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Oscillation results for a certain class of fourth-order nonlinear delay differential equations

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Abstract:

In this work, we study the oscillation of the fourth order neutral differential equations with delay argument. By means of generalized Riccati transformation technique, we obtain new oscillation criteria for oscillation of this equation. An example is given to clarify the main results in this paper.

Keywords: Oscillation; Fourth order; Neutral delay; Differential equations.

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1. Introduction

In this paper, we are concerned with the oscillation of the fourth-order nonlinear neutral differential equations with delay argument

$$(1.1)\left[r\left(t\right)\left(\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)'''\right)^{\alpha}\right]'+q\left(t\right)f\left(x\left(g\left(t\right)\right)\right)=0,\quad t\geq t_{0},$$

where $r,\ p,\ q,\ \tau,\ g\in C\left([t_{0},\infty),\mathbf{R}\right)$, $r\left(t\right)$ and $q\left(t\right)$ are positive, $0\leq p\left(t\right)\leq p<1$, $\alpha\geq1$ is a quotient of odd positive integers, $r'\left(t\right)>0$, $\tau'\left(t\right)>0$, $\tau\left(t\right)\leq t$, $g\left(t\right)\leq t$, $\lim_{t\rightarrow\infty}\tau\left(t\right)=\infty$, $\lim_{t\rightarrow\infty}g\left(t\right)=\infty$, the function f satisfies the following condition

$$(1.2)\qquad f\in C\left(\mathbf{R},\mathbf{R}\right),\ f\left(x\right)/x^{\alpha}\geq k>0\ \text{ for }\ x=0,$$

where k a constant and

$$(1.3)\qquad \int_{t_{0}}^{\infty}r^{-1/\alpha}\left(t\right)dt<\infty.$$

We define the corresponding function $\upsilon\left(t\right)=x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)$. By a solution of Eq. (1.1), we mean a non-trivial real function $x(t)\in C\left([t_{x},\infty)\right)$, $t_{x}\geq t_{0}$, which has the properties $x\left(t\right),\ \upsilon(t),\ \upsilon'(t),\ \upsilon''(t)$ and $r\left(t\right)\left[\upsilon'''\left(t\right)\right]^{\alpha}$ are continuously differentiable for all $t\in[t_{x},\infty)$ and satisfies (1.1) on $[t_{x},\infty)$. We consider only those solutions $x\left(t\right)$ of (1.1) which satisfy $\sup\{|x(t)|:t\geq L\}>0$ for any $L\geq t_{x}$. A solution of Eq. (1.2) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

In models concerning chemical, biological and physical phenomena, fourth-order differential equations naturally appear; see [1]. In the past decade, there has been increasing interest in obtaining oscillation and nonoscillation criteria of different classes of third and fourth-order differential equations. The works [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] improved the oscillation criteria for second-order equations with delay and advanced argument. For even-order delay equations, papers [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] developed the oscillation criteria. Whereas, the results in [31, 32, 33, 34, 35, 36] dealt with the issue of oscillation of equations of odd-order.

Agarwal et al. [14] studied the oscillatory behavior of the delay equation

$$\left[r_{3}\left(t\right)\left(\left[r_{2}\left(t\right)\left(\left[r_{1}\left(t\right)\left(x'\left(t\right)\right)^{\alpha_{1}}\right]'\right)^{\alpha_{2}}\right]'\right)^{\alpha_{3}}\right]'\pm q\left(t\right)f\left(x\left(g\left(t\right)\right)\right)=0,$$

where $\int^{\infty}r_{i}^{1/\alpha_{i}}\left(\vartheta\right)\ d\vartheta<\infty,\ i=1,2,3$. Grace et al. [16] established new criteria for oscillation of delay equation

$$\left(r\left(t\right)\left(x'\left(t\right)\right)^{\alpha}\right)'''+q\left(t\right)f\left(x\left(g\left(t\right)\right)\right)=0,$$

Wu [30] and Kamo and Usami [24] studied the oscillation behavior of delay equation

$$\left(r(t)\left|x''(t)\right|^{\alpha-1}x''(t)\right)'' + q\left|x(t)\right|^{\beta-1}x(t) = 0,$$

where α, $\beta > 0$.

For neutral delay equations, Li et al. [19] studied oscillatory properties of solutions of even-order equations

$$\left(x(t) + p(t)\,x(\tau(t))\right)^{(n)} + q(t)\,x(g(t)) = 0,$$

relating oscillation of higher-order equations to that of a pair of associated first-order delay differential equations. Based on the comparison with first-order delay equations, Moaaz et al. [28] established an oscillation criterion for neutral equations

$$\left(r(t)\left(\left(x(t) + p(t)\,x(\tau(t))\right)^{(n-1)}\right)^{\alpha}\right)' + q(t)\,x^{\alpha}(g(t)) = 0,$$

where $\int^{\infty} r^{1/\alpha}(\vartheta)\ d\vartheta = \infty$..

