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Some characterizations of frames in $\ell^2(I;H)$ and topological applications

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Abstract

We propose in this article some characterizations of the notion of frame in $\ell^2(I;H)$. The first one is general, and depends on a procedure of inserting a family of vectors instead of x in the definition of a frame. This allows us to define the analysis, synthesis and frame operator on the space $\ell^2(I;H)$ instead of H. The second one is specific to $\ell^2(I;\mathbf{C}^k)$ and relate it to the freeness of the finite set of components of the frame. The third one concerns normalised tight frames in $\ell^2(I;\mathbf{C}^k)$. Afterwards, we give an example of a frame in $\ell^2(I;\mathbf{C}^2)$ using another sufficient condition in dimension 2. We conclude with some topological applications of these characterizations.

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1. Introduction

Frames introduced by Duffin and Shaeffer in [1] have recently received great attention due to their wide range of applications in both mathematics and engineering science.

The classical definition of the notion of a frame is that of a family of vectors in a Hilbert space H that satisfy a double inequality involving the scalar product and the norm maps. Specifically :

Definition 1. *We say that a family $u = \{u_n\}_{n\in I}$ with u_n in H is a frame if*

$$\exists A, B > 0, \forall x \in H, A\|x\|^2 \leq \sum_{n\in I} |\langle u_n, x\rangle|^2 \leq B\|x\|^2$$

Remark 1. *Bessel families are the families for which only the second inequality holds.*

Frames have been generalized in different directions and notably in the setting of Hilbert C*-modules.

In the definition of the notion of a frame given above, it could be observed that the variable x belongs to a type of "test space" H over which we are testing the frame inequality.

Therefore, the definition of this notion could in theory be extendend in many directions provided that we supply a good notion of a mathematical object which is somehow "dual" to that of a family of vectors, or generalizations thereof.

Taking the "duality" map to be :

$$(\{u_n\}_{n\in I}, x) \mapsto \{\langle u_n, x\rangle\}_{n\in I},$$

where x belongs to the test space H, we get the classical notion of frame.

In the following section, we investigate the duality map :

$$(\{u_n\}_{n\in I}, \{v_n\}_{n\in J}) \mapsto \{\langle u_n, v_m\rangle\}_{(n,m)\in I\times J},$$

where $\{v_n\}_{n\in J}$ belongs to the test space $\ell^2(J; H)$.

Proposition 3 of the next section says that generalizing the definition of a frame naturally through this duality doesn't bring any novelty. Nevertheless, this proposition allows us to define the analysis and frame operators on the spaces $\ell^2(I \times J, H)$; therefore enlarging the domains of these operators as they are classically defined.

The next sections are concerned with the case $H = \mathbf{C}^k$ or $H = \mathbf{C}^2$. We find some expressions of the functions N (later defined) and prove some characterizations of frames and normalized tight frames.

Finally, we answer some questions related to the topological nature of the set of frames or normalized tight frames inside $H = \mathbf{C}^k$.

2. Test space: from vectors to family of vectors

Let I,J be countably infinite sets and $\mathcal{H} = \{H_i\}_{i\in I}$ a family of Hilbert spaces.

We set : $\ell^2(I;\mathcal{H}) = \{u = \{u_i\}_{i\in I}; u_i \in H_i, \sum_{i\in I} ||u_i||^2_{H_i} < \infty\}$

We endow $\ell^2(I;\mathcal{H})$ with the pointwise scalar product :
$\langle u, v\rangle = \sum_{i\in I}\langle u_i, v_i\rangle_{H_i}$

We recall the proof of the following proposition.

Proposition 1. $\ell^2(I;\mathcal{H})$ *is a Hilbert space.*

Proof.

