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A new approach for Volterra functional integral equations with non-vanishing delays and fractional Bagley-Torvik equation

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Abstract:

A numerical technique for Volterra functional integral equations (VFIEs) with non-vanishing delays and fractional Bagley-Torvik equation is displayed in this work. The technique depends on Bernstein polynomial approximation. Numerical examples are utilized to evaluate the accurate results. The findings for examples figs and tables show that the technique is accurate and simple to use.

Keywords: Integral equations; Numerical solution; Fractional Bagley-Torvik equation.

MSC (2020): 65K10, 26A33, 49K15.

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1. Introduction

The following general Volterra functional integral equation (VFIE) is considered

$$\begin{aligned}
 y(t) &= g(t) + b(t)y(\theta(t)) + \int_{t_0}^t k_1(t, s)y(s)ds \\
 &+ \int_{t_0}^{\theta(t)} k_2(t, s)y(s)ds, \quad t \in I := (t_0, T] = (t_0, t_f], \\
 (1.1) \quad y(t) &= \phi(t), \quad t \in I_\theta := [\theta(t_0), t_0], \quad t_f = T,
 \end{aligned}$$

where $0 < q < 1$ ($\theta(t)$ is the function which related to q), and $k_1(t, s)$ and $k_2(t, s)$ are assumed to be continuous functions on their respective domains $D := \{(t, s) : t_0 \leq s \leq t \leq T, t \in I\}$ and $D_\theta := \{(t, s) : \theta(t_0) \leq s \leq \theta(t), t \in I\}$ (for more explanation see [1]).

The hp variant of the discontinuous Galerkin technique for the numerical solution of delay differential equations (DDEs) with nonlinear vanishing delays was introduced in [2]. In [3], the existence, uniqueness of solutions for VFIEs with the time delay were analyzed. Huang et al. [4] studied the superconvergence of the discontinuous Galerkin solutions for DDEs with proportional vanishing at $t = 0$.

Additionally many papers manage the Bezier curves. In [5] and [6], authors utilized the Bezier curves in approximating functions. For solving differential equations (DEs) numerically, authors in [7] proposed the utilization of Bezier curves. Also, to solve delay differential equation, the Bezier control points strategy is utilized (see [8]). Some other uses of the Bezier functions are found in (see [9]). In the present work, we use the proposed method in [9] for solving VFIE.

The outline of this sequel is as follow: In Section 2, problem statement is stated. Explanation of the problem is explained in Section 3. Also, error estimation based on residual error function is presented in Section 4. Some numerical results are provided in Section 5. Also, a remark is stated about fractional Bagley-Torvik equation. Finally, Section 6 will give a conclusion briefly.

2. Problem Statement

For finding an approximate solution of Eq. (1.1), one may consider the following $y(t)$:

$$(2.1) \quad y(t) \cong y_n(t) = \sum_{i=0}^n c_i B_{i,n}(t), \quad t_0 \leq t \leq t_f, \quad n \geq 1,$$

where

$$B_{r,n}\left(\frac{t-t_0}{h}\right) = \binom{n}{r} \frac{1}{h^n} (t_f-t)^{n-r} (t-t_0)^r, \quad t_0 \leq t \leq t_f, \quad 0 \leq i \leq n,$$

$$h = t_f - t_0,$$

and $c_i, i = 0, 1, \dots, n$ are the unknown control points. In this paper, one may utilize Bezier curve for solving VFIE. This technique is applied in [10], [11] for solving optimal control of problems (OCPs) and some linear OCPs with pantograph delays. The convergence of the proposed technique is proven in this sequel.

By substituting $y(t)$ in (1.1), one may define

$$(2.2) \quad f_{objective} = \sum_{i=0}^n c_i^2, \quad t \in [t_0, t_f],$$

Now, our goal is to solve the following problem to find the values c_r , for $r = 0, 1, \dots, n$.

$$(2.3) \quad \begin{aligned} \min \quad & f_{objective} \\ \text{s.t.} \quad & y(t) = g(t) + b(t)y(\theta(t)) + \int_{t_0}^t k_1(t,s)y(s)ds \\ & + \int_{t_0}^{\theta(t)} k_2(t,s)y(s)ds, \quad t \in I := (t_0, T], \\ & y(t) = \phi(t), \quad t \in I_\theta := [\theta(t_0), t_0], \end{aligned}$$

where "s.t." is abbreviation of "such that".

