not weak 0-simple semigroup. See the following example.

Example 2.8. *Let $B = \{\beta_1, \beta_2, ...\}$, that can be finite or infinite. Define:*

$$\beta_i.\beta_i = \beta_i^2 = \beta_i, \qquad \forall i \geq 1;$$
$$\beta_i.\beta_k = \beta_k.\beta_i = \beta_i, \quad \forall i, k \in \mathbf{N}, i < k.$$

Obviously, B is a semigroup. Now let T is a weak 0-simple semigroup, which θ is its zero. Consider $S = T \cup B$, with the following operation:

If two elements are in T or B, then the operation is the same operation T and B respectively. Also,

$$\beta_i.t = t.\beta_i = \beta_i, \qquad \forall \beta_i \in B, \forall t \in T, t \neq \theta;$$
$$\beta_i.\theta = \theta.\beta_i = \theta, \qquad \forall \beta_i \in B.$$

Clearly, S is a semigroup, such that $S^m = T^m \cup B$, for every $m \geq 1$. Since T is a weak 0-simple semigroup, there exists α in T, such that $T = T^2 \cup \{\alpha\}$. Therefore

$$S = T \cup B = S^2 \cup \{\alpha\},$$

which $\alpha \notin S^2$. Now if T is a semigroup as Example 2.6, which the chain of subsemigroups of T is not static, then we have $S^i = S^{i+1} \cup \{\alpha^i\}$, for every $i \geq 1$. Then

$$S = S^{i+1} \cup \{\alpha^1, \alpha^2, ..., \alpha^i\}, \quad \alpha^j \in S^j \setminus S^{j+1}, \quad \forall j \geq 1.$$

Therefore S satisfies in condition (iii) of Theorem 2.3, but S is not weak 0-simple semigroup, because $B \cup \{\theta\}$ is an ideal of S, which is not as a power of S.

Also if T is a semigroup as Example 2.7, which the chain of subsemigroups of T is static, then S satisfies in condition (ii) of Theorem 2.3, but S is not weak 0-simple semigroup, because $B \cup \{\theta\}$ is an ideal of S, which is not as a power of S.

Recall from [2] that an *equivalence* $\mathcal{J}$ on S is defined by the rule that $a\mathcal{J}b$, for $a, b \in S$, if and only if $S^1aS^1 = S^1bS^1$. Also, the $\mathcal{J}$-class containing the element a is denoted by J_a.

Theorem 2.9. *Let S is a weak 0-simple semigroup. Then one of the following statements holds:*

(i) $J_a = \{a\}$, *for every* $a \in S$.

(ii) *There exists* $n \geq 1$ *such that for every* $a, b \in S^n \setminus \{0\}$, $a\mathcal{J}b$ *and for every* $a \in S \setminus S^n$, $J_a = \{a\}$.

Proof. By Theorem 2.3, one of the following cases holds:

Case 1. $S^2 = S$, and so S is a 0-simple. Since S^1aS^1 is a non zero ideal of S, for every $a \in S \setminus \{0\}$, $S^1aS^1 = S$, and so $a\mathcal{J}b$, for every $a, b \in S^2 \setminus \{0\} = S \setminus \{0\}$. In this case $S \setminus S^2 = \emptyset$. Hence (ii) is satisfied.

Case 2. There exists $n \geq 2$, such that $S^{n-1} \neq S^n$ and $S^n = S^{n+1}$, that is,

$$S^{n+1} = S^n \subset S^{n-1} \subset \ldots \subset S^2 \subset S.$$

Now $S^1aS^1 \subseteq S^1S^nS^1 \subseteq S^n$, for every $a \in S^n \setminus \{0\}$, and so by assumption $S^1aS^1 = S^n$. Therefore $a\mathcal{J}b$, for every $a, b \in S^n \setminus \{0\}$. Also $S \setminus S^n = \{\alpha, \alpha^2, \ldots, \alpha^{n-1}\}$, such that $\alpha^i \in S^i \setminus S^{i+1}$, $1 \leq i \leq n-1$. Hence $S^1\alpha^iS^1 \neq S^1\alpha^jS^1$, for every $1 \leq i \neq j \leq n-1$, and so $\alpha^i\mathcal{J}\alpha^j$, for every $1 \leq i \neq j \leq n-1$. Therefore $J_a = \{a\}$, for every $a \in S \setminus S^n$, and so (ii) is satisfied.