- $\ell^2(I;H)$ is a vector space thanks to $|a+b|^2 \leq 2(|a|^2 + |b|^2)$ and the fact that H_i are vector spaces.
- $\langle\cdot,\cdot\rangle$ is a scalar product because each $\langle\cdot,\cdot\rangle_{H_i}$ is.
- Let's show that $\ell^2(I,H)$ is complete. Let $(u^{(n)})_{n\in\mathbf{N}}$ be a Cauchy sequence in $\ell^2(I,\mathcal{H})$. Let $\epsilon > 0$. Let $N \in \mathbf{N}$ such that $\forall n, m \geq N, ||u^{(n)} - u^{(m)}||^2 \leq \epsilon$. Therefore, for $i \in I$ fixed, we have : $\forall n, m \geq N, ||u_i^{(n)} - u_i^{(m)}||^2_{H_i} \leq \epsilon$, so $\{u_i^{(n)}\}_{n\in\mathbf{N}}$ is a Cauchy sequence in H_i, so it converges to a certain $u_i \in H_i$, H_i being complete. Moreover, we have for every finite subset P of I, $\forall n, m \geq N, \sum_{i\in I\cap P} ||u_i^{(n)} - u_i^{(m)}||^2_{H_i} \leq \epsilon$, and passing to the limit $n \to +\infty$: $\forall m \geq N, \sum_{i\in I\cap P} ||u_i - u_i^{(m)}||^2_{H_i} \leq \epsilon$, valid for each finite subset P, so by setting $u = \{u_i\}_{i\in I}$ we have : $\forall m \geq N, ||u - u^{(m)}||^2 \leq \epsilon$, which proves the convergence of $(u^{(n)})_{n\in\mathbf{N}}$ to $u \in \ell^2(I,\mathcal{H})$

$\square$

Remark 2. $\ell^2(I;\mathcal{H})$ *is a (vector) subspace of the space of all families of vectors of* $\mathcal{H}$ *indexed by* $i \in I$.

We fix the family $\{H_i\}_{i\in I}$ such that each H_i is equal to the same Hilbert space H.
We define $\ell^2(J;H) := \ell^2(J;\mathcal{H})$. **It will be our test space in this section.**
The next proposition asserts that the natural generalization of the notion of a frame obtained by changing the test space to $\ell^2(J,H)$; doesn't bring anything new. It is more generally formulated in the context of frames associated with measurable spaces.

Proposition 2. *Let X be a σ-finite measure space with positive measure μ and $\{u_x\}_{x\in X}$ a family in H such that $\forall f \in H$, $x \mapsto \langle u_x, f\rangle$ is a measurable function on X. The following are equivalent :*

1. *There exists constants $A, B > 0$ such that*
$$\forall f \in H : A\|f\|^2 \leq \int_X |\langle u_x, f\rangle|^2 d\mu \leq B\|f\|^2$$
(i.e. $\{u_x\}_{x\in X}$ is a frame associated with the measure space (X,μ))
2. *There exists constants $A, B > 0$ such that for each σ-finite measure space (Y,ν) such that ν is a non-zero positive measure, and for all $f = \{f_y\}_{y\in Y}$ such that $(x, y) \mapsto \langle u_x, f_y\rangle$ is a measurable function on $X \times Y$ and $f \in \ell^2(Y,\nu;H)$, we have*
$$A\int_Y \|f_y\|^2 d\nu \leq \int_{X\times Y} |\langle u_x, f_y\rangle|^2 d\mu \leq B\int_Y \|f_y\|^2 d\nu$$

Proof. $(1 \Rightarrow 2)$ This part follows immediately by integration and Tonelli's theorem.
$(2 \Rightarrow 1)$ Take $f \in H$ and $D \subset Y$ a measurable subset of Y with $0 < \nu(D) < +\infty$. Then $g = \frac{1_D}{\nu(D)} \in \ell^2(Y)$ satisfy $\|g\|_2 = 1$. Applying the hypothesis to $f_y = g(y)f$ yields the result. $\square$

In the case of classical frames, this specializes to :

Proposition 3. *Suppose that $u = \{u_n\}_{n\in I}$ with $u_n \in H$. Then the following are equivalent.*

1. *There exists $A, B > 0$ such that*
$$\forall x \in H, A\|x\|^2 \leq \sum_{n\in I} |\langle u_n, x\rangle|^2 \leq B\|x\|^2$$
(i.e. $\{u_k\}_{k\in\mathbf{N}}$ is a frame)

2. *There exists $A, B > 0$ such that for all countably infinite sets J and $v = \{v_k\}_{k\in J} \in \ell^2(J;H)$*

$$A\sum_{k\in J}\|v_k\|^2 \leq \sum_{n\in I,k\in J}|\langle u_n, v_k\rangle|^2 \leq B\sum_{k\in J}\|v_k\|^2$$

Proof. $(1 \Rightarrow 2)$ Immediate through summation.
$(2 \Rightarrow 1)$ Suppose there exists constants $A, B > 0$ such that :

$$A\sum_{k\in J}\|v_k\|^2 \leq \sum_{n\in I,k\in J}|\langle u_n, v_k\rangle|^2 \leq B\sum_{k\in J}\|v_k\|^2$$