3. Explanation of the problem

Now, the following problem is considered

$$\begin{aligned}
 (Dy)(t) &= y(t) - g(t) + b(t)y(\theta(t)) \\
 &+ \int_0^t k_1(t,s)y(s)ds + \int_0^{\theta(t)} k_2(t,s)y(s)ds, \\
 y_n(t) &= \sum_{i=0}^n c_i B_{i,n}(t), \quad 0 \leq t \leq t_f, \\
 y(t) &= \phi(t), \quad t \leq 0.
 \end{aligned}$$

Without loss of generality, one may consider $t_0 = 0$. Firstly, we have

$$\begin{aligned}
 (Dy_n)(t) &= \sum_{i=0}^n c_i B_{i,n}(t) - g(t) \\
 &+ b(t) \sum_{i=0}^n c_i B_{i,n}(\theta(t)) \\
 &+ \int_0^t k_1(t,s) \sum_{i=0}^n c_i B_{i,n}(s) ds \\
 &+ \int_0^{\theta(t)} k_2(t,s) \sum_{i=0}^n c_i B_{i,n}(s) ds \\
 &= \sum_{i=0}^n c_i \left(B_{i,n}(t) + b(t) B_{i,n}(\theta(t)) \right. \\
 &+ \int_0^t k_1(t,s) B_{i,n}(s) ds \\
 &+ \left. \int_0^{\theta(t)} k_2(t,s) B_{i,n}(s) ds \right) \\
 &- g(t) \\
 &= \sum_{i=0}^n c_i \alpha_i(t) - g(t),
 \end{aligned}$$

where $\alpha_i = \left(B_{i,n}(t) + b(t) B_{i,n}(\theta(t)) \right.$

$+ \int_0^t k_1(t, s)B_{i,n}(s)ds + \int_0^{\theta(t)} k_2(t, s)B_{i,n}(s)ds \Big), i = 0, 1, 2, \dots, n$. Now, the following real function is defined

$$J = J(c_0, c_1, \dots, c_n) = \int_0^{t_f} (Dy_n)^2(t)dt,$$

according to the above equations, we have

$$\begin{aligned} & \min J(C) \\ & \text{s.t. } I(C) = \sum_{i=0}^n c_i \left(\sum_{k=0}^n c_k B_{k,n} \left(-\frac{j}{n} \right) \right) - \phi \left(-\frac{j}{n} \right), \\ (3.1) \quad & j = 0, 1, \dots, n, \end{aligned}$$

where $C = (c_0, c_1, \dots, c_n)$. There are various techniques for solving this problem, such as the Newton’s method, the gradient descent, the conjugate-gradient technique and the Lagrange technique. Here, the Lagrange-multiplier method is considered to solve the problem (3.1). By the Lagrange function, one may have the following form of

$$L = J(C) + \mu I(C),$$

Hence

$$\begin{aligned} \frac{\partial L}{\partial c_i} &= 2 \int_0^{t_f} (Dy_n)(t) \frac{\partial (Dy_n)(t)}{\partial c_i} dt + \mu \frac{\partial I(C)}{\partial c_i} \\ &= 2 \left(\sum_{j=0}^n c_j \int_0^{t_f} \alpha_j(t) \alpha_i(t) dt \right. \\ &\quad \left. - \int_0^{t_f} g(t) \alpha_i(t) dt \right) \\ &\quad + \mu \sum_{k=0}^n c_k B_{k,n} \left(-\frac{i}{n} \right), \\ (3.2) \quad \frac{\partial L}{\partial \mu} &= I(C), \end{aligned}$$

As we known, a necessary condition (3.1) is that

$$(3.3) \quad \begin{cases} \frac{\partial L}{\partial c_i} = 0, & 0 \leq i \leq n, \\ \frac{\partial L}{\partial \mu} = 0, \end{cases}$$