Actually, we have greatly less results for fourth order neutral differential equations than those can be found in the literature on the oscillation of solutions of differential equations of first, second orders, also even-order delay equations. So, the main objective of this paper is to shed light on the class of fourth order neutral equations (1.1), and through study the oscillation criteria of solutions of this equation. In order to discuss our main results, we need the following lemmas

Lemma 1.1. *[3, Lemma 2.1] Assume that $\alpha \geq 1$ is a quotient of odd positive integers, A, B, U and V are constants, $AB \geq 0$ and $V > 0$. Then*

$$A^{(\alpha+1)/\alpha} - (A-B)^{(\alpha+1)/\alpha} \leq \frac{1}{\alpha}B^{1/\alpha}\left((\alpha+1)A - B\right) \tag{1.4}$$

and

$$Uy - Vy^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}U^{\alpha+1}V^{-\alpha}, \quad y \geq 0. \tag{1.5}$$

Lemma 1.2. *[37] If the function x satisfies $x^{(i)}(t) > 0$, $i = 0, 1, ..., n$, and $x^{(n+1)}(t) < 0$, then*

$$\frac{x(t)}{t^n/n!} \geq \frac{\lambda x'(t)}{t^{n-1}/(n-1)!},$$

for all $\lambda = (0,1)$.

Lemma 1.3. *[2, Lemma 2.2.3] Let* $x \in C^n([t_0,\infty),(0,\infty))$. *Assume that* $x^{(n)}(t)$ *is of fixed sign and not identically zero on* $[t_0,\infty)$ *and that there exists a* $t_1 \geq t_0$ *such that*
$x^{(n-1)}(t)\,x^{(n)}(t) \leq 0$ *for all* $t \geq t_1$. *If* $\lim_{t\to\infty} x(t) \neq 0$, *then for every* $\mu \in (0,1)$ *there exists* $t_\mu \geq t_1$ *such that*
$\mathrm{x}(t) \geq \frac{\mu}{(n-1)!} t^{n-1} \left| x^{(n-1)}(t) \right|$ *for* $t \geq t_\mu$.

2. Main results

In this section, we will establish some oscillation criteria for solutions of the Eq. (1.1). For simplicity, denote by $S^{\oplus}$ the set of all eventually positive solutions of the equation (1.1). Also, we introduce the following notations:
$\eta_0(t) = \int_t^\infty r^{-1/\alpha}(s)\,ds,\ \eta_i(t) = \int_t^\infty \eta_{i-1}(s)\,ds,\ \eta_0(t) = \eta(t),\ i = 1,2,$
and
$\beta = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}.$

Lemma 2.1. *If* $x(t)$ *is an eventually positive three times continuously differentiable function such that* $r(t)(\upsilon'''(t))^\alpha$ *is continuously differentiable and* $(r(t)(\upsilon'''(t))^\alpha)' \leq 0$ *for large* t, *then one of the following cases holds for large* t,

$$\begin{array}{ll}
(\mathbf{C}_1) & \upsilon(t) > 0, \upsilon'(t) > 0, \upsilon''(t) > 0 \text{ and } \upsilon'''(t) > 0, \\
(\mathbf{C}_2) & \upsilon(t) > 0, \upsilon'(t) > 0, \upsilon''(t) < 0 \text{ and } \upsilon'''(t) > 0, \\
(\mathbf{C}_3) & \upsilon(t) > 0, \upsilon'(t) < 0, \upsilon''(t) > 0 \text{ and } \upsilon'''(t) < 0, \\
(\mathbf{C}_4) & \upsilon(t) > 0, \upsilon'(t) > 0, \upsilon''(t) > 0 \text{ and } \upsilon'''(t) < 0.
\end{array}$$

The proof is immediate and hence is omitted.

Lemma 2.2. *Let* $x(t) \in S^{\oplus}$ *and the corresponding* $\upsilon(t)$ *satisfy* $(\mathbf{C}_1)$. *Then there exist a positive function* $\rho_1 \in C^1([t_0,\infty))$ *such that for some* $\mu \in (0,1)$,

$$R_1 := \int_{t_0}^{\infty} \left(\Psi(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\rho_1'(s))^{\alpha+1}}{\mu^{\alpha} s^{2\alpha} \rho_1^{\alpha}(s)} \right) ds < \infty, \tag{2.1}$$

where

$$\Psi(t) = k\rho_1(t) q(t) (1 - p(g(t)))^{\alpha} \left(\frac{g(t)}{t} \right)^{3\alpha}. \tag{2.2}$$

Proof. Assume that $x(t) \in S^{\oplus}$. It follow from the facts $r'(t) > 0$ and $(r(t)(\upsilon'''(t))^{\alpha})' \leq 0$ that $\upsilon^{(4)}(t) < 0$. Since $\tau(t) \leq t$ and $\upsilon'(t) > 0$, we get

$$\begin{aligned} x(t) &= \upsilon(t) - p(t)x(\tau(t)) \\ &\geq (1 - p(t))\upsilon(t). \end{aligned}$$

From equation (1.1), we see that

$$\begin{aligned} (r(t)(\upsilon'''(t))^{\alpha})' &= -q(t) f(x(g(t))) && (2.3) \\ &\leq -kq(t)(1 - p(g(t)))^{\alpha} \upsilon^{\alpha}(g(t)). && (2.4) \end{aligned}$$