By choosing v such that $v_m = x$ for only one fixed index m and 0 for other indices, we obtain :

$$A\|x\|^2 \leq \sum_{n\in I}|\langle u_n, x\rangle|^2 \leq B\|x\|^2$$

. □

Based on this characterization, we can define, for a frame $u \in \ell^2(I;H)$ with bounds $A, B > 0$:

$$\begin{array}{ll} T : \ell^2(I;H) & \to \ell^2(I\times J) \\ v = \{v_m\}_{m\in I} & \mapsto \{\langle v_m, u_n\rangle\}_{(m,n)\in I\times J} \\ T^* : \ell^2(I\times J) & \to \ell^2(I;H) \\ \{\lambda_{m,n}\}_{(m,n)\in I\times J} & \mapsto \{\sum_{n\in J}\lambda_{m,n}u_n\}_{m\in I} \\ S = T^*T : \ell^2(I;H) & \to \ell^2(I;H) \\ v = \{v_m\}_{m\in I} & \mapsto \{\sum_{n\in J}\langle v_m, u_n\rangle u_n\}_{m\in I} \end{array}$$

called respectively the analysis, synthesis and frame operators.
We have :

$$A\|v\|^2 \leq \langle Sv, v\rangle = \sum_{(m,n)\in I^2}|\langle v_m, u_n\rangle|^2 \leq B\|v\|^2$$

which shows in particular that S is an invertible, positive definite operator on $\ell^2(I,H)$.
In fact S is a diagonal operator, so its importance compared to the classical frame operator is not yet clear.
Finally let's set up some notation.
We set, for $u = \{u_n\}_{n\in I}$ with $u_n \in H$ and $x \in H\backslash\{0\}$:

$$N(u,x) = \frac{\sum_{n\in I}|\langle u_n, x\rangle|^2}{||x||^2}$$

We also set, for $u = \{u_n\}_{n\in I}$ with $u_n \in H$ and $v \in \ell^2(J;H)\backslash\{0\}$:

$$N(u,v) = \frac{\sum_{n\in I, m\in J} |\langle u_n, v_m\rangle|^2}{||v||^2} = \frac{\sum_{n\in I, m\in J} |\langle u_n, v_m\rangle|^2}{\sum_{m\in J} ||v_m||^2}$$

3. Rewriting of N in the case $H = \mathbf{C}^k$

We will be writing in this section scalar products which may live in different spaces.

Definition 2. *Let $u \in \ell^2(I;\mathbf{C}^k)$. We define the Gramian matrix of $\{u^i\}_{i=1\cdots k}$ as the matrix U whose i,j-th component is $U_{i,j} = \langle u^i, u^j\rangle$. This matrix is well defined because $u \in \ell^2(I,\mathbf{C}^k)$ implies that each component u^i is in $\ell^2(I)$.*

Proposition 4. *For $u, v \in \ell^2(I;\mathbf{C}^k)\backslash\{0\}$ and $x \in H\backslash\{0\}$, we have*

$$N(u,x) = \frac{\sum_{i,j=1}^k \langle u^i, u^j\rangle x^j \overline{x^i}}{||x||^2} = \frac{|| \sum_{i=1}^k \overline{x^i} u^i ||^2_{\ell^2(I)}}{||x||^2} \tag{3.1}$$

$$N(u,v) = \frac{\sum_{i,j=1}^k \langle u^i, u^j\rangle \langle v^j, v^i\rangle}{\sum_{j=1}^k ||v^j||^2} = \frac{\mathrm{Tr}\ (UV)}{\mathrm{Tr}\ (V)} \tag{3.2}$$

where U and V denote the Gramian matrices of $\{u^i\}_{i=1\cdots k}$ and $\{v^i\}_{i=1\cdots k}$ respectively.