For simplification, one may define

$$(3.4) \quad (f, g) = \int_0^{t_f} f(t)g(t)dt,$$

Combining Eqs. (3.2)-(3.4), we have

$$(3.5) \quad \begin{aligned} \sum_{j=0}^n 2c_j(\alpha_j, \alpha_i) + \mu \sum_{k=0}^n c_k B_{k,n}\left(-\frac{i}{n}\right) &= 2(g, \alpha_i), \\ 0 \leq i \leq n, \\ \sum_{i=0}^n c_i B_{i,n}(t) &= \phi(t), \quad t \leq 0, \end{aligned}$$

we can rewrite Eq. (3.5) as the following form

$$\sum_{i=0}^n c_i B_{i,n}\left(-\frac{j}{n}\right) = \phi\left(-\frac{j}{n}\right), \quad -n \leq j \leq 0.$$

Now, we write

$$(3.6) \quad \begin{aligned} G\tilde{C} &= F, \\ G &= \begin{bmatrix} 2(\alpha_0, \alpha_0) & 2(\alpha_1, \alpha_0) & \dots & 2(\alpha_n, \alpha_0) & \phi(0) \\ 2(\alpha_0, \alpha_1) & 2(\alpha_1, \alpha_1) & \dots & 2(\alpha_n, \alpha_1) & \phi\left(-\frac{1}{n}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2(\alpha_0, \alpha_n) & 2(\alpha_1, \alpha_n) & \dots & 2(\alpha_n, \alpha_n) & \phi\left(-\frac{n-1}{n}\right) \\ B_{0,n}(0) & B_{1,n}(0) & \dots & B_{n,n}(0) & 0 \end{bmatrix}, \\ \tilde{C} &= [c_0, c_1, \dots, c_n, \mu]^T, \\ F &= [2(g, \alpha_0), \dots, 2(g, \alpha_n), \phi(0)]^T. \end{aligned}$$

the unique solution of linear system (3.6) is $y_n(t) = \sum_{i=0}^n c_i B_{i,n}(t)$ is called an optimal control approximation solution (OCAS). If $\lim_{n \rightarrow \infty} \int_0^{t_f} (Dy_n)^2(t)dt = 0$ then the OCAS converges to the exact solution.

4. Error estimation based on residual error function

Now, some theorems are stated for the error estimation.

Theorem 4.1. *Let $y(t)$ is a continuous exact solution defined on $[0, t_f]$, and $y_n(t)$ is an OCAS of this problem. If there exists $P_n(t) = \sum_{i=0}^n c_i B_{i,n}(t)$, $c_i \in R$ such that for any $t \in [0, t_f]$, $\lim_{n \rightarrow \infty} P_n(t) = y(t)$, then*

$$\lim_{n \rightarrow \infty} \int_0^{t_f} (Dy_n)^2(t) dt = 0.$$

Proof. By the Weierstrass theorem it follows that there exists the sequence of polynomials, $P_n(t) = \sum_{i=0}^n c_i B_{i,n}(t)$ such that $\lim_{n \rightarrow \infty} P_n(t) = y(t)$, that is

$$(4.1) \quad \lim_{n \rightarrow \infty} \int_0^{t_f} (DP_n)^2(t) dt = 0,$$

Hence, one may have

$$(4.2) \quad 0 \leq \int_0^{t_f} (Dy_n)^2(t) dt \leq \int_0^{t_f} (DP_n)^2(t) dt,$$

$$(4.3) \quad 0 \leq \lim_{n \rightarrow \infty} \int_0^{t_f} (Dy_n)^2(t) dt \leq \lim_{n \rightarrow \infty} \int_0^{t_f} (DP_n)^2(t) dt,$$

By Eqs.(4.1) and (4.3), the proof is completed. \square

Now, the approximate solution obtained by this technique can be stated as:

$$y_n(t) = \sum_{i=0}^n c_i B_{i,n}(t).$$

Presently, in the following theorem, an upper bound for the error estimation of the proposed technique can be proven.