Case 3. $S^n \neq S^{n+1}$, for every $n \geq 1$. That is,

$$\ldots \subset S^n \subset S^{n-1} \subset \ldots \subset S^2 \subset S.$$

In this case, by Theorem 2.3(iii), S is an infinite monogenic semigroup with zero, and so $S^1aS^1 \neq S^1bS^1$, for every $a \neq b \in S$, that is $J_a = \{a\}$, for every $a \in S$. Therefore (i) is satisfied. □

The converse of Theorem 2.9 is not true, see the following example.

Example 2.10. *Consider S with the following table:*

	0	a	b	c
0	0	0	0	0
a	0	b	0	0
b	0	0	a	0
c	0	0	0	0

Indeed, $S^2 = \{0, a, b\}$, $a\mathcal{J}b$ *and* $J_c = \{c\}$. *Therefore condition* (ii) *of Theorem 2.9, for* $n = 2$ *is satisfied. But* $I = \{0, c\}$ *is a non zero ideal of* S, *which is not equals with no powers of* S, *because* $S^n = S^{n+1}$, *for every* $n \geq 2$.

Theorem 2.11. *Let* S *is a weak* 0*-simple semigroup. Then for every* $0 \neq a \in S$, *there exists* $m \geq 1$, *such that* $SaS = S^m$.

Proof. Let $0 \neq a \in S$. If $SaS = \{0\}$, then $I = \{\alpha | S\alpha S = \{0\}\}$ is a non-zero ideal of S. Since S is a weak 0-simple semigroup, there exists $n \geq 1$, such that $I = S^n$. But $SIS = \{0\}$, by definition I. Then $S^{n+2} = \{0\}$, which is a contradiction. Therefore $SaS \neq \{0\}$. Since SaS is an ideal of S, there exists $m \geq 1$, such that $SaS = S^m$. □

Theorem 2.12. *The semigroup* S *is weak* 0*-simple if and only if for every* $n \geq 1$, $S^n \neq \{0\}$, *and for every* $0 \neq a \in S$ *there exists* $m \geq 1$, *such that* $SaS = S^m$.

Proof. Necessity. It is obvious by definition and Theorem 2.11.

Sufficiency. Let I is a non zero ideal of S. Then for every $0 \neq a \in I$, there exists $m_a \in \mathbf{N}$, such that $S^1aS^1 = S^{m_a}$. Therefore $I = \cup_{a \in I} S^{m_a}$. Let $m = min\{m_a | a \in I\}$. Then $I = S^m$, and so S is a weak 0-simple semigroup. □

Lemma 2.13. *Let* S *is a semigroup with zero, such that for every* $0 \neq a \in S$, *there exists* $m \geq 1$, *such that* $SaS = S^m$. *If* A *is a non-empty and non-zero subset of* S, *then there exists* $m \geq 1$, *such that* $SAS = S^m$.

Proof. By assumption we have for every $0 \neq a \in A$, there exists $m_a \geq 1$, such that $SaS = S^{m_a}$. Thus $SAS = \cup_{a \in A} SaS = \cup_{0 \neq a \in A} S^{m_a}$. Let $m = min\{m_a | a \in A\}$. Then $\cup_{0 \neq a \in A} S^{m_a} = S^m$, and so $SAS = S^m$, as required. □

Theorem 2.14. *Let S is a weak 0-simple semigroup, such that $S^{n+1} = S^n$, for $n \geq 1$. Then S^n is a 0-simple semigroup.*

Proof. We must show the only non-zero ideal of S^n is S^n. Let A is a non-zero ideal of S^n. Since S is a weak 0-simple semigroup, by Theorem 2.11, for every $0 \neq a \in S$ there exists $m \geq 1$, such that $SaS = S^m$. Thus $SAS = S^k$, for $k \geq 1$, by Lemma 2.13. Therefore $S^nAS^n = S^{k+2n-2}$. Since $k + 2n - 2 \geq n$, from equality $S^n = S^{n+1}$, we obtain $S^nAS^n = S^{k+2n-2} = S^n$. But $S^n = S^nAS^n \subseteq A$, because A is an ideal of S. Thus $S^n = A$, that is, the only non-zero ideal of S^n is S^n, and so S^n is a 0-simple semigroup, as required. □

Notice that, since in a finite weak 0-simple semigroup, the chain of powers of S is static, thus in these semigroups there exists a power of S, such that is a 0-simple semigroup.

