On I -statistically ϕ -convergence

Shyamal Debnath^{1*} orcid.org/0000-0003-2804-6564
Chiranjib Choudhury² orcid.org/0000-0002-5607-9884

Tripura University, Dept. of Mathematics, Agartala, TR, India

¹ shyamalnitamath@gmail.com ; ² chiranjib.mathematics@tripurauniv.in

Received: 13 March 2020 | Accepted: 24 March 2021

Abstract:

In this paper we investigate the notion of I -statistical ϕ -convergence and introduce I_s - ϕ limit points and I_s - ϕ cluster points of real number sequence and also studied some of its basic properties.

Keywords: I_s - ϕ limit point; I_s - ϕ cluster point; I_s - ϕ statistically convergent.

MSC (2020): 40A35.

Cite this article as (IEEE citation style):

S. Debnath and C. Choudhury, "On I -statistically ϕ -convergence", *Proyecciones (Antofagasta, On line)*, vol. 40, no. 3, pp. 593-604, 2021, doi: 10.22199/issn.0717-6279-4036

Article copyright: © 2021 Shyamal Debnath and Chiranjib Choudhury. This is an open access article distributed under the terms of the Creative Commons License, which permits unrestricted use and distribution provided the original author and source are credited.

*Corresponding author

1. Introduction

In 1951 Fast [7] and Steinhaus [22] introduced the concept of statistical convergence independently and established a relation with summability. Later on it was further investigated from sequence space point of view by Fridy [9], Salat [23] and many others. Some applications of statistical convergence in number theory and mathematical analysis can be found in [1, 2, 15, 16, 26].

The notion of I -convergence is a generalization of the statistical convergence which was introduced by Kostyrko et al. [14]. They used the notion of an ideal I of subsets of the set N to define such a concept. For an extensive view of this article we refer [5, 13, 24].

The idea of I -convergence was further extended to I -statistical convergence by Savas and Das [19]. Later on more investigation in this direction was done by Savas and Das [20], Debnath and Debnath [3], Mursaleen et al. [17], Et et al. [6] and many others [10, 11, 27, 28]. In [19], Savas and Das introduced the I -statistical convergence and I - λ -statistical convergence and the relation between them. Also they studied these concept in the notion of $[V, \lambda]$ -summability method.

In many literature, statistical convergence of any real sequence is defined relatively to absolute value. While, we know that the absolute value of real numbers is a special case of an Orlicz function [18] i.e. a function $\phi : R \rightarrow R$ such that it is even, non-decreasing on R^+ , continuous on R , and satisfying

$$\phi(x) = 0 \iff x = 0 \text{ and } \phi(x) \rightarrow \infty \text{ as } x \rightarrow \infty.$$

Rao and Ren [18] describe that Orlicz functions have important roles and applications in many areas such as economics, stochastic problems etc.

An Orlicz function $\phi : R \rightarrow R$ is said to satisfy the Δ_2 condition, if there exists an $M > 0$ such that $\phi(2x) \leq M \cdot \phi(x)$, for every $x \in R^+$.

Example 1.1:

- (i) A function $\phi : R \rightarrow R$ with $\phi(x) = |x|$ is an Orlicz function.
- (ii) A function $\phi : R \rightarrow R$ with $\phi(x) = x^3$ is not an Orlicz function.
- (iii) A function $\phi : R \rightarrow R$ with $\phi(x) = x^2$ is an Orlicz function satisfying the Δ_2 condition.
- (iv) A function $\phi : R \rightarrow R$ with $\phi(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the Δ_2 condition.

The idea of ϕ -convergence was introduced by Khusnussaadah and Supama [12] and later on it was generalized to statistically ϕ -convergence by Supama [25]. Further, it was generalized to Lacunary statistically ϕ -convergence by Savas and Debnath [21].

Recently in [4] Debnath and Rakshit have introduced I -statistical convergence as a sequential limit concept. In the present paper we generalized it and introduced I_S - ϕ convergence as a sequential limit concept and extend this concept in a natural way to define a I_S - ϕ analogue of the set of limit points and cluster points of a real number sequence. We mainly investigate their relationship and also make some observations about the above mentioned class.