Theorem 4.2. *Let $y(t) \in C^{n+1}[0, t_f]$ is the exact solution of explained problem, $y_n(t) = \sum_{i=0}^n c_i B_{i,n}(t)$ is the solution with degree n , hence*

$$\|y(t) - y_n(t)\|_\infty \leq \frac{M}{(n+1)!} \max_{i=0,1,\dots,n} |\tilde{c}_i|,$$

$$M = \max_{0 \leq t \leq t_f} |y^{(n+1)}(t)|, \quad \tilde{c}_k = \frac{y^{(k)}(0)}{k!} - c_k,$$

Proof. See [12]. □

Here, the residual error function of the numerical solution is defined $y_n(t)$ as

$$R_n(t) = L\{y_n(t)\} - g(t),$$

and

$$e_n(t) = y(t) - y_n(t),$$

where $y(t)$ is exact solution. Hence

$$\begin{aligned} L\{e_n(t)\} &= L\{y(t)\} - L\{y_n(t)\} = -R_n(t), \\ \sum_{i=0}^n c_i B_{i,n}(t) &= \phi(t), \quad t \leq 0. \end{aligned}$$

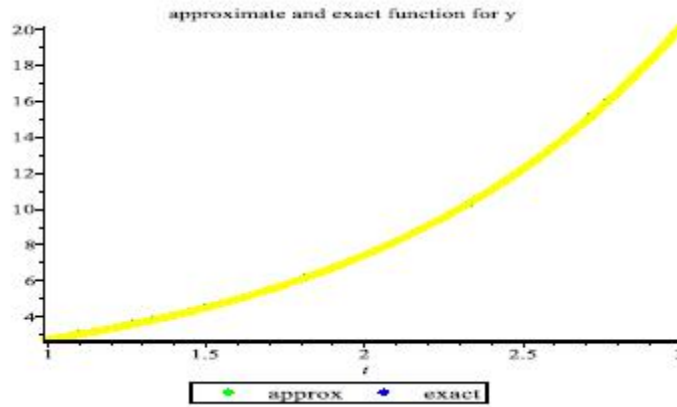
5. Numerical applications

In this section, some results are given to demonstrate the quality of the sated technique in approximating the solution of delay Volterra integral equations (1.1).

Example 1. Now, the following proportional delay VFIE is considered

$$\begin{aligned} y(t) &= (e^t + e^{1-t} - e - 1) + y(qt) + \int_1^t e^{-t} y(s) ds \\ &\quad - \int_1^{qt} y(s) ds, \quad t \in [0, 3], \\ y(t) &= e^t, \quad t \leq 1, \end{aligned}$$

where $y_{exact}(t) = e^t$. One may obtain $y(t) = 0.04102190678t^5 - 0.0987753295t^4 + 0.4132317318t^3 + 0.3108745122t^2 + 1.055479444t + 0.9964495623$ with this technique by $n = 5$. The approximate solution for $y(t)$ is shown in Fig. 1.



Example 2. The following vanishing delay Volterra integral equation (VIE) is considered (see [13])

$$y(t) = \frac{1}{2}(1 + e^{-qt}) - \int_0^t y(s)ds + \frac{1}{2} \int_0^{qt} y(s)ds,$$

where $q = 0.2$, and $y_{exact}(t) = e^{-t}$. One may obtain $y(t) = 1 - 0.99941777t + 0.4953779080t^2 - 0.15359822t^3 + 0.0255175232t^4$ with this method by $n = 4$. The approximate solution for $y(t)$ is shown in Fig. 2. Table 1 demonstrates the errors of the this technique where $\|error\|_\infty$ in [13] and this paper are respectively 2×10^{-4} (for $n = 4, m = 1$) and 0.0.