Now, we define a generalized Riccati substitution by
$\omega(t) := \rho_1(t) \frac{r(t)(\upsilon'''(t))^{\alpha}}{\upsilon^{\alpha}(t)}$.
Then $\omega(t) > 0$. By differentiating and using (2.4), we obtain

$$\begin{aligned} \omega'(t) &\leq \frac{\rho_1'(t)}{\rho_1(t)} \omega(t) - k\rho_1(t) q(t) (1 - p(g(t)))^{\alpha} \frac{\upsilon^{\alpha}(g(t))}{\upsilon^{\alpha}(t)} && (2.5) \\ &\quad - \alpha\rho_1(t) \frac{r(t)(\upsilon'''(t))^{\alpha}}{\upsilon^{\alpha+1}(t)} \upsilon'(t). && (2.6) \end{aligned}$$

From Lemma 1.2, we have that $\upsilon(t) \geq \frac{t}{3}\upsilon'(t)$, and hence,

$$\frac{\upsilon(g(t))}{\upsilon(t)} \geq \frac{g^3(t)}{t^3}. \tag{2.7}$$

It follows from Lemma 1.3 that

$$\upsilon'(t) \geq \frac{\mu}{2} t^2 \upsilon'''(t), \tag{2.8}$$

for all $\mu \in (0, 1)$ and every sufficiently large t. Thus, by (2.6), (2.7) and (2.8), we get

$$\omega'(t) \leq \frac{\rho_1'(t)}{\rho_1(t)}\omega(t) - k\rho_1(t)\, q(t)\,(1-p(g(t)))^{\alpha}\left(\frac{g(t)}{t}\right)^{3\alpha} \tag{2.9}$$

$$-\alpha\mu \frac{t^2}{2r^{1/\alpha}(t)\,\rho_1^{1/\alpha}(t)}\omega^{\frac{\alpha+1}{\alpha}}(t). \tag{2.10}$$

Using the inequality (1.5) with $U = \frac{\rho_1'(t)}{\rho_1(t)}, V = \frac{\alpha\mu t^2}{2r^{1/\alpha}(t)\rho_1^{1/\alpha}(t)}$ and $y = \omega$, we get

$$\omega'(t) \leq -\Psi(t) + \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(t)\,(\rho_1'(t))^{\alpha+1}}{\mu^{\alpha}t^{2\alpha}\rho_1^{\alpha}(t)}.$$

This implies that

$$\int_{t_1}^{t}\left(\Psi(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(s)\,(\rho_1'(s))^{\alpha+1}}{\mu^{\alpha}s^{2\alpha}\rho_1^{\alpha}(s)}\right)ds \leq \omega(t_1).$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $x(t) \in S^{\oplus}$ and the corresponding $\upsilon(t)$ satisfy $(\mathbf{C}_2)$. Then there exist a positive function $\rho_2 \in C^1([t_0,\infty))$ such that*

$$R_2 := \int_{t_0}^{\infty}\left(\Phi(s) - \frac{(\rho_2'(s))^2}{4\rho_2(s)}\right)ds < \infty, \tag{2.11}$$

where

$$\Phi(t) = \rho_2(t)\int_t^{\infty}\left(\frac{k}{r(u)}\int_u^{\infty} q(s)\,\frac{g^{\alpha}(s)}{s^{\alpha}}\,(1-p(g(s)))^{\alpha}\,ds\right)^{1/\alpha} du.$$

(2.12)

Proof. In view of the proof of Lemma 2.2, it follows that (2.4) holds. Integrating (2.4) from t to u, we obtain

$$r(u)\,(\upsilon'''(u))^{\alpha} - r(t)\,(\upsilon'''(t))^{\alpha} \leq -k\int_t^u q(s)\,(1-p(g(s)))^{\alpha}\,\upsilon^{\alpha}(g(s))\,ds.$$

(2.13)

From Lemma 1.2, we get that $\upsilon(t) \geq t\upsilon'(t)$, and hence,

$$\upsilon(g(t)) \geq \frac{g(t)}{t}\upsilon(t). \tag{2.14}$$

For (2.13), letting $u \to \infty$ and using (2.14), we get
$r(t)(\upsilon'''(t))^{\alpha} \geq k\upsilon^{\alpha}(t)\int_t^{\infty} q(s)\frac{g^{\alpha}(s)}{s^{\alpha}}(1-p(g(s)))^{\alpha}ds$.
Integrating the former inequality again from t to ∞, we get

$$\upsilon''(t) \leq -\upsilon(t)\int_t^{\infty}\left(\frac{k}{r(u)}\int_u^{\infty} q(s)\frac{g^{\alpha}(s)}{s^{\alpha}}(1-p(g(s)))^{\alpha}ds\right)^{1/\alpha}du.$$

(2.15)