2. Definitions and Preliminaries

Definition 2.1. [9] If K is a subset of the positive integers N , then K_n denotes the set $\{k \in K : k \leq n\}$. The natural density of K is given by $D(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$.

Definition 2.2. [9] A sequence (x_n) is said to be statistically convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : |x_k - x_0| \geq \varepsilon\}$ has natural density zero. x_0 is called the statistical limit of the sequence (x_n) and we write $st\text{-}\lim_{n \rightarrow \infty} x_n = x_0$.

Definition 2.3. [8] If $(x_{k(j)})$ be a subsequence of a sequence $x = (x_n)$ and density of $K = \{k(j) : j \in N\}$ is zero then $(x_{k(j)})$ is called a thin subsequence. Otherwise $(x_{k(j)})$ is called a non-thin subsequence of x .

x_0 is said to be a statistical limit point of a sequence (x_n) , if there exist a non-thin subsequence of (x_n) which converges to x_0 .

Let x denotes the set of all statistical limit points of the sequence (x_n) .

Definition 2.4. [8] x_0 is said to be a statistical cluster point of a sequence $x = (x_n)$, provided that for each $\varepsilon > 0$, the density of the set $\{k \in N : |x_k - x_0| < \varepsilon\}$ is not equal to 0.

Let x denotes the set of all statistical cluster points of the sequence (x_n) .

Definition 2.5. [14] Let X is a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal on X if and only if

- (i) $\emptyset \in I$;
- (ii) for each $A, B \in I$ implies $A \cup B \in I$;
- (iii) for each $A \in I$ and $B \subset A$ implies $B \in I$.

Definition 2.6. [14] Let X is a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if

- (i) $\emptyset \notin \mathcal{F}$;
- (ii) for each $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$;
- (iii) for each $A \in \mathcal{F}$ and $B \supset A$ implies $B \in \mathcal{F}$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. The filter $\mathcal{F} = \mathcal{F}(I) = \{X - A : A \in I\}$ is called the filter associated with the ideal I . A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if $I \supset \{\{x\} : x \in X\}$.

Definition 2.7. [14] Let $I \subset P(N)$ be a non-trivial ideal on N . A sequence (x_n) is said to be I -convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : |x_k - x_0| \geq \varepsilon\}$ belongs to I . x_0 is called the I -limit of the sequence (x_n) and we write $I\text{-}\lim_{n \rightarrow \infty} x_n = x_0$.

Definition 2.8. [14] x_0 is said to be I -limit point of a sequence $x = (x_n)$ provided that there is a subset $K = \{k_1 < k_2 < \dots\} \subset N$ such that $K \notin I$ and $\lim x_{k_i} = x_0$.

Let $I(x)$ denotes the set of all I -limit points of the sequence x .

Definition 2.9. [14] x_0 is said to be I -cluster point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ the set $\{k \in N : |x_k - x_0| < \varepsilon\} \notin I$.

Let $I(x)$ denotes the set of all I -cluster points of the sequence x .

Definition 2.10. [19] A sequence $x = (x_n)$ is said to be I -statistically convergent to x_0 if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in N : \frac{1}{n} |\{k \leq n : |x_k - x_0| \geq \varepsilon\}| \geq \delta\right\} \in I.$$

x_0 is called I -statistical limit of the sequence (x_n) and we write, $I\text{-st } \lim x_n = x_0$.

Definition 2.11. [12] Let $\phi : R \rightarrow R$ be an Orlicz function. A sequence $x = (x_n)$ is said to be ϕ -convergent to x_0 if $n\lim \phi(x_n - x_0) = 0$.

In this case, x_0 is called the ϕ -limit of (x_n) and denoted by $\phi\text{-}\lim x = x_0$.

Remark 2.12. If we take $\phi(x) = |x|$, then ϕ -convergent concepts coincide with usual convergence. Also it is easy to check, if $x = (x_n)$ is ϕ -convergent to x_0 , then any of its subsequence is ϕ -convergent to x_0 as well.

Definition 2.13. [25] Let $\phi : R \rightarrow R$ be an Orlicz function. A sequence $x = (x_n)$ is said to be statistically ϕ -convergent to x_0 if for each $\varepsilon > 0$, $n \lim \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \varepsilon\}| = 0$. x_0 is called the statistical ϕ -limit of the sequence (x_n) and we write $S - \phi \lim x = x_0$ or $x_k \rightarrow x_0 (S - \phi)$. We shall also use $S - \phi$ to denote the set of all statistically ϕ -convergent sequences.