For a semigroup S, the set of all idempotents of S is denoted by $E(S)$.

Lemma 2.15. *Let S is a weak 0-simple semigroup, such that $E(S) \neq \{0\}$. Let $SeS = S^m$, for $0 \neq e \in E(S)$ and $m \in \mathbf{N}$. Then the chain of powers of S is static, for every $k \geq m$, and so S^m is 0-simple.*

Proof. Since S is a weak 0-simple semigroup, for $0 \neq e \in E(S)$ there exists $m \geq 1$, such that $SeS = S^m$, by Theorem 2.11. Indeed $e \in S^n$, for every $n \geq 1$. Then $S^m = SeS \subseteq S^k$, for every $k \geq 3$. If $m = 1$ or $m = 2$, then obviously the converse of inclusion is satisfied, and so $S^m = S^k$, for every $k \geq 3$. Thus $S^m = S^{m+1}$. Now if $m \geq 3$, since above inclusion is satisfied, for every $k \geq 3$, thus for $k = m + 1$, is satisfied too, that is $S^m \subseteq S^{m+1}$, and so $S^m = S^{m+1}$. Therefore the chain of powers of S is static, for every $k \geq m$, and so S^m is 0-simple, by Theorem 2.14. □

Lemma 2.16. *Let S is a weak 0-simple semigroup, such that $E(S) \neq \{0\}$. Let $SE(S)S = S^m$, for $m \in \mathbf{N}$. Then the chain of powers of S is static, for every $k \geq m$, and so S^m is 0-simple.*

Proof. Since S is a weak 0-simple semigroup, there exists $m \geq 1$ such that $SE(S)S = S^m$, by Lemma 2.13. Since $E(S) \subseteq S^n$, for every $n \geq 1$, therefore $S^m = SE(S)S \subseteq S^k$, for every $k \geq 3$. Now by using a similar argument as in the proof of Lemma 2.15, we obtain the result. □

Recall from [2] that an element a of semigroup S is called *regular* if $a = axa$, for some $x \in S$. If all elements of S are regular, we say that S is a regular semigroup.

Corollary 2.17. *Every regular weak* 0*-simple semigroup is a* 0*-simple semigroup.*

Proof. Let S is a regular weak 0-simple semigroup. Then for every $a \in S$, there exists $x \in S$, such that $a = axa$, and so $a = a(xa)(xa)$. Since $xa \in E(S)$, therefore $a = a(xa)(xa) \in SE(S)S$, and so $S \subseteq SE(S)S$. Obviously, the converse of inclusion is satisfied. Thus $S = SE(S)S$, and so S is 0-simple, by Lemma 2.16, as required. □

3. Completely Weak (0-) Simple Semigroups

In this section by using the natural partial order relation on the idempotents of a semigroup, we define completely weak simple semigroups and completely weak 0-simple semigroups. Then we prove some results of these semigroups and give equivalences of them.

Among idempotents in an arbitrary semigroup, there is a natural partial order relation defined by the rule that $e \leq f$ if and only if $ef = fe = e$. It is easy to verify that the given relation is a partial order relation. If S is a semigroup with zero, then the defining properties of a zero element immediately imply that 0 is the unique minimum idempotent. The idempotents that are minimal within the set of non-zero idempotents are called *primitive*. Thus a primitive idempotent e has the property that

$$ef = fe = f \neq 0 \Longrightarrow e = f.$$

Recall from [2] that the semigroup S is called *completely simple* if it is simple and $(E(S), \leq)$ has a primitive idempotent. Also, the semigroup S$iscalledcompletely0-simpleifitis0-simpleand$(E(S),$\leq$) has a primitive idempotent.

Definition 3.1. *We say that the semigroup S is completely weak simple if S is a weak simple and $(E(S), \leq)$ has a primitive idempotent.*

Definition 3.2. *We say that the semigroup S is completely weak* 0*-simple if S is a weak* 0*-simple and $(E(S), \leq)$ has a primitive idempotent.*

Since in every finite weak simple semigroup, every element has a power which is idempotent, $E(S) \neq \emptyset$. Also since $|E(S)| < \infty$, $E(S)$ has a primitive idempotent. Therefore every finite weak simple semigroup is completely weak simple.