Now, we define

$$w(t) = \theta(t)\frac{\upsilon'(t)}{\upsilon(t)}.$$

Then $w(t) > 0$ for $t \geq t_1 \geq t_0$. By differentiating the last inequality and using (2.15), we find

$$w'(t) = \frac{\theta'(t)}{\theta(t)}w(t) + \theta(t)\frac{\upsilon''(t)}{\upsilon(t)} - \theta(t)\left(\frac{\upsilon'(t)}{\upsilon(t)}\right)^2$$

$$\leq \frac{\theta'(t)}{\theta(t)}w(t) - \theta(t)\int_t^{\infty}\left(\frac{k}{r(u)}\int_u^{\infty} q(s)\frac{g^{\alpha}(s)}{s^{\alpha}}(1-p(g(s)))^{\alpha}ds\right)^{1/\alpha}du$$

$$-\frac{1}{\theta(t)}w^2(t). \tag{2.16}$$

Thus, we obtain

$$w'(t) \leq -\Phi(t) + \frac{(\theta'(t))^2}{4\theta(t)}.$$

Then, we get

$$\int_{t_1}^{t}\left(\Phi(s)-\frac{(\theta'(t))^2}{4\theta(t)}\right)ds\le w(t_1).$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that $p(t)=p$ (constant). If $x(t)\in S^{\oplus}$ and the corresponding $\upsilon(t)$ satisfy ($\mathbf{C}_3$), then there exists an odd integer $n\ge 0$ such that*

$$R_3:=\int_{t_0}^{\infty}\left(k\lambda^{\alpha}\eta_2^{\alpha}(s)\,q(s)-\alpha\beta\frac{\eta_1(s)}{\eta_2(s)}\right)ds\le 1, \tag{2.17}$$

where $\lambda=\sum_{r=0}^{n}(-1)^r p^r<1$.

Proof. Let $x(t)$ is a positive solution of equation (1.1). Since $r(t)(\upsilon'''(t))^{\alpha}$ nonincreasing, we have that $r(s)(\upsilon'''(s))^{\alpha}\le r(t)(\upsilon'''(t))^{\alpha}$ for all $s\ge t\ge t_1$. This yields

$\upsilon'''(s)\le[r(t)(\upsilon'''(t))^{\alpha}]^{1/\alpha}\frac{1}{r^{1/\alpha}(s)}$.

By integrating this inequality three times from t to ∞ and using ($\mathbf{C}_3$), we get

$$(-1)^{i+1}\upsilon^{(i)}(t)\le\left[r(t)(\upsilon'''(t))^{\alpha}\right]^{1/\alpha}\eta_{2-i}(t),\ i=0,1,2. \tag{2.18}$$

New, we define

$\tau^0(t)=t,\tau^j(t)=\tau(\tau^{j-1}(t)),\ j=1,2,\dots$.

Then, we obtain

$\mathrm{x}(t)=\upsilon(t)-px(\tau(t))=\upsilon(t)-p\upsilon(\tau(t))+p^2x(\tau^2(t))$,

for all sufficiently large t. Repeating this procedure and the monotonicity of υ, we obtain that there exists an odd integer $n\ge 0$ such that $\tau^{n+1}(g(t))\ge t_1\ge t_0$ and

$\mathrm{x}(g(t))=\sum_{r=0}^{n}(-1)^r p^r\upsilon(\tau^r(g(t)))+p^{n+1}x(\tau^{n+1}(g(t)))\ge\lambda\upsilon(g(t))$,

where $\lambda>0$. Hence, from (1.1), we see that

$$\left[r(t)(\upsilon'''(t))^{\alpha}\right]'\le-k\lambda^{\alpha}q(t)\upsilon^{\alpha}(g(t)) \tag{2.19}$$

$$\le-k\lambda^{\alpha}q(t)\upsilon^{\alpha}(t). \tag{2.20}$$

Next, we define

$$\psi(t)=\frac{r(t)(\upsilon'''(t))^{\alpha}}{\upsilon^{\alpha}(t)}. \tag{2.21}$$

Thus, we see that $\psi(t) < 0$ and satisfies
$\psi'(t) = \frac{[r(t)(v'''(t))^{\alpha}]'}{v^{\alpha}(t)} - \alpha\frac{r(t)(v'''(t))^{\alpha}}{v^{\alpha+1}(t)}v'(t)$.
Hence, from (2.20) and [(2.18) with $i = 1$], we have

$$\psi'(t) \leq -k\lambda^{\alpha}q(t) - \alpha\eta_1(t)\psi^{\frac{1+\alpha}{\alpha}}(t). \tag{2.22}$$

From (2.18) with $i = 0$, we have

$$\eta_2^{\alpha}(t)\psi(t) \geq -1. \tag{2.23}$$

Multiplying (2.22) by $\eta_2^{\alpha}(t)$ and integrating from t_1 to t, we obtain

$$\eta_2^{\alpha}(t)\psi(t) - \eta_2^{\alpha}(t_1)\psi(t_1) \leq -\alpha\int_{t_1}^{t}\eta_1(s)\eta_2^{\alpha-1}(s)\psi(s)\,ds$$
$$-k\lambda^{\alpha}\int_{t_1}^{t}\eta_2^{\alpha}(s)q(s)\,ds - \alpha\int_{t_1}^{t}\eta_1(s)\eta_2^{\alpha}(s)\psi^{\frac{1+\alpha}{\alpha}}(s)\,ds,$$

which with (2.23) gives

$$\begin{aligned} 1 + \eta_2^{\alpha}(t_1)\psi(t_1) \geq\ & k\lambda^{\alpha}\int_{t_1}^{t}\eta_2^{\alpha}(s)q(s)\,ds \\ & +\alpha\int_{t_1}^{t}\eta_1(s)\eta_2^{\alpha-1}(s)\left[\psi(s) + \eta_2(s)\psi^{\frac{1+\alpha}{\alpha}}(s)\right]ds. \end{aligned}$$