Remark 2.14. If we take $\phi(x) = |x|$, then $S - \phi$ convergence concepts coincide with statistically convergence.

Definition 2.15. Let $\phi : R \rightarrow R$ be an Orlicz function. A sequence $x = (x_n)$ is said to be $I - \phi$ convergent to x_0 if for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in N : \phi(x_k - x_0) \geq \varepsilon\}$ belongs to I .

x_0 is called the $I - \phi$ limit of the sequence (x_n) and we write $I - \phi \lim x = x_0$.

Remark 2.16. If we take $\phi(x) = |x|$, then $I - \phi$ convergence concepts coincide with I -convergence.

Throughout the paper we consider I as an admissible ideal and ϕ is an Orlicz function.

3. Main Results

Definition 3.1. Let $\phi : R \rightarrow R$ be an Orlicz function. A sequence $x = (x_n)$ is said to be I -statistically ϕ -convergent to x_0 if for every $\varepsilon > 0$ and every $\delta > 0$,

$$\left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \varepsilon\}| \geq \delta\right\} \in I.$$

x_0 is called I -statistical ϕ -limit of the sequence (x_n) and we write, $I_S - \phi \lim x_n = x_0$.

Remark 3.2. If we take $\phi(x) = |x|$, then $I_S - \phi$ convergent coincide with I_S -convergent. Thus $I_S - \phi$ convergence is a generalization of I_S convergence.

Theorem 3.3. Let $\phi : R \rightarrow R$ be an Orlicz function. For any sequence (x_n) , $S - \phi \lim x_n = x_0$ implies $I_S - \phi \lim x_n = x_0$.

Proof. Let $S - \phi \lim x_n = x_0$. Then for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \leq n : \phi(x_k - x_0) \geq \varepsilon\}$ has natural density zero.

i.e., $\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \varepsilon\}| = 0$

So for every $\varepsilon > 0$ and $\delta > 0$, $\left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \varepsilon\}| \geq \delta\right\}$ is a finite set and therefore belongs to I , as I is an admissible ideal.

Hence $I_S - \phi \lim x_n = x_0$.

But the converse is not true. $\square$

Example 3.4. Let $\phi(x) = |x|$ and $I = \zeta$ be the class of $A \subset N$ that intersect a finite number of Δ_j 's where $N = \cup_{j=1}^{\infty} \Delta_j$ and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$.

Let $x_n = \frac{1}{n}$ and so $\lim_{n \rightarrow \infty} x_n = 0$. Put $\epsilon_n = \frac{1}{n}$ for $n \in N$.

Now define a sequence (y_n) by $y_n = x_j$ if $n \in \Delta_j$

Let $\eta > 0$. Choose $\nu \in N$ such that $\epsilon_\nu < \eta$. Then

$A(\eta) = \{n : y_n \geq \eta\} \subset \Delta_1 \cup \dots \cup \Delta_\nu \in \zeta$.

Now, $\{k \leq n : y_k \geq \eta\} \subseteq \{n \in N : y_k \geq \eta\}$ i.e., $\frac{1}{n} |\{k \leq n : y_k \geq \eta\}| \leq |\{n \in N : y_k \geq \eta\}|$ so for any $\delta > 0$, $\left\{n \in N : \frac{1}{n} |\{k \leq n : y_k \geq \eta\}| \geq \delta\right\} \subseteq \{n \in N : y_k \geq \eta\} \in \zeta$.

Therefore (y_n) is ζ -statistically ϕ -convergent to 0.

But (y_n) is not a statistically ϕ -convergent.

Theorem 3.5. Let $\phi : R \rightarrow R$ be an Orlicz function. For any sequence (x_n) , $I - \phi \lim x_n = x_0$ implies $I_S - \phi \lim x_n = x_0$.

Proof. The proof is obvious. But the converse is not true. $\square$

Example 3.6. Let $\phi(x) = |x|$. If we take $I = I_f$ then sequence (x_n) , where

$$x_n = \begin{cases} 0, & n = k^2, k \in N \\ 1, & \text{otherwise} \end{cases}$$

is I -statistically ϕ -convergent to 1. But (x_n) is not I - ϕ convergent.