In every finite weak 0-simple semigroup, if $E(S) \neq \{0\}$, then by using a similar argument as in the finite weak simple semigroup, we can conclude that every finite weak 0-simple semigroup is completely weak 0-simple.

Notice that if S is a completely weak 0-simple semigroup, then S is not an infinite monogenic semigroup. Now similar to Theorem 2.3 we have the following theorem.

Theorem 3.3. *Let S is a completely weak 0-simple semigroup. Then one of the following statements holds:*

- (i) *$S^2 = S$, and so S is a completely 0-simple semigroup.*
- (ii) *There exists $n \geq 2$, such that $S^{n-1} \neq S^n$ and $S^n = S^{n+1}$. In this case, $S^n \setminus \{0\}$ is a group, and S^n is a completely 0-simple semigroup.*

Recall from [2] that an *equivalence* $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$, for $a, b \in S$, if and only if $S^1a = S^1b$. Similarly, an *equivalence* $\mathcal{R}$ on S is defined by the rule that $a\mathcal{R}b$, for $a, b \in S$, if and only if $aS^1 = bS^1$, and the join $\mathcal{L} \vee \mathcal{R}$ is denoted by $\mathcal{D}$. Also the $\mathcal{L}$-class [$\mathcal{R}$-class, $\mathcal{D}$-class] containing the element a is denoted by L_a [R_a, D_a].

Lemma 3.4. *Let S is a completely weak 0-simple semigroup and e is a primitive idempotent. Then,*

- (i) $R_e = eS \setminus \{0\}$.
- (ii) $L_e = Se \setminus \{0\}$.

Proof. We prove (i), the proof (ii) is similar. If $a \in R_e$, it is clear that $a \neq 0$. Since e is a left identity element for R_e, $ea = a \in eS \setminus \{0\}$, and so $R_e \subseteq eS \setminus \{0\}$. Now let $a = es \in eS \setminus \{0\}$. Then $ea = e^2s = es = a$. Since S is a weak 0-simple semigroup, there exist $t, z \in S$, such that $e = zat$, by Theorem 2.11. Now by using a similar argument as in the proof of [2, Lemma 3.2.4], we obtain the result. □

Lemma 3.5. *Let S is a completely weak 0-simple semigroup. Then there exists $n \geq 1$, such that for every $a_1, a_2, ..., a_n \in S$, if $a_1 a_2 ... a_n \neq 0$, then*

(*i*) $R_{a_1 a_2 ... a_n} = (a_1 a_2 ... a_n S) \setminus \{0\}$.

(*ii*) $L_{a_1 a_2 ... a_n} = (S a_1 a_2 ... a_n) \setminus \{0\}$.

Proof. (*i*) Let e is a primitive idempotent. Then there exists $n \geq 1$, such that $SeS = S^n$, by Theorem 2.11. Now let $a_1, a_2, ..., a_n \in S$, such that $a_1 a_2 ... a_n \neq 0$ and $u \in (a_1 a_2 ... a_n S) \setminus \{0\}$. Since $SeS = S^n$, there exist $z, t \in S$, such that $a_1 a_2 ... a_n = zet$, and so for some $v \in S$, $u = zev$. Thus $e\mathcal{R}et$ and $e\mathcal{R}ev$, by Lemma 3.4, and so $ev\mathcal{R}et$. Since $\mathcal{R}$ is a left congruence, we obtain $zev\mathcal{R}zet$, that is $u\mathcal{R}a_1 a_2 ... a_n$, and so $u \in R_{a_1 a_2 ... a_n}$. Therefore $(a_1 a_2 ... a_n S) \setminus \{0\} \subseteq R_{a_1 a_2 ... a_n}$, hence

$$[(a_1 a_2 ... a_n S) \setminus \{0\}] \setminus \{a_1 a_2 ... a_n\} \subseteq [R_{a_1 a_2 ... a_n} \setminus \{a_1 a_2 ... a_n\}]. \tag{3.1}$$

Now let $c \in R_{a_1 a_2 ... a_n} \setminus \{a_1 a_2 ... a_n\}$. Then there exists $x \in S$, such that $c = a_1 a_2 ... a_n x$. Thus $c \in [(a_1 a_2 ... a_n S) \setminus \{0\}] \setminus \{a_1 a_2 ... a_n\}$, and so

$$R_{a_1 a_2 ... a_n} \setminus \{a_1 a_2 ... a_n\} \subseteq [(a_1 a_2 ... a_n S) \setminus \{0\}] \setminus \{a_1 a_2 ... a_n\}. \tag{3.2}$$

Therefore $R_{a_1 a_2 ... a_n} \setminus \{a_1 a_2 ... a_n\} = [(a_1 a_2 ... a_n S) \setminus \{0\}] \setminus \{a_1 a_2 ... a_n\}$, by (1) and (2). Hence $R_{a_1 a_2 ... a_n} = a_1 ... a_n S \setminus \{0\}$, as required.