Table 1: The errors of the this method for Example 2

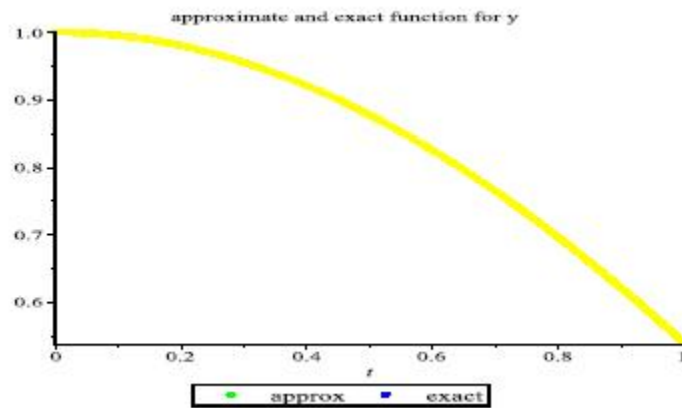
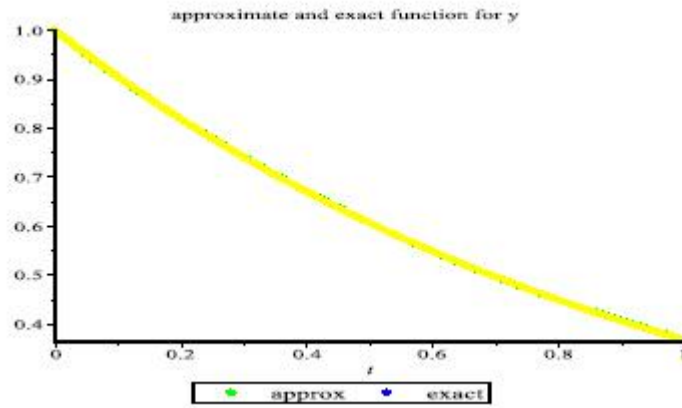
t	error of y
0.1	0.00002353764412
0.2	0.0000128515059
0.3	7.4×10^{-11}
0.4	0.000003726205
0.5	8×10^{-11}
0.6	0.00000360418
0.7	3×10^{-11}
0.8	0.0000116301
0.9	0.0000206027
1.0	0.0

Example 3. The following vanishing delay VIE is considered (see [13])

$$y(t) = f(t) + \int_0^t y(s)ds + \int_0^{qt} e^{-t}y(s)ds,$$

$$f(t) = \cos(t) - \sin(t) - e^{-t}\sin(0.5t),$$

where $q = 0.5$, and exact solution is $y(t) = \cos(t)$. One may obtain $y(t) = 1 + 0.000271524t - 0.502368133t^2 + 0.006222416t^3 + 0.03617649887t^4$ with this technique by $n = 4$. The approximate solution for $y(t)$ is shown in Fig. 3. Table 2 demonstrates the errors of the this technique.



Remark 5.1. The fractional differential equations have drawn expanding consideration as a result of their vital applications in science, material science, and designing (see [14, 15, 16, 17, 18, 19, 20]). Generally, the solving of most fractional differential equations are definitely not simple. The Bagley-Torvik equation was introduced by the authors of [21]. In different papers, it has been examined. This equation has been concentrated both diagnostically and, numerically in [22]. These techniques included expansion formula for fractional derivative [23], Haar wavelet, pseudo-spectral scheme [24], Bessel collocation technique [25], Taylor collocation method [26]. Mekkaoui and Hammouch [27] solved the Bagley-Torvik equation by the fractional iteration method (FIM). In addition, the stability of the Bagley-Torvik equation is given [28]. Mashayekhi and Razzaghi [29] stated a new numerical technique by utilizing hybrid functions approximation for solving the fractional Bagley-Torvik equation.

In this remark, one may utilize Bezier curves technique for solving fractional Bagley-Torvik equation.

5.1. Basic Preliminaries

Definition 5.1. The caputo’s fractional derivative of order α is stated in [25]

$$D^\alpha f(t) = \frac{1}{\Gamma(n_1 - \alpha)} \int_0^t (t - s)^{-\alpha-1+n_1} f^{(n_1)}(s) ds,$$

$$n_1 - 1 \leq \alpha \leq n_1, n_1 \in \mathbf{N},$$

where $\alpha > 0$ and n_1 is the smallest integer greater than α .

Table 2: The errors of the this method for Example 3

t	error of y
0.1	0.91459×10^{-5}
0.2	6.634×10^{-7}
0.3	0.000007129
0.4	0.69327×10^{-5}
0.5	$1.000000000 \times 10^{-10}$
0.6	0.0000072877
0.7	0.0000077604
0.8	0.0000013243
0.9	0.0000122423