Theorem 3.7. Let $\phi : R \rightarrow R$ be an Orlicz function. If each subsequence of (x_n) is I_S - ϕ convergent to ξ then (x_n) is also I_S - ϕ convergent to ξ .

Proof. Suppose (x_n) is not I_S - ϕ convergent to ξ , then there exists $\varepsilon > 0$ and $\delta > 0$ such that $A = \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - \xi) \geq \varepsilon\}| \geq \delta\right\} \notin I$.

Since I is admissible ideal, so A must be an infinite set.

Let $A = \{n_1 < n_2 < \dots < n_m < \dots\}$. Let $y_m = x_{n_m}$ for $m \in N$. Then $(y_m)_{m \in N}$ is a subsequence of (x_n) which is not I -statistically ϕ -convergent to ξ , a contradiction. Hence the theorem.

But the converse is not true. We can easily show this from example 3.6. $\square$

Theorem 3.8. Let $\phi : R \rightarrow R$ be an Orlicz function. Let (x_n) and (y_n) be two real sequences then

$I_S - \phi \lim x_n = x_0$ and $I_S - \phi \lim y_n = y_0$ implies
 $I_S - \phi \lim (x_n + y_n) = x_0 + y_0$.

Proof. We have $A_1 = \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\right\} \in \mathcal{F}(I)$ and $A_2 = \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(y_k - y_0) \geq \frac{\varepsilon}{2}\}| < \frac{\delta}{2}\right\} \in \mathcal{F}(I)$.

Since $A_1 \cap A_2 \neq \emptyset$, therefore for all $n \in A_1 \cap A_2$ we have,
 $\frac{1}{n} |\{k \leq n : \phi(x_k + y_k - x_0 - y_0) \geq \varepsilon\}|$
 $\leq \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \frac{\varepsilon}{2}\}| + \frac{1}{n} |\{k \leq n : \phi(y_k - y_0) \geq \frac{\varepsilon}{2}\}| < \delta$.
 i.e, $\left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k + y_k - x_0 - y_0) \geq \varepsilon\}| < \delta\right\} \in \mathcal{F}(I)$.

Hence $I_S - \phi \lim (x_n + y_n) = x_0 + y_0$. $\square$

Definition 3.9. A sequence $x = (x_n)_{n \in N}$ of elements of X is said to be I^* -statistical ϕ -convergent to $\xi \in X$ if and only if there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \in \mathcal{F}(I)$, such that $S - \phi \lim x_{m_k} = \xi$. We write, $I_S^* - \phi \lim x_n = \xi$.

Theorem 3.10. If $I_S^* - \phi \lim x_n = \xi$ then $I_S - \phi \lim x_n = \xi$.

Proof. Let $I_S^* - \phi \lim x_n = \xi$. By assumption there exist a set $H \in I$ such that for $M = N \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$ we have $S - \phi \lim x_{m_k} = \xi$

i.e, $\lim_{n \rightarrow \infty} \frac{1}{n} |\{m_k \leq n : \phi(x_{m_k} - \xi) \geq \varepsilon\}| = 0$

So for any $\delta > 0$, $\left\{n \in N : \frac{1}{n} |\{m_k \leq n : \phi(x_{m_k} - \xi) \geq \varepsilon\}| \geq \delta\right\} \in I$, since I is an admissible ideal.

Now $A(\varepsilon, \delta) = \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - \xi) \geq \varepsilon\}| \geq \delta\right\}$
 $\subset H \cup \left\{n \in N : \frac{1}{n} |\{m_k \leq n : \phi(x_{m_k} - \xi) \geq \varepsilon\}| \geq \delta\right\} \in I$
 i.e, $I_S - \phi \lim x_n = \xi$.

But the converse may not be true. $\square$

Example 3.11. we have $\zeta_S - \phi \lim_{n \rightarrow \infty} y_n = 0$.