Proof. We set $u_n = (u_n^1, \cdots, u_n^k)$, $v_n = (v_n^1, \cdots, v_n^k)$ and $x = (x^1, \cdots, x^k)$. So $\langle u_n, x\rangle = \sum_{i=1}^k u_n^i \overline{x^i}$ and $\langle u_n, v_m\rangle = \sum_{i=1}^k u_n^i \overline{v_m^i}$.
We then have

$$\begin{aligned} \sum_{n\in I} |\langle u_n, x\rangle|^2 &= \sum_{n\in I} \sum_{i,j=1}^k u_n^i \overline{x^i}\,\overline{u_n^j} x^j = \sum_{i,j=1}^k \sum_{n\in I} (u_n^i \overline{u_n^j})(\overline{x^i} x^j) \\ &= \sum_{i,j=1}^k (\sum_{n\in I} u_n^i \overline{u_n^j}) x^j \overline{x^i} = \sum_{i,j=1}^k \langle u^i, u^j\rangle x^j \overline{x^i} \\ &= || \sum_{i=1}^k \overline{x^i} u^i ||^2_{\ell^2(I)} \end{aligned}$$

In the same way, we find that

$$\begin{aligned} \sum_{n,m\in I} |\langle u_n, v_m\rangle|^2 &= \sum_{n,m\in I} \sum_{i,j=1}^k u_n^i \overline{v_m^i}\,\overline{u_n^j} v_m^j \\ &= \sum_{i,j=1}^k \sum_{n,m\in I} (u_n^i \overline{u_n^j})(\overline{v_m^i} v_m^j) \\ &= \sum_{i,j=1}^k (\sum_{n\in I} u_n^i \overline{u_n^j})(\sum_{m\in I} v_m^j \overline{v_m^i}) \\ &= \sum_{i,j=1}^k \langle u^i, u^j\rangle \langle v^j, v^i\rangle \end{aligned}$$

We also have

$$\begin{aligned}\sum_{n\in I}||u_n||^2 &= \sum_{n\in I}\sum_{i=1}^k |u_n^i|^2\\ &= \sum_{i=1}^k\sum_{n\in I}|u_n^i|^2\\ &= \sum_{i=1}^k ||u^i||^2\end{aligned}$$

Finally, the equality involving the trace is a direct calculation □

4. Second and third characterizations

Remark 3. *If $u \in \ell^2(I,H)\backslash\{0\}$, then it is necessarily a Bessel family due to the Cauchy-Schwarz inequality. Therefore we have :*

$$u \text{ is a frame } \Leftrightarrow \exists K > 0, \forall x \in H\backslash\{0\}, K \leq N(u,x)$$

We have the following characterization

Theorem 1. *Suppose $u \in \ell^2(I;\mathbf{C}^k)\backslash\{0\}$. Then :*

$$u \text{ is a frame} \Leftrightarrow \{u^i\}_{i=1\cdots k} \text{ is free}$$

Proof. ($\Rightarrow$) First let's suppose that u is a frame. For the sake of contradiction, suppose that $\{u^i\}_{i=1\cdots k}$ is dependant. So there exists $\lambda_1\cdots,\lambda_k$ not all zero such that $\sum_{i=1}^k \lambda_i u^i = 0$. We then have $N(u,\overline{\lambda}) = 0$ using Nux, which is a contradiction. Hence, $\{u^i\}_{i=1\cdots k}$ is free.
($\Leftarrow$) Suppose that $\{u^i\}_{i=1\cdots k}$ is free. In this case, the nonnegative continuous function $N(u,x)$ of the second variable restricted to the unit sphere of $\mathbf{C}^k$ has a global minimum, the unit sphere being compact. This minimum is nonnegative and different from 0 because $\{u^i\}_{i=1\cdots k}$ is free. Let's denote it by $K > 0$. We then have for $x \in H\backslash\{0\}$: $N(u,x) = N(u,\frac{x}{||x||}) \geq K$ by homogeneity of N in the second variable. □

Remark 4. *This theorem has been proved in [3] in the case of a finite family $u = \{u_n\}_{n=1\cdots m}$ (see proposition 1.4.3 p.19).*

Another short proof of the direction ($\Leftarrow$) of the theorem is given below. We denote by U the Gramian matrix of $\{u^i\}_{i=1\cdots k}$.
Suppose that $\{u^i\}_{i=1\cdots k}$ is free. So the Gramian matrix is invertible, i.e. $\det(U) \neq 0$. The matrix U is a positive semidefinite matrix, so the condition $\det(U) \neq 0$ is equivalent to the fact that all the eingenvalues of U satisfy $\lambda > 0$. Now, using (3.2), we have :

$$\forall v \in \ell^2(I,\mathbf{C}^k)\backslash\{0\} : N(u,v) = \frac{\text{Tr }(UV)}{\text{Tr }(V)} \geq \lambda_k(U) > 0$$

where V denotes the Gramian matrix of $\{v^i\}_{i=1\cdots k}$ and $\lambda_k(U)$ is the smallest eigenvalue of U (see [4] for a trace inequality from which this one could be easily derived).