Using the inequality (1.5) with $U = 1, V = \eta_2(s)$ and $y = -\psi$, we get
$\psi(s) + \eta_2(s)\psi^{\frac{1+\alpha}{\alpha}}(s) \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\eta_2^{-\alpha}(s)$.
Hence, it follows that

$$\begin{aligned} \int_{t_1}^{t}\left(k\lambda^{\alpha}\eta_2^{\alpha}(s)q(s) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{\eta_1(s)}{\eta_2(s)}\right)ds \ & \leq\ 1 + \eta_2^{\alpha}(t_1)\psi(t_1) \\ & \leq\ 1. \end{aligned}$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Assume that $x(t) \in S^{\oplus}$ and the corresponding $v(t)$ satisfy* ($\mathbf{C}_4$). *Then*

$$R_4 := \int_{t_0}^{\infty}\left(\frac{k\widetilde{\mu}^{\alpha}}{2^{\alpha}}q(s)\left[\eta_0(s)(1 - p(g(s)))g^2(s)\right]^{\alpha} - \frac{\alpha\beta}{r^{1/\alpha}(s)\eta_0(s)}\right)ds \leq 1, \tag{2.24}$$

for some $\widetilde{\mu} \in (0,1)$.

Proof. In view of the proofs of Lemma 2.2 and Lemma 2.4, we have that (2.4) and (2.18) with $i = 2$ hold, respectively. The inequality (2.18) with $i = 2$ yields

$$\eta_0^\alpha(t) \frac{r(t)(\upsilon'''(t))^\alpha}{(\upsilon''(t))^\alpha} \geq -1 \tag{2.25}$$

From Lemma 1.3, we have that $\upsilon(t) \geq \frac{\widetilde{\mu}}{2} t^2 \upsilon''(t)$ for all $\widetilde{\mu} \in (0,1)$ and every sufficiently large t. Thus, there exists a $t_2 \geq t_1$ such that

$$\frac{\upsilon(g(t))}{\upsilon''(g(t))} \geq \frac{\widetilde{\mu}}{2} g^2(t), \tag{2.26}$$

for $t \geq t_2$. Next, we define

$$\varphi(t) = \frac{r(t)(\upsilon'''(t))^\alpha}{(\upsilon''(t))^\alpha}. \tag{2.27}$$

We note that $\varphi(t) < 0$ for $t \geq t_1$. By differentiating (2.27) and using (2.4) and (2.26), we obtain

$$\varphi'(t) \leq -\frac{k\widetilde{\mu}^\alpha}{2^\alpha} q(t)(1 - p(g(t)))^\alpha g^{2\alpha}(t) \left(\frac{\upsilon''(g(t))}{\upsilon''(t)}\right)^\alpha - \alpha \frac{1}{r^{1/\alpha}(t)} \varphi^{\frac{\alpha+1}{\alpha}}(t).$$

Since, $g(t) \leq t$ and $\upsilon'''(t) < 0$, we have that $\upsilon''(g(t)) \geq \upsilon''(t)$, and hence

$$\varphi'(t) \leq -\frac{k\widetilde{\mu}^\alpha}{2^\alpha} q(t)(1 - p(g(t)))^\alpha g^{2\alpha}(t) - \frac{\alpha}{r^{1/\alpha}(t)} \varphi^{\frac{\alpha+1}{\alpha}}(t). \tag{2.28}$$

Multiplying (2.28) by $\eta_0^\alpha(t)$ and integrating from t_2 to t, and using (2.25), we obtain

$$\begin{aligned} 1 + \eta_0^\alpha(t_2)\varphi(t_2) \geq\ & \frac{k\widetilde{\mu}^\alpha}{2^\alpha} \int_{t_2}^t q(s)\eta_0^\alpha(s)(1 - p(g(s)))^\alpha g^{2\alpha}(s)\,ds \\ & + \alpha \int_{t_2}^t \frac{\eta_0^{\alpha-1}(s)}{r^{1/\alpha}(s)} \left(\varphi(s) + \eta_0(s)\varphi^{\frac{1+\alpha}{\alpha}}(s)\right) ds. \end{aligned}$$

By following the same steps in Lemma 2.4, we get that

$$\begin{aligned} \int_{t_2}^t \left(\frac{k\widetilde{\mu}^\alpha}{2^\alpha} q(s)\eta_0^\alpha(s)(1 - p(g(s)))^\alpha g^{2\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{r^{1/\alpha}(s)\eta_0(s)} \right) ds & \\ \leq 1 + \eta_0^\alpha(t_2)\varphi(t_2), & \\ \leq 1. & \end{aligned}$$

This completes the proof of Lemma 2.5. □

Theorem 2.1. *Assume that* $p(t) = p$ *(constant). If there exist positive functions* $\rho, \theta \in C^1([t_0, \infty))$ *and odd integer* $n \geq 0$ *such that* $R_1 = R_2 = \infty$, $R_3 > 1$ *and* $R_4 > 1$, *for some* $\mu, \widetilde{\mu} \in (0,1)$, *where* R_1, R_2, $R_3\$andR_4$ *are defined by (2.1), (2.11), (2.17) and (2.24), respectively, then every solution of (1.1) is oscillatory.*

In the next theorem, we establish new oscillation results for equation (1.1) by using the integral averaging technique due to Philos [38].