Suppose that $\zeta_S^* - \phi \lim_{n \rightarrow \infty} y_n = 0$. Then there exist a set $H \in \zeta$ such that for $M = N \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$ we have $S - \phi \lim y_{m_k} = 0$. By definition of ζ there exist a $p \in N$ such that $H \subset \Delta_1 \cup \dots \cup \Delta_p$. But then $\Delta_{p+1} \subset M$, so for infinitely many $m_k \in \Delta_{p+1}$, $D\{m_k \in \Delta_{p+1} : y_{m_k} \geq \eta\} = 2^{-(p+1)} > 0$ for $0 < \eta < \frac{1}{p+1}$ i.e, $D\{m_k \in \Delta_{p+1} : y_{m_k} \geq \eta\} \neq 0$, which is a contradicts $S - \phi \lim y_{m_k} = 0$.

Hence $\zeta_S^* - \phi \lim_{n \rightarrow \infty} y_n \neq 0$.

Definition 3.12. An element x_0 is said to be an $I_S - \phi$ limit point of a sequence $x = (x_n)$ provided that for each $\varepsilon > 0$ there is a set $M = \{m_1 < m_2 < \dots\} \subset N$ such that $M \notin I$ and $S - \phi \lim x_{m_k} = x_0$. $I_S - \phi(x)$ denotes the set of all $I_S - \phi$ limit points of the sequence (x_n) .

Definition 3.13. An element x_0 is said to be an $I_S - \phi$ cluster point of a sequence $x = (x_n)$ if for each $\varepsilon > 0$ and $\delta > 0$

$$\left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \varepsilon\}| < \delta\right\} \notin I.$$

$I_S - \phi(x)$ denotes the set of all $I_S - \phi$ cluster points of the sequence (x_n) .

Theorem 3.14. For any sequence $x = (x_n)$, $I_S - \phi(x)$ is closed.

Proof. Let y_0 be a limit point of the set $I_S - \phi(x)$ then for any $\varepsilon > 0$, $I_S - \phi(x) \cap B(y_0, \varepsilon) \neq \emptyset$, where $B(y_0, \varepsilon) = \{z \in R : \phi(z - y_0) < \varepsilon\}$

Let $z_0 \in I_S - \phi(x) \cap B(y_0, \varepsilon)$ and choose $\varepsilon_1 > 0$ such that $B(z_0, \varepsilon_1) \subseteq B(y_0, \varepsilon)$.

Then we have $\{k \leq n : \phi(x_k - z_0) \geq \varepsilon_1\} \supseteq \{k \leq n : \phi(x_k - y_0) \geq \varepsilon\} \Rightarrow \frac{1}{n} |\{k \leq n : \phi(x_k - z_0) \geq \varepsilon_1\}| \geq \frac{1}{n} |\{k \leq n : \phi(x_k - y_0) \geq \varepsilon\}|$

Now for any $\delta > 0$, $\left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - z_0) \geq \varepsilon_1\}| < \delta\right\} \subseteq \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - y_0) \geq \varepsilon\}| < \delta\right\}$

Since $z_0 \in I_S - \phi(x)$ therefore, $\left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - y_0) \geq \varepsilon\}| < \delta\right\} \notin I$. i.e, $y_0 \in I_S - \phi(x)$. Hence the theorem. $\square$

Theorem 3.15. For any sequence $x = (x_n)$, $I_S - \phi(x) \subseteq I_S - \phi(x)$.

Proof. Let $x_0 \in I_S - \phi(x)$. Then there exist a set $M = \{m_1 < m_2 < \dots\} \notin I$ such that, $S - \phi \lim x_{m_k} = x_0 \Rightarrow \lim_{k \rightarrow \infty} \frac{1}{k} |\{m_i \leq k : \phi(x_{m_i} - x_0) \geq \varepsilon\}| = 0$.

Take $\delta > 0$, so there exist $k_0 \in N$ such that for $n > k_0$ we have,
 $\frac{1}{n} |\{m_i \leq n : \phi(x_{m_i} - x_0) \geq \varepsilon\}| < \delta$.

Let $A = \left\{n \in N : \frac{1}{n} |\{m_i \leq n : \phi(x_{m_i} - x_0) \geq \varepsilon\}| < \delta\right\}$.