Definition 3. *We say that a family $u = \{u_n\}_{n\in I}$ in H is a normalised tight frame if*

$$\forall x \in H\backslash\{0\}, \frac{\sum_{n\in I} |\langle u_n, x\rangle|^2}{||x||^2} = 1$$

We have the following characterization

Theorem 2. *Suppose $u \in \ell^2(I; \mathbf{C}^k)\backslash\{0\}$. Then*

$$u \text{ is a normalised tight frame} \Leftrightarrow U = Id$$

where U is the Gramian matrix of $\{u^i\}_{i=1\cdots k}$

Proof. ($\Rightarrow$) Suppose u is a normalised tight frame. Then : $\forall x \in \mathbf{C}^k\backslash\{0\} : N(u, x) = 1$. By choosing $x_i = 1$ for one index i and 0 otherwise, we have : $\forall i = 1\cdots k : ||u^i|| = 1$. Next, by picking two indices i and j and choosing $x_i = \frac{1}{\sqrt{2}}$ and $x_j = \frac{1}{\sqrt{2}}$ and 0 otherwise, we have $\Re(\langle u^i, u^j\rangle) = 0$. Choosing this time $x_i = \frac{1}{\sqrt{2}}$ and $x_j = \frac{i}{\sqrt{2}}$ and 0 otherwise, we have $\Im(\langle u^i, u^j\rangle) = 0$ and so $\langle u^i, u^j\rangle = 0$. This means that $U = Id$.
($\Leftarrow$) Suppose $U = Id$. Then by Pythagoras' theorem,

$$\forall x \in \mathbf{C}^k\backslash\{0\} : N(u, x) = \frac{|| \sum_{i=1}^k \overline{x^i} u^i ||^2}{||x||^2} = \frac{||x||^2}{||x||^2} = 1$$

which means that u is a normalised tight frame □

Remark 5. *This theorem has been proved in [3] in the case of a finite family $u = \{u_n\}_{n=1\cdots m}$ (see proposition 1.4.7 p.21).*

If $k = 2$, a necessary and sufficient condition for $u \in \ell^2(I; \mathbf{C}^2)\backslash\{0\}$ to be a frame is that the determinant of the Gramian matrix is (strictly) positive, that is $||u^1||||u^2|| > |\langle u^1, u^2\rangle|$. In the next section, we are going to prove in another way a sufficient condition for $u \in \ell^2(I; \mathbf{C}^2)\backslash\{0\}$ to be a frame.

5. Sufficient condition for a family in $\mathbf{C}^2$ to be a frame

Proposition 5. *Let $u \in \ell^2(I; \mathbf{C}^2)\backslash\{0\}$. We set $u = (u^1, u^2)$. A sufficient condition for u to be a frame is*

$$\min(||u^1||^2, ||u^2||^2) > |\langle u^1, u^2\rangle|$$

Proof. We have for all non-zero v :

$$N(u,v) = \frac{\sum_{i,j=1}^2 \langle u^i,u^j\rangle\langle v^j,v^i\rangle}{\sum_{j=1}^2 ||v^j||^2} = \frac{||u^1||^2||v^1||^2+||u^2||^2||v^2||^2+2Re(\langle u^1,u^2\rangle\langle v^2,v^1\rangle)}{||v^1||^2+||v^2||^2}$$

If $||u^1||^2 \leq ||u^2||^2$, we have :

$$\frac{||u^1||^2||v^1||^2+||u^2||^2||v^2||^2}{||v^1||^2+||v^2||^2} = \frac{||u^1||^2(||v^1||^2+||v^2||^2)}{||v^1||^2+||v^2||^2} + \frac{||v^2||^2(||u^2||^2-||u^1||^2)}{||v^1||^2+||v^2||^2} \geq ||u^1||^2 = \min(||u^1||^2, ||u^2||^2)$$

By symmetry, the same lower bound is obtained if $||u^1||^2 \geq ||u^2||^2$. By using the inequality

$$2|Re(a\langle v^2,v^1\rangle)| \leq 2|a\langle v^2,v^1\rangle| \leq 2|a|||v^1||||v^2|| \leq |a|(||v^1||^2+||v^2||^2)$$

valid for all $a \in \mathbf{C}$, we then obtain

$$\frac{||u^1||^2||v^1||^2+||u^2||^2||v^2||^2+2Re(\langle u^1,u^2\rangle\langle v^2,v^1\rangle)}{||v^1||^2+||v^2||^2} \geq \min(||u^1||^2, ||u^2||^2) - |\langle u^1,u^2\rangle|$$