(*ii*) It is similar to the proof of (*i*). □

Theorem 3.6. *Let S is a completely weak 0-simple semigroup. Then there exists $n \geq 1$, such that S^n is completely 0-simple.*

Proof. Since S is a completely weak 0-simple semigroup, by Lemma 3.5, there exists $n \geq 1$, such that for every $a_1, a_2, ..., a_n \in S$, if $a_1 a_2 ... a_n \neq 0$, then

(*i*) $R_{a_1 a_2 ... a_n} = (a_1 a_2 ... a_n S) \setminus \{0\}$.

(*ii*) $L_{a_1 a_2 ... a_n} = (S a_1 a_2 ... a_n) \setminus \{0\}$.

Let $x = a_1 ... a_n$ and $y = b_1 ... b_n$ are a non-zero elements of S^n. Then $a_1 ... a_n S b_1 ... b_n \neq \{0\}$, since otherwise, if $a_1 ... a_n S b_1 ... b_n = \{0\}$, then, by

Theorem 2.11, there exist $m_1, m_2 \in \mathbf{N}$, such that $Sa_1...a_nS = S^{m_1}$ and $Sb_1...b_nS = S^{m_2}$. Then

$$S^{m_1+m_2} = (Sa_1...a_nS)(Sb_1...b_nS) = S(a_1...a_nS^2b_1...b_n)S$$
$$\subseteq S(a_1...a_nSb_1...b_n)S = \{0\},$$

and so $S^{m_1+m_2} = \{0\}$, which is a contradiction. Thus $a_1...a_nSb_1...b_n \neq \{0\}$, and so there exists $u \in S$, such that

$$(a_1...a_n)u(b_1...b_n) = d \neq 0.$$

Since $R_{b_1b_2...b_n} = (b_1...b_nS) \setminus \{0\}$, there exists $k \in S$, such that $b_1...b_nk = b_1...b_n$. Thus $b_1...b_nk^m = b_1...b_n$, for every $m \geq 1$. Also since $R_{a_1a_2...a_n} = (a_1a_2...a_nS) \setminus \{0\}$ and $d \in (a_1a_2...a_nS) \setminus \{0\}$, there exists $l \in S$, such that

$$a_1...a_n = dl = (a_1...a_n)u(b_1...b_n)l = (a_1...a_n)u(b_1...b_n)k^nl = dk^nl.$$

Therefore $(a_1...a_n, d) \in \mathcal{R}^{S^n}$. Similarly, we can show $(d, b_1...b_n) \in \mathcal{L}^{S^n}$, and so

$$(a_1...a_n, b_1...b_n) \in \mathcal{L}^{S^n} \circ \mathcal{R}^{S^n} = \mathcal{D}^{S^n}.$$

Thus $S^n \setminus \{0\}$ and $\{0\}$ are $\mathcal{D}^{S^n}$-classes, in S^n. Since $E(S) \setminus \{0\} \subseteq S^n\{0\}$ and S has a primitive idempotent, $\mathcal{D}^{S^n}$-class $S^n \setminus \{0\}$ has a regular element, and so $S^n \setminus \{0\}$ is regular. Since $\{0\}$ is regular, tS^n is regular. Let $a \in S^n$. Then there exists $x \in S^n$, such that $a = axa$, and so $a = a(xa)(xa)$. Since $xa \in E(S)$, therefore $a = a(xa)(xa) \in S^nE(S)S^n$, and so $S^n \subseteq S^nE(S)S^n$. Obviously, the converse of inclusion is satisfied. Thus $S^n = S^nE(S)S^n$, and so $S^n = S^{n+1}$, that is S^n is 0-simple. Since S has a primitive idempotent and $E(S) \subseteq S^n$, S^n is completely 0-simple. □

By using a similar argument as in the proof of Theorem 3.6, for every completely weak simple semigroup S, there exists $n \geq 1$, such that S^n is completely simple and $E(S) \subseteq S^n$. Since every idempotent is primitive, in every completely simple semigroup, thus every idempotent is primitive, in every completely weak simple semigroup.