5.2. Function approximation

Utilizing Bezier curves, this technique is to approximate the solutions $f(t)$ where $f(t)$ is given in Eq. (5.1). Define the Bezier polynomials of degree n that approximate over the interval $t \in [t_0, t_f]$ as follows:

$$f \approx P^n f = \sum_{r=0}^n c_r B_{r,n} \left(\frac{t-t_0}{h} \right) = C^T B(t), \quad (5.1)$$

where $h = t_f - t_0$, $C^T = [c_0, c_1, \dots, c_n]^T$,

$$B^T(t) = [B_{0,n}(t), B_{1,n}(t), \dots, B_{n,n}(t)]^T, \quad (5.2)$$

$$B_{r,n} \left(\frac{t-t_0}{h} \right) = \binom{n}{r} \frac{1}{h^n} (t_f - t)^{n-r} (t - t_0)^r,$$

is the Bernstein polynomial with degree n for $t \in [t_0, t_f]$, and c_r is the control point [9].

5.3. Error bound for the Bezier curve

The error bound is presented for the Bezier curve now.

Theorem 5.1. Let $f \in H^\mu(0, 1)$ with $\mu \geq 0$, then one may have

$$\|f - P^n f\|_{L^2(0,1)} \leq CM^{-\mu} |f|_{H^{0;\mu;M;n}(0,1)}, \quad (5.3)$$

where

$$H^\mu(0, 1) = W^{\mu,2}(0, 1) = \{f \in L^2(0, 1) | \partial^\alpha f \in L^2(0, 1), |\alpha| \leq \mu\}.$$

Proof. See [29]. □

Now, one may derive the operator I^α for $B(t)$ in Eq. 11 given by

$$I^\alpha B(t) = \bar{B}(t, \alpha), \quad (5.4)$$

where

$$\bar{B}(t, \alpha) = [I^\alpha B_{0,n}(t), \dots, I^\alpha B_{n,n}(t)]^T. \quad (5.5)$$

5.4. Problem statement

One may introduce the following Bagley-Torvik fractional equation

$$(5.6) \quad AD^{(2)}f(t) + BD^{(\frac{3}{2})}f(t) + Cf(t) = g(t).$$

In this sequel, one may have the following problems:

5.4.1. Problem (a)

Bagley-Torvik equations in Eq. 22 with the conditions given by:

$$(5.7) \quad f(0) = f_0, f'(0) = f'_0,$$

where f_0 and f'_0 are given.

5.4.2. Problem (b)

Bagley-Torvik equations in Eq. 22 with the following conditions

$$(5.8) \quad f(0) = f_0, f(1) = f_1,$$

where f_0 and f_1 are given.

For more explanation about these problems, one can study [29].

5.5. Error bound for problem (a)

Now, the error bound E^n is found for problem (a).

Theorem 5.2. Let $f \in H^\mu(0, 1)$ with $\mu \geq 0$, we have

$$(5.9) \quad \begin{aligned} \|E_n\|_{L^2(0,1)} &\leq \left(cM^{-\mu}n^{-\mu} \|f^{(\mu)}\|_{L^2(0,1)} \right) \\ &\times \left(A + \frac{B}{\Gamma(\frac{3}{2})} + \frac{C}{\Gamma(3)} \right). \end{aligned}$$

Proof. See [29].

□

5.6. Numerical applications

Now, for demonstrating the applicability and accuracy of this technique some examples are solved. The package of Maple version 14 has been used to solve the test problems.

Example 4. Now, one may consider Eqs. 22 and 24 such that (see [29])

$$A = 1, B = \frac{8}{17}, C = \frac{13}{51}, f_0 = 0, f_1 = 0,$$

$$g(t) = \frac{t^{-\frac{1}{2}}}{89250\sqrt{\pi}}(48p(t) + 7\sqrt{t}q(t))$$

where

$$p(t) = 16000t^4 - 32480t^3 + 2128t^2 - 4746t + 189,$$

$$q(t) = 3250t^5 - 9425t^4 + 264880t^3 - 44.$$

the exact solution is $f(t) = t^5 - \frac{29t^4}{10} + \frac{76t^3}{25} - \frac{339t^2}{250} + \frac{27t}{125}$.

With $n = 5$, the obtained $f(t)$ is $0.216t(1-t)^4 - 0.492t^2(1-t)^3 + 0.2680000000t^3(1-t)^2 - 0.024t^4(1-t)$ with $c_0 = 0$, $c_1 = 0.04320000000000003$, $c_2 = -0.04920000000000008$, $c_3 = 0.02680000000000010$, $c_4 = -0.004800000000000036$, and $c_5 = 0$. The obtained error is zero (see Table 3). The graphs of approximated and exact solution $f(t)$ are plotted in Fig. 4.