Also, $A \supset M / \{m_1 < m_2 < \dots < m_{k_0}\}$. Since I is an admissible ideal and $M \notin I$, therefore $A \notin I$. So by definition of $I_S - \phi$ cluster point $x_0 \in I_S - \phi(x)$.

Hence the theorem. $\square$

Theorem 3.16. *If $x = (x_n)$ and $y = (y_n)$ be two sequences such that $\{n \in N : x_n \neq y_n\} \in I$, then (i) $I_S - \phi(x) = I_S - \phi(y)$ and (ii) $I_S - \phi(x) = I_S - \phi(y)$.*

Proof. (i) Let $x_0 \in I_S - \phi(x)$. So by definition there exist a set $K = \{k_1 < k_2 < k_3 < \dots\}$ of N such that $K \notin I$ and $S - \phi \lim x_{k_n} = x_0$.

Since $\{n \in K : x_n \neq y_n\} \subset \{n \in N : x_n \neq y_n\} \in I$, therefore

$K' = \{n \in K : x_n = y_n\} \notin I$ and $K' \subseteq K$.

So we have $S - \phi \lim y_{k'_n} = x_0$.

This shows that $x_0 \in I_S - \phi(y)$ and therefore $I_S - \phi(x) \subseteq I_S - \phi(y)$.

By symmetry $I_S - \phi(y) \subseteq I_S - \phi(x)$.

Hence $I_S - \phi(y) = I_S - \phi(x)$.

(ii) Let $x_0 \in I_S - \phi(\Gamma_x)$. So by definition for each $\varepsilon > 0$ the set, $A = \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(x_k - x_0) \geq \varepsilon\}| < \delta\right\} \notin I$.

Let $B = \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(y_k - x_0) \geq \varepsilon\}| < \delta\right\}$. We have to prove that $B \notin I$.

Suppose $B \in I$. So, $B^c = \left\{n \in N : \frac{1}{n} |\{k \leq n : \phi(y_k - x_0) \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{F}(I)$.

By hypothesis the set $C = \{n \in N : x_n = y_n\} \in \mathcal{F}(I)$.

Therefore $B^c \cap C \in \mathcal{F}(I)$. Also it is clear that $B^c \cap C \subset A^c \in \mathcal{F}(I)$, i.e., $A \in I$, which is a contradiction.

Hence $B \notin I$ and the result established. $\square$

References

- [1] R. C. Buck, "The measure theoretic approach to density", *American journal of mathematics*, vol. 68, no. 4, pp. 560–580, 1946, doi: 10.2307/2371785
- [2] R. C. Buck, "Generalized asymptotic density", *American journal of mathematics*, vol. 75, no. 2, pp. 335–346, 1953, doi: 10.2307/2372456
- [3] S. Debnath and J. Debnath, "On I-statistically convergent sequence spaces defined by sequences of Orlicz functions using matrix transformation", *Proyecciones (Antofagasta)*, vol. 33, no. 3, pp. 277–285, 2014, doi: 10.4067/S0716-09172014000300004
- [4] S. Debnath and D. Rakshit, "On I-statistical convergence", *Iranian journal mathematical sciences and informatics*, vol. 13, no. 2, pp. 101–109, 2018, doi: 10.7508/ijmsi.2018.13.009
- [5] K. Dems, "On I-cauchy sequences", *Real analysis exchange*, vol. 30, no. 1, pp. 123–128, 2005. [On line]. Available: <https://bit.ly/3x5zfyQ>
- [6] M. Et, A. Alotaibi, and S. A. Mohiuddine, "On (m, I)-statistical convergence of order α ", *Scientific world journal*, Art. ID 535419, 2014, doi: 10.1155/2014/535419
- [7] H. Fast, "Sur la convergence statistique", *Colloquium mathematicae*, vol. 2, no. 3-4, pp. 241–244, 1951. [On line]. Available: <https://bit.ly/3gdlrMS>
- [8] J. A. Fridy, "Statistical limit points", *Proceeding American Mathematical Society*, vol. 111, no. 4, pp. 1187–1192, 1993, doi: 10.1090/S0002-9939-1993-1181163-6
- [9] J. A. Fridy, "On statistical convergence", *Analysis*, vol. 5, no. 4, pp. 301–313, 1985, doi: 10.1524/anly.1985.5.4.301
- [10] M. Gürdal and H. Sari, "Extremal A-statistical limit points via ideals", *Journal of the Egyptian Mathematical Society*, vol. 22, no. 1, pp. 55–58, 2014, doi: 10.1016/j.joems.2013.06.005
- [11] M. Gürdal and M. O. Özgür, "A generalized statistical convergence via moduli", *Electronic journal mathematical analysis & applications*, vol. 3, no. 2, pp. 173–178, 2015. [On line]. Available: <https://bit.ly/3mSMJcI>
- [12] N. Khusnussaadah and S. Supama, "Completeness of sequence spaces generated by an Orlicz function", *EKSAKTA: journal of sciences and data analysis*, vol. 1, no. 1, pp. 1–14, 2019. doi: 10.20885/eksakta.vol19.iss1.art1