The quantity $\min(||u^1||^2, ||u^2||^2) - |\langle u^1,u^2\rangle|$ being strictly positive, the characterization of section 2 shows that u is a frame □

Example 1. *Let's give an example of a frame in $\mathbf{C}^2$ using the previous criterion.*
We'll define two families of complex numbers (sequences of numbers for simplicity) u^1 and u^2 that satisfy the previous criterion and that are square summable.
We take $u_n^1 = \frac{1}{n}e^{2\pi ian}$ and $u_n^2 = \frac{1}{n}e^{2\pi ibn}$ with a,b reals such that $a-b$ is not an integer. u^1 and u^2 are square summable with sum $\frac{\pi^2}{6}$.
$|\langle u^1,u^2\rangle| = |\sum_{n\in\mathbf{N}} \frac{1}{n^2}e^{2\pi i(a-b)n}| < \sum_{n\in\mathbf{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$ by the strict triangle inequality, the equality case being excluded because $\{e^{2\pi i(a-b)n}\}_{n\in\mathbf{N}}$ are not positively proportional due to the conditions required on a and b.
From this, we deduce that $u=(u^1,u^2)$ is a frame in $\mathbf{C}^2$.

6. Topological properties of frames in $\ell^2(I,\mathbf{C}^k)$

Proposition 6. *Let's denote by $\mathcal{F}$ the set of frames in $\ell^2(I,\mathbf{C}^k)$.*
Then $\mathcal{F}$ is open, path-connected and dense.

Proof. $\mathcal{F}$ is open because theorem 1 implies that $\mathcal{F} = (\det\circ G)^{-1}(\mathbf{R}^*)$, where the continuous function G sends $u \in \ell^2(I,H)$ to the Gramian matrix of $\{u^i\}_{i=1\cdots k}$.
Let's prove the path-connectedness of $\mathcal{F}$. Let $u,v \in \ell^2(I,\mathbf{C}^k)$ be two

frames. We take $\{a^i\}_{i=1\cdots k}$ a finite independant set of families in $\ell^2(I)$ satisfying $span(\{a^i\}_{i=1\cdots k}) \cap span(\{u^i\}_{i=1\cdots k} \cup \{v^i\}_{i=1\cdots k}) = \{0\}$ (this is possible because $\ell^2(I)$ is infinite dimensional).
We define the continuous paths

$$\begin{cases} \gamma_1 & : [0,1] \to \ell^2(I, \mathbf{C}^k) \\ \gamma_2 & : [0,1] \to \ell^2(I, \mathbf{C}^k) \end{cases}$$

by $\gamma_1(t) = tu + (1-t)a$ and $\gamma_2(t) = tv + (1-t)a$ respectively.
We have $\gamma_1(1) = u$ and $\gamma_2(1) = v$.
We are going to show that $\forall t \in [0,1] : \gamma_1(t)$ and $\gamma_2(t)$ are frames, which will prove this part of the proposition after composing γ_1 with the inverse path of γ_2. By symmetry, we'll show it only for γ_1.
Consider $\lambda_1, \cdots, \lambda_k \in \mathbf{C}$ such that : $\sum_{i=1}^k \lambda_i \gamma_1^i(t) = 0$. It follows that $\sum_{i=1}^k \lambda_i t u^i = -\sum_{i=1}^k \lambda_i (1-t) a^i \in span(\{a^i\}_{i=1\cdots k}) \cap span(\{u^i\}_{i=1\cdots k}) = \{0\}$. Now, since $\{u^i\}_{i=1\cdots k}$ is free, we have $\forall i = 1\cdots k : \lambda_i = 0$.
$\forall i = 1 \cdots k : a_t^i = u^i + t(v^i - u^i)$. Let $A(t)$ be the determinant of the Gramian of $\{a_t^i\}_{i=1\cdots k}$. Clearly, $A(t)$ is a polynomial in t which satisfies $A(0) \neq 0$ and $A(1) \neq 0$ (theorem 1), so $\deg A \geq 1$ or $A(t)$ is a nonzero constant. Therefore $A(t) \neq 0$ except for a finite number of t's. We choose a continuous path $\gamma : [0,1] \to \mathbf{C}$ such that $\gamma(0) = 0$, $\gamma(1) = 1$ and $\gamma(t)$ avoids the roots of A for all $t \in [0,1]$. Thus $b_t : [0,1] \to a_{\gamma(t)}$ is a continuous path from u to v. To prove the density of $\mathcal{F}$, let's consider $u \in \ell^2(I, \mathbf{C}^k)$. Let's take a finite independant set of families in $\ell^2(I)$: $\{a^i\}_{i=1\cdots k}$ (this is possible because $\ell^2(I)$ is infinite dimensional). For each real number t, we define $v_t \in \ell^2(I, \mathbf{C}^k)$ by : $\forall i = 1 \cdots k : v_t^i = u^i + t(a^i - u^i)$. Let $V(t)$ be the determinant of the Gramian of $\{v_t^i\}_{i=1\cdots k}$. Clearly, $V(t)$ is a polynomial in t which satisfies $V(1) \neq 0$ (theorem 1). Therefore $V(t) \neq 0$ except for a finite number of t's. Moreover,