Theorem 2.2. *Assume that there exist positive functions* $\rho, \theta \in C^1([t_0, \infty))$, $H_i, h_i \in C(D, \mathbf{R})$ *for* $i = 1, 2, 3, 4$, *where* $D = \{(t,s) \in \mathbf{R}^2 : t \geq s \geq t_0\}$, *such that* $H_i(t,t) = 0\$fort \geq t_0$, $H_i(t,s) > 0$ *for* $t > s \geq t_0$, H_i *has a nonpositive continuous partial derivative* $\partial H_i / \partial s$ *satisfying*

$\rho(s) \frac{\partial H_1(t,s)}{\partial s} + \rho'(s) H_1(t,s) = h_1(t,s) H_1^{\frac{\alpha}{\alpha+1}}(t,s)$,

$\theta(s) \frac{\partial H_2(t,s)}{\partial s} + \theta'(s) H_2(t,s) = h_2(t,s) \sqrt{H_2(t,s)}$,

$\rho_3(s) \frac{\partial H_3(t,s)}{\partial s} + \rho_3'(s) H_3(t,s) = -h_3(t,s) H_3^{\frac{\alpha}{\alpha+1}}(t,s)$

and

$\rho_4(s) \frac{\partial H_4(t,s)}{\partial s} + \rho_4'(s) H_4(t,s) = -h_4(t,s) H_4^{\frac{\alpha}{\alpha+1}}(t,s)$.

If

$$\limsup_{t\to\infty} \frac{1}{H_1(t,t_0)} \int_{t_0}^{t} \left(H_1(t,s) \Psi(s) - \frac{2^\alpha}{(\alpha+1)^{\alpha+1}} \frac{r(s) \left[h_1^{\alpha+1}(t,s) \right]_+}{\mu^\alpha s^{2\alpha} \rho^\alpha(s)} \right) ds = \infty, \tag{2.29}$$

$$\limsup_{t\to\infty} \frac{1}{H_2(t,t_0)} \int_{t_0}^{t} \left(H_2(t,s) \Phi(s) - \frac{\left[h_2^2(t,s) \right]_+}{4\theta(s)} \right) ds = \infty, \tag{2.30}$$

$$\limsup_{t\to\infty} \frac{1}{H_3(t,t_0)} \int_{t_0}^{t} \left(k\lambda^\alpha H_3(t,s) \rho_3(s) q(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{\left[h_3^{\alpha+1}(t,s) \right]_+}{\eta_1^\alpha \rho_3^\alpha(s)} \right) ds = \infty \tag{2.31}$$

and

$$\limsup_{t\to\infty}\frac{1}{H_4(t,t_0)}\int_{t_0}^{t}\left(\Omega(t,s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{rh_4^{\alpha+1}}{\rho_4^{\alpha}}\right)ds=\infty, \tag{2.32}$$

where $\Omega(t,s)=\frac{k\widetilde{\mu}^{\alpha}}{2^{\alpha}}H_4(t,s)\rho_4(s)q(s)(1-p(g(s)))^{\alpha}g^{2\alpha}(s)$, *then every solution of (1.1) is oscillatory.*

Proof. Assume that equation (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is an eventually positive solution. By Lemma 2.1, we observe that $\upsilon(t)$ has one of the four cases $(\mathbf{C}_1)-(\mathbf{C}_4)$ for $t\geq t_1$. For Case $(\mathbf{C}_1)$, proceeding as a proof of Lemma 2.2, we have that (2.10) holds. Multiplying (2.10) by $H_1(t,s)$ and integrating the resulting inequality from t_1 to t, we find

$$\int_{t_1}^{t}H_1(t,s)\Psi(s)\,ds \leq H_1(t,t_1)\omega(t_1)+\int_{t_1}^{t}\left(\frac{\partial H_1(t,s)}{\partial s}+\frac{\rho'(s)}{\rho(s)}H_1(t,s)\right)\omega(s)\,ds \tag{2.33}$$

$$-\int_{t_1}^{t}\frac{\alpha\mu s^2}{2r^{1/\alpha}(s)\rho^{1/\alpha}(s)}H_1(t,s)\omega^{\frac{\alpha+1}{\alpha}}(s)\,ds \tag{2.34}$$

$$\leq H_1(t,t_1)\omega(t_1)+\int_{t_1}^{t}\frac{h(t,s)}{\rho(s)}H_1^{\frac{\alpha}{\alpha+1}}(t,s)\omega(s)\,ds \tag{2.35}$$

$$-\int_{t_1}^{t}\frac{\alpha\mu s^2}{2r^{1/\alpha}(s)\rho^{1/\alpha}(s)}H_1(t,s)\omega^{\frac{\alpha+1}{\alpha}}(s)\,ds. \tag{2.36}$$