- [13] P. Kostyrko, M. Macaj, T. Šalát and M. Szeziak, "I -convergence and extremal I-limit points", *Mathematica slovac*, vol. 55, no. 4, pp. 443-464, 2005. [On line]. Available: <https://bit.ly/3tsVT24>
- [14] P. Kostyrko, W. Wilczy ski, and T. Šalát "I-convergence", *Real analysis. exchange*, vol. 26, no. 2, pp. 669-686, 2001. [On line]. Available: <https://bit.ly/3e9CnBb>
- [15] M. Mamedov and S. Pehlivan, "Statistical cluster points and Turnpike theorem in non convex problems", *Journal mathematical analysis applications*, vol. 256, no. 2, pp. 686-693, 2001, doi: 10.1006/jmaa.2000.7061
- [16] J. Sandor, D. S. Mitrinovic, and B. Crstici, *Handbook of number theory*, vol. 1. Dordrecht: Kluwer, 1996.
- [17] M. Mursaleen, S. Debnath, and D. Rakshit, "I-statistical limit superior and I-statistical limit inferior", *Filomat*, vol. 31, no. 7, pp. 2103-2108, 2017, doi: 10.2298/FIL1707103M
- [18] M. M. Rao and Z. D. Ren, *Applications of Orlicz spaces*. New York, NY: M. Dekker, 2002.
- [19] E. Savas and P. Das, "A generalized statistical convergence via ideals", *Applied mathematics letters*, vol. 24, no. 6, pp. 826-830, 2011, doi: 10.1016/j.aml.2010.12.022
- [20] P. Das and E. Savas, "On I-statistically pre-Cauchy sequences", *Taiwanese journal of mathematics*, vol. 18, no. 1, pp. 115-126, 2014, doi: 10.11650/tjm.18.2014.3157
- [21] E. Savas and S. Debnath, "Lacunary statistically ϕ -convergence", *Note di matematica*, vol. 39, no. 2, pp. 111-119, 2019, doi: 10.1285/i15900932v39n2p111
- [22] H. Steinhaus, "Sur la convergence ordinaire et la convergence asymptotique", *Colloquium mathematicum*, vol. 2, pp. 73-74, 1951.
- [23] T. Šalát, "On statistically convergent sequences of real numbers", *Mathematica slovac*, vol. 30, no. 2, pp. 139-150, 1980. [On line]. Available: <https://bit.ly/3spGBtp>
- [24] T. Šalát, B. C. Tripathy and M. Ziman, "On some properties of I-convergence", *Tatra mountains mathematical publications*, vol. 28, pp. 279-286, 2004. [On line]. Available: <https://bit.ly/3mVxq2U>
- [25] S. Supama, "On statistically ϕ -convergence", *Pre-print*.
- [26] B C. Tripathy, "On statistically convergent and statistically bounded sequences", *Bulletin of Malaysian Mathematical Sciences Society*, vol. 20, pp. 31-33, 1997.

- [27] U. Yamancı and M. Gürdal, "I-statistical convergence in 2-normed space", *Arab Journal Mathematical Scientific*, vol. 20, no. 1, pp. 41-47, 2014, doi: 10.1016/j.ajmsc.2013.03.001
- [28] U. Yamancı and M. Gürdal, "I-statistically pre-Cauchy double sequences", *Global journal mathematical analysis*, vol. 2, no. 4, pp. 297-303, 2014, doi: 10.14419/gjma.v2i4.3135