Lemma 3.7. *For a regular semigroup S, the following statements are equivalent:*

- (i) *S has only one idempotent;*
- (ii) *S is cancellative;*
- (iii) *S is a group.*

Proof. $(i) \Rightarrow (ii)$. Since S is regular, for every $a \in S$, there exists $x \in S$, such that $a = axa$. Then ax and xa are idempotent. Thus for every $a \in S$, there exists $x \in S$ such that $ax = xa = e$ and $ae = ea = a$. Now let $ac = bc$, for $a, b, c \in S$. Thus there exists $x \in S$ such that $cx = e$. Therefore $acx = bcx$ implies that $a = b$. Similarly, $ca = cb$ implies that $a = b$, and so S is cancellative, as required.

$(ii) \Rightarrow (iii)$. Since S is regular and cancellative, for every $a \in S$, there exists $x \in S$ such that $axa = a$ and $xax = x$. Let $xa = e$. Then $xax = x$ implies that $xaxe = xe$, and so $ax = e$. Therefore $ax = xa = e$, and so $a^2x = a$ and $ax^2 = x$. Now $a^2xb = ab$, implies that $a(xb) = b$, for every $b \in S$. Also $bax^2 = bx$ implies that $bax = b$, and so $(bx)a = b$, for every $b \in S$. Therefore $aS = Sa = S$, for every $a \in S$, that is S is a group.

$(iii) \Rightarrow (i)$. It is obvious. □

Corollary 3.8. (i) *Let S is a weak simple semigroup, such that $|E(S)| = 1$. Then there exists $n \geq 1$, such that S^n is a group.*
(ii) *Let S is a completely weak simple semigroup, such that is cancellable. Then there exists $n \geq 1$, such that S^n is a group.*

Proof. (i). Let S is a weak simple semigroup, such that $|E(S)| = 1$. Then S has a primitive element, and so S is a completely weak simple semigroup. Since Theorem 3.6 is also true, for every completely weak simple semigroup, therefore there exists $n \geq 1$, such that S^n is completely simple, and so S^n is regular. Since $E(S) \subseteq S^n$ and $|E(S)| = 1$, S^n is a group, by Lemma 3.7.

(ii). Since Theorem 3.6 is also true, for every completely weak simple semigroup, therefore there exists $n \geq 1$, such that S^n is completely simple, and so S^n is regular. Since S is cancellative, S^n is also cancellative, and so S^n is a group, by Lemma 3.7. □

Notice that every theorem that said for (completely) weak 0-simple semigroup, satisfies for (completely) weak simple semigroup, too.

Recall from [2] that for a and b in semigroup S, $L_a \leq L_b$ ($R_a \leq R_b$) if $S^1a \subseteq S^1b$ ($aS^1 \subseteq bS^1$). Thus we may regard $S/\mathcal{L}$ ($S/\mathcal{R}$) as a partially ordered set. The semigroup S satisfies *condition* min_L (min_R) according to the partially ordered set $S/\mathcal{L}$ ($S/\mathcal{R}$) satisfies the minimal condition.

In a semigroup S, an ideal minimal within the set of all non-zero ideals is called 0-*minimal*.

A semigroup S is called *group-bound* if every element a in S has a power a^n $(n \geq 1)$ lying in a subgroup of S.

Theorem 3.9. *Let S is a weak 0-simple semigroup. Then the following statements are equivalent:*

(*i*) *S is a completely weak 0-simple semigroup;*

(*ii*) *there exists $n \geq 1$, such that S^n is completely 0-simple;*

(*iii*) *there exists $n \geq 1$, such that S^n is group-bound and simple;*

(*iv*) *there exists $n \geq 1$, such that S^n is simple and satisfies in conditions $min_{R^{S^n}}$ and $min_{L^{S^n}}$;*

(*v*) *there exists $n \geq 1$, such that S^n is simple and contains at least one left 0-minimal ideal that is simple and contains at least one right 0-minimal ideal that is simple.*

Proof. (*i*) $\Rightarrow$ (*ii*). Since S is a weak 0-simple semigroup, there exists $n \geq 1$, such that S^n is completely 0-simple, by Theorem 3.6, and so (*ii*) is satisfied.