By Lemma 1.1, if we set $U=\frac{h}{\rho}H_1^{\frac{\alpha}{\alpha+1}}$, $V=\frac{\alpha\mu s^2}{2r^{1/\alpha}\rho^{1/\alpha}}H_1$ and $y=\omega$, then we obtain

$$\frac{h}{\rho}H_1^{\frac{\alpha}{\alpha+1}}\omega-\frac{\alpha\mu s^2}{2r^{1/\alpha}\rho^{1/\alpha}}H_1\omega^{(\alpha+1)/\alpha}\leq\frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{rh_1^{\alpha+1}}{\mu^{\alpha}s^{2\alpha}\rho^{\alpha}},$$

which with (2.36) gives

$$\frac{1}{H_1(t,t_1)}\int_{t_1}^{t}\left(H_1(t,s)\Psi(s)-\frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left[h_1^{\alpha+1}(t,s)\right]_+}{\mu^{\alpha}s^{2\alpha}\rho^{\alpha}(s)}\right)ds\leq\omega(t_1),$$

which contradicts (2.29). In the Case $(\mathbf{C}_2)$, as in the proof of Lemma 2.3, we get that (2.16) holds. Multiplying (2.16) by $H_2(t,s)$ and integrating the resulting from t_2 to t, we obtain

$$\int_{t_2}^{t}H_2(t,s)\Phi(s)\,ds \leq H_2(t,t_2)w(t_2)+\int_{t_2}^{t}\left(\frac{\partial H_2(t,s)}{\partial s}+\frac{\theta'(s)}{\theta(s)}H_2(t,s)\right)w(s)\,ds$$

$$
\begin{aligned}
&-\int_{t_2}^{t}\frac{1}{\theta(s)}H_2(t,s)\,w^2(s)\,ds\\
\le\ & H_2(t,t_2)\,w(t_2)+\int_{t_2}^{t}\frac{h_2(t,s)}{\theta(s)}\sqrt{H_2(t,s)}w(s)\,ds\\
&-\int_{t_2}^{t}\frac{1}{\theta(s)}H_2(t,s)\,w^2(s)\,ds
\end{aligned}
$$

and so,

$$\frac{1}{H_2(t,t_2)}\int_{t_2}^{t}\left(H_2(t,s)\Phi(s)-\frac{h_2^2(t,s)}{4\theta(s)}\right)ds\le w(t_2),$$

which contradicts (2.30). Now, let Case $(\mathbf{C}_3)$ holds. Proceeding as a proof of Lemma 2.4, we have that (2.18), (2.20) and (2.22) hold. Next, we define $\overline{\psi}(t)=\rho_3(t)\,\psi(t)$ where $\psi(t)$ defined as (2.21). Then, we see that $\overline{\psi}(t)<0$ and satisfies $\overline{\psi}'(t)=(\rho_3'(t)/\rho_3(t))\,\overline{\psi}(t)+\rho_3(t)\,\psi'(t)$. Hence, from (2.22), we find

$$\overline{\psi}'(t)\le\frac{\rho_3'(t)}{\rho_3(t)}\overline{\psi}(t)-k\lambda^{\alpha}\rho_3(t)\,q(t)-\frac{\alpha\eta_1(t)}{\rho_3^{1/\alpha}(t)}\overline{\psi}^{\frac{1+\alpha}{\alpha}}(t)\,.$$

Multiplying this inequality by $H_3(t,s)$ and integrating the resulting from t_3 to t, we obtain

$$\int_{t_3}^{t}k\lambda^{\alpha}H_3(t,s)\,\rho_3(s)\,q(s)\,ds \tag{2.37}$$

$$\le H_3(t,t_3)\,\overline{\psi}(t_3)+\int_{t_3}^{t}\left(\frac{\partial H_3(t,s)}{\partial s}+\frac{\rho_3'(s)}{\rho_3(s)}H_3(t,s)\right)\overline{\psi}(s)\,ds \tag{2.38}$$

$$-\int_{t_3}^{t}\frac{\alpha\eta_1(s)}{\rho_3^{1/\alpha}(s)}H_3(t,s)\,\overline{\psi}^{\frac{1+\alpha}{\alpha}}(s)\,ds \tag{2.39}$$

$$\le H_3(t,t_3)\,\overline{\psi}(t_3)-\int_{t_3}^{t}\frac{h_3(t,s)}{\rho_3(s)}H_3^{\frac{\alpha}{\alpha+1}}(t,s)\,\overline{\psi}(s)\,ds \tag{2.40}$$

$$-\int_{t_3}^{t}\frac{\alpha\eta_1(s)}{\rho_3^{1/\alpha}(s)}H_3(t,s)\,\overline{\psi}^{\frac{1+\alpha}{\alpha}}(s)\,ds. \tag{2.41}$$