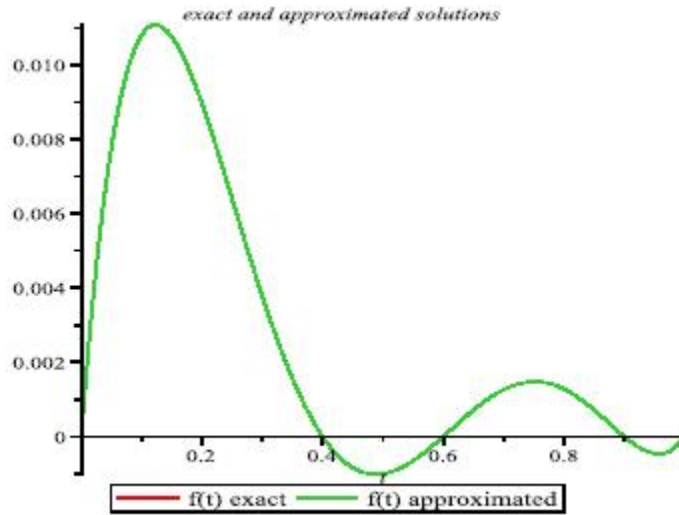


Table 3: The errors of the this method for Example 4

t	error of f
0.1	0.0
0.2	0.0
0.3	0.0
0.4	0.0
0.5	0.0
0.6	0.0
0.7	0.0
0.8	0.0
0.9	0.0

Example 5. One may consider Eqs. 22 and 24 such that (see [29])

$$f(0) = 0, f(1) = 0, g(t) = \frac{4\sqrt{t}}{\sqrt{\pi}} + t(t - 1),$$

$$A = 0, B = C = 1,$$

where the exact solution is $f(t) = t^2 - t$.
 With $n = 5$, the obtained $f(t)$ is $-t(1-t)^4 - 3t^2(1-t)^3 - 3t^3(1-t)^2 - 1t^4(1-t)$ with $c_0 = 0, c_1 = -0.2000000000000002, c_2 = -0.2999999999999996, c_3 = -0.3000000000000004, c_4 = -0.1999999999999999, c_5 = 0$. The obtained error is zero (see Table 4). The graphs of approximated and exact solution $f(t)$ are plotted in Fig. 5.

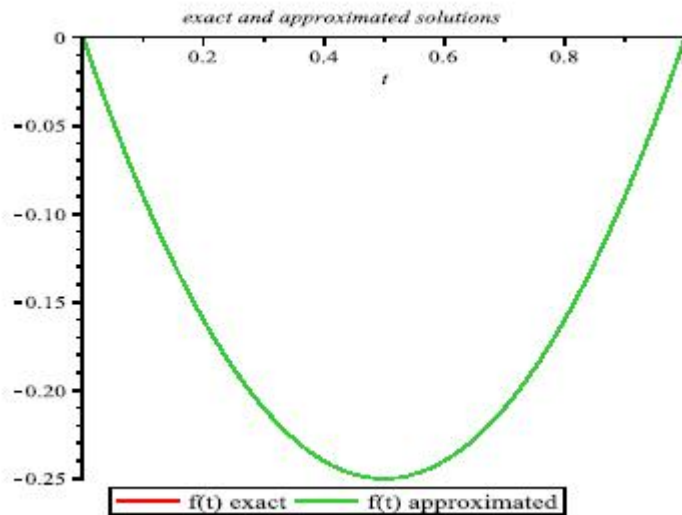


Table 4: The errors of the this method for Example 5

t	error of f
0.1	0.0
0.2	0.0
0.3	0.0
0.4	0.0
0.5	0.0
0.6	0.0
0.7	0.0
0.8	0.0
0.9	0.0

6. Conclusions

The aim of this sequel is to improve an effective and accurate technique for solving VFIE and fractional Bagley-Torvik equation. The Bezier curve strategy is utilized to obtain the approximate solution of this problem.

Some results are included to explain the validity of this technique.

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