(*ii*) $\Rightarrow$ (*i*). Since $E(S) \subseteq S^n$, for every $n \geq 1$, S has a primitive idempotent, and so S completely weak 0-simple, as required.

Statements (*ii*), (*iii*), (*iv*) and (*v*) are equivalent, by [2, Theorem 3.2.11]. $\square$

A semigroup S is called *completely regular* if every element a of S lies in a subgroup of S (see [2]).

Similarly, we can show the following theorem for weak simple semigroup.

Theorem 3.10. *Let S is a weak simple semigroup. Then the following statements are equivalent:*

(*i*) *S is a completely weak simple semigroup;*

(*ii*) *there exists $n \geq 1$, such that S^n is completely simple;*

(*iii*) *there exists $n \geq 1$, such that S^n is completely regular and simple;*

(*iv*) *there exists* $n \geq 1$, *such that* S^n *is simple and satisfies in conditions* $min_{R^{S^n}}$ and $min_{L^{S^n}}$;

(*v*) *there exists* $n \geq 1$, *such that* S^n *is simple and contains at least one left minimal ideal that is simple and contains at least one right minimal ideal that is simple.*

Recall from [2] that a semigroup S has the *weak cancelation* property if for all a, b, c in S, $ca = cb$ and $ac = bc$ imply that $a = b$.

Theorem 3.11. *Let* S *is a semigroup, such that if* $A^n = S^n$, *for ideal* A *of* S *and* $n \geq 1$, *then there exists* $m \geq 1$, *such that* $A = S^m$. *Then the following statements are equivalent:*

(*i*) S *is a completely weak simple semigroup;*

(*ii*) *there exists* $n \geq 1$, *such that* S^n *is completely simple;*

(*iii*) *there exists* $n \geq 1$, *such that* S^n *is regular and has the weak cancelation property;*

(*iv*) *there exists* $n \geq 1$, *such that* S^n *is regular and for every* $a, b \in S^n$, $aba = a$ *implies* $bab = b$;

(*v*) *there exists* $n \geq 1$, *such that* S^n *is regular and every idempotent in* S *is primitive.*

Proof. Statements (*ii*), (*iii*), (*iv*) and (*v*) are equivalent, by [2, Theorem 3.2.11].

(*i*) $\Rightarrow$ (*ii*). Since Theorem 3.6 is also true, for every completely weak simple semigroup, therefore there exists $n \geq 1$, such that S^n is completely simple, as required.

(*ii*) $\Rightarrow$ (*i*). It is sufficient to show S is weak simple. Let A is an ideal of S. Then $S^n A^n S^n \subseteq A^n$, that is A^n is an ideal of S^n. Thus $A^n = S^n$, because S^n is simple. Now by assumption, there exists $m \geq 1$, such that $A = S^m$. Thus the powers of S, are the only ideals of S, and so S is weak simple. □

Theorem 3.12. *Let* S *is a semigroup, such that if* $A^n = S^n$, *for ideal* A *of* S *and* $n \geq 1$, *then there exists* $m \geq 1$, *such that* $A = S^m$. *Then the following statements are equivalent:*

(*i*) *S is a completely weak simple semigroup;*

(*ii*) *there exists $n \geq 1$, such that S^n is completely simple;*

(*iii*) *there exists $n \geq 1$, such that S^n is completely regular and for every $x, y \in S^n$, $xx^{-1} = (xyx)(xyx)^{-1}$;*

(*iv*) *there exists $n \geq 1$, such that S^n is completely regular and simple.*

Proof. $(i) \Leftrightarrow (ii)$ is satisfied, by Theorem 3.11.

Statements (ii), (iii) and (iv) are equivalent, by [2, Proposition 4.1.2]. $\square$

References

[1] P. M. Higgins, *Techniques of semigroup theory*. Oxford: Oxford University, 1992.

[2] J. M. Howie, *Fundamentals of semigroup theory*. London: Oxford University Press, 1995.

[3] M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categorie*. Berlin: Walter de Gruyter, 2000.

Leila Nouri
Department of Mathematics,
University of Sistan and Baluchestan,
Zahedan,
Iran
e-mail: Leila_Nouri@math.usb.ac.ir
and

Hossein Mohammadzadeh Saany
Department of Mathematics,
University of Sistan and Baluchestan,
Zahedan,
Iran
e-mail: hmsdm@math.usb.ac.ir
Corresponding author