By Lemma 1.1, if we set $U=\frac{h_3}{\rho_3}H_3^{\frac{\alpha}{\alpha+1}}$, $V=\frac{\alpha\eta_1}{\rho_3^{1/\alpha}}H_3$ and $y=-\overline{\psi}$, then we obtain

$$-\frac{h_3}{\rho_3}H_3^{\frac{\alpha}{\alpha+1}}\overline{\psi}-\frac{\alpha\eta_1}{\rho_3^{1/\alpha}}H_3\overline{\psi}^{(\alpha+1)/\alpha}\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{h_3^{\alpha+1}}{\eta_1^{\alpha}\rho_3^{\alpha}},$$

which with (2.41) gives

$$\frac{1}{H_3(t,t_3)}\int_{t_3}^{t}\left(k\lambda^{\alpha}H_3(t,s)\,\rho_3(s)\,q(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{h_3^{\alpha+1}(t,s)}{\eta_1^{\alpha}\rho_3^{\alpha}(s)}\right)ds\leq\overline{\psi}(t_3),$$
(2.42)

which contradicts (2.31). Assume that Case ($\mathbf{C}_4$) holds. Proceeding as a proof of Lemma 2.5, we get that (2.28) holds. Next, we define $\overline{\varphi}(t)=\rho_4(t)\,\varphi(t)$ where $\varphi(t)$ defined as (2.27). Then, we note that $\overline{\varphi}(t)<0$ and $\overline{\varphi}'(t)=(\rho_4'(t)/\rho_4(t))\,\overline{\varphi}(t)+\rho_4(t)\,\varphi'(t)$. Thus, from (2.28), we have

$$\begin{aligned}\overline{\varphi}'(t) \quad &\leq \quad \frac{\rho_4'(t)}{\rho_4(t)}\overline{\varphi}(t)-\frac{k\widetilde{\mu}^{\alpha}}{2^{\alpha}}\rho_4(t)\,q(t)\,(1-p(g(t)))^{\alpha}g^{2\alpha}(t)\\ &\quad -\frac{\alpha}{r^{1/\alpha}(t)\,\rho_4^{1/\alpha}(t)}\overline{\varphi}^{\frac{\alpha+1}{\alpha}}(t).\end{aligned}$$

Multiplying this inequality by $H_4(t,s)$ and integrating the resulting from t_4 to t, we obtain

$$\begin{aligned}&\int_{t_4}^{t}\frac{k\widetilde{\mu}^{\alpha}}{2^{\alpha}}H_4(t,s)\,\rho_4(s)\,q(s)\,(1-p(g(s)))^{\alpha}g^{2\alpha}(s)\,ds\\ &\qquad\leq H_4(t,t_4)\,\overline{\varphi}(t_4)+\int_{t_4}^{t}\left(\frac{\partial H_4(t,s)}{\partial s}+\frac{\rho_4'(s)}{\rho_4(s)}H_4(t,s)\right)\overline{\varphi}(s)\,ds\\ &\qquad\quad-\int_{t_4}^{t}\frac{\alpha}{r^{1/\alpha}(s)\,\rho_4^{1/\alpha}(s)}H_4(t,s)\,\overline{\varphi}^{\frac{\alpha+1}{\alpha}}(s)\,ds\end{aligned}$$

and so,

$$\int_{t_4}^{t}\frac{k\widetilde{\mu}^{\alpha}}{2^{\alpha}}H_4(t,s)\,\rho_4(s)\,q(s)\,(1-p(g(s)))^{\alpha}g^{2\alpha}(s) \tag{2.43}$$

$$\leq H_4(t,t_4)\,\overline{\varphi}(t_4)-\int_{t_4}^{t}\frac{h_4(t,s)}{\rho_4(s)}H_4^{\frac{\alpha}{\alpha+1}}(t,s)\,\overline{\varphi}(s)\,ds \tag{2.44}$$

$$-\int_{t_4}^{t}\frac{\alpha}{r^{1/\alpha}(s)\,\rho_4^{1/\alpha}(s)}H_4(t,s)\,\overline{\varphi}^{\frac{\alpha+1}{\alpha}}(s)\,ds. \tag{2.45}$$

By Lemma 1.1, if we set $U=\frac{h_4}{\rho_4}H_4^{\frac{\alpha}{\alpha+1}}$, $V=\frac{\alpha}{r^{1/\alpha}\rho_4^{1/\alpha}}H_4$ and $y=-\overline{\varphi}$, then we find

-h$\frac{4}{\rho_4 H_4^{\frac{\alpha}{\alpha+1}}\overline{\varphi}-\frac{\alpha}{r^{1/\alpha}\rho_4^{1/\alpha}}H_4\overline{\varphi}^{\frac{\alpha+1}{\alpha}}\leq\frac{1}{(\alpha+1)^{\alpha+1}}\frac{rh_4^{\alpha+1}}{\rho_4^{\alpha}}},$

which with (2.45) gives

$$\frac{1}{H_4(t,t_4)}\int_{t_4}^{t}\left(\Omega(t,s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)\left[h_4^{\alpha+1}(t,s)\right]_+}{\rho_4^{\alpha}(s)}\right)ds\le\overline{\varphi}(t_4),$$

(2.46)

which contradicts (2.31). □

Example 2.1. *Consider a delay differential equation*

$$\left[e^t\left(\left[x(t)+\frac{1}{2}x\left(\frac{t}{2}\right)\right]'''\right)\right]'+\delta e^t x\left(\frac{t}{2}\right)=0,\quad t\ge t_0, \tag{2.47}$$

where $\delta>0$ is a constant. We note that

$\eta_i(t)=e^{-t},\ i=0,1,2.$

If we now set $\rho_1(t)=\rho_2(t)=1$, then we get that $R_1=R_2=\infty$. Also, we see that $R_3=R_4>1$ if $\delta>1/4$. Thus, By Theorem 2.1, every solution of equation (2.47) is oscillatory for $\delta>1/4$.

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