$$||v_t - u||^2_{\ell^2(I,\mathbf{C}^k)} = \sum_{i=1}^k ||v_t^i - u^i||^2_{\ell^2(I)} = \sum_{i=1}^k |t|^2 ||a^i - u^i||^2_{\ell^2(I)} \to 0 \text{ when t } \to 0$$

Hence, there exists $t \in \mathbf{R}$ such that v_t is near enough to u and $V(t) \neq 0$, hence v_t is a frame. □

Proposition 7. *Let's denote by $\mathcal{F}^*$ the set of normalised tight frames in $\ell^2(I, \mathbf{C}^k)$.*
Then $\mathcal{F}^*$ is closed and path-connected.

Proof. $\mathcal{F}^*$ is closed because theorem 2 implies that $\mathcal{F}^* = G^{-1}(Id)$, where the continuous function G sends $u \in \ell^2(I,H)$ to the Gramian matrix of $\{u^i\}_{i=1\cdots k}$.

Let's prove the path-connectedness of $\mathcal{F}^*$. Let $u, v \in \ell^2(I, \mathbf{C}^k)$ be two normalised tight frames. We take $\{a^i\}_{i=1\cdots k}$ a finite orthonormal set of families in $\ell^2(I)$ such that a^i is orthogonal to u^j and v^j for each i and j (this is possible because $\ell^2(I)$ is infinite dimensional).

We define the continuous paths

$$\begin{cases} \gamma_1 & : [0,1] \to \ell^2(I, \mathbf{C}^k) \\ \gamma_2 & : [0,1] \to \ell^2(I, \mathbf{C}^k) \end{cases}$$

by $\gamma_1(t) = \frac{tu+(1-t)a}{\sqrt{2t^2-2t+1}}$ and $\gamma_2(t) = \frac{tv+(1-t)a}{\sqrt{2t^2-2t+1}}$ respectively.

We have $\gamma_1(1) = u$ and $\gamma_2(1) = v$.

We are going to show that $\forall t \in [0,1] : \gamma_1(t)$ and $\gamma_2(t)$ are normalised tight frames, which will prove this part of the proposition after composing γ_1 with the inverse path of γ_2. By symmetry, we'll show it only for γ_1. But we have obviously that $\langle \gamma_1(t)^i, \gamma_1(t)^i \rangle = 1$ and $\langle \gamma_1(t)^i, \gamma_1(t)^j \rangle = 0$, which shows that the Gramian matrix of $\gamma_1(t)$ is Id, so we can conclude using theorem 2. □

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References

[1] R. Duffin and A. Schaffer, "A class of non-harmonic Fourier series", *Transactions of the American Mathematical Society,* vol. 72, pp. 341-366, 1952. doi: 10.1090/S0002-9947-1952-0047179-6

[2] P. G. Casazza, "The art of frame theory", *Taiwanese Journal of Mathematics,* vol. 4, no. 2, pp. 129-201, 2000. doi: 10.11650/twjm/1500407227

[3] O. Christensen, *An introduction to frames and Riesz Bases,* 2nd ed. Springer, 2016.

[4] H. Richter, "Zur Abschatzung von Matrizennormen", *Mathematische Nachrichten,* vol. 18, no. 1-6, pp. 178-187, 1958. doi: 10.1002/mana.19580180121

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