Proyecciones (Antofagasta. On line) | vol. 39, n. 6 (2020) | pp. 1537-1554.

10.22199/issn.0717-6279-2020-06-0092

PROYECCIONES
Journal of Mathematics

ISSN 0717-6279 (On line)

On generalizations of graded second submodules

Mashhoor Refai[1] orcid.org/0000-0001-7436-5385
Rashid Abu-Dawwas[2] orcid.org/0000-0001-8998-7590

[1]President of Princess Sumaya University for Technology, Jordan.
m.refai@psut.edu.jo
[2]Yarmouk University, Dept. of Mathematics, Jordan.
rrashid@yu.edu.jo

Received: February 2019 | Accepted: April 2020

Abstract:

Let G be a group with identity e, R be a commutative G-graded ring with unity 1 and M be a G-graded R-module. In this article, we introduce and study two generalizations of graded second submodules, namely, graded 2-absorbing second submodules and graded strongly 2- absorbing second submodules. Also, we introduce and study the concept of graded quasi 2-absorbing second submodules, that is a generalization for graded strongly 2-absorbing second submodules.

Keywords: Graded second submodules; Graded 2-absorbing submodules; Graded strongly 2-absorbing submodules; Graded 2-absorbing second submodules; Graded strongly 2-absorbing second submodules; Graded quasi 2-absorbing second submodules.

MSC (2020): 13A02, 16W50.

Cite this article as (IEEE citation style):

M. Refai and R. Abu-Dawwas, "On generalizations of graded second submodules", *Proyecciones (Antofagasta, On line)*, vol. 39, no. 6, pp. 1537-1554, Dec. 2020, doi: 10.22199/issn.0717-6279-2020-06-0092.

Article copyright: © 2020 Mashhoor Refai and Rashid Abu-Dawwas. This is an open access article distributed under the terms of the Creative Commons License, which permits unrestricted use and distribution provided the original author and source are credited.

1. Introduction

Throughout this article, G will be a group with identity e and R will be a commutative ring with a nonzero unity 1. R is said to be G-graded if $R = \bigoplus_{g\in G} R_g$ with $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$ where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. Consider $supp(R, G) = \{g \in G : R_g = 0\}$. If $x \in R$, then x can be written as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also, $h(R) = \bigcup_{g\in G} R_g$. Moreover, it has been proved in [21] tha R_e is a subring of R and $1 \in R_e$.

Let I be an ideal of a graded ring R. Then I is said to be graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$ where $x_g \in I$ for all $g \in G$. Let R be a G-graded ring and I be a graded ideal of R. Then R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$.

Assume that M is a left R-module. Then M is said to be G-graded if $M = \bigoplus_{g\in G} M_g$ with $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$ where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. Also, we consider $supp(M, G) = \{g \in G : M_g = 0\}$. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover, $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g\in G} x_g$ where $x_g \in N$ for all $g \in G$. Let M be a G-graded R-module and N be a graded R-submodule of M. Then M/N is a graded R-module by $(M/N)_g = (M_g + N)/N$ for all $g \in G$.

Lemma 1.1. *([16]) Let R be a G-graded ring and M be a G-graded R-module.*

1. *If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals of R.*
2. *If N and K are graded R-submodules of M, then $N + K$ and $N \bigcap K$ are graded R-submodules of M.*

3. *If N is a graded R-submodule of M, $r \in h(R)$, $x \in h(M)$ and I is a graded ideal of R, then Rx, IN and rN are graded R-submodules of M. Moreover, $(N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.*

Also, it has been proved in [17] that if N is a graded R-submodule of M, then $Ann_R(N) = \{r \in R : rN = \{0\}\}$ is a graded ideal of R.

Graded prime submodules have been introduced by Atani in [12]. A proper graded R-submodule N of M is said to be graded prime if whenever $r \in h(R)$ and $m \in h(M)$ such that $rm \in N$, then either $m \in N$ or $r \in (N :_R M)$. Graded prime submodules have been widely studied by several authors, for more details one can look in [1], [2], [4] and [8].

Let M and S be two G-graded R-modules. An R-homomorphism $f : M \to S$ is said to be graded R-homomorphism if $f(M_g) \subseteq S_g$ for all $g \in G$ (see [21]). Graded second submodules have been introduced by Ansari-Toroghy and Farshadifar in [9]. A nonzero graded R-submodule N of M is said to be graded second if for each $a \in h(R)$, the graded R-homomorphism $f : N \to N$ defined by $f(x) = ax$ is either surjective or zero. In this case, $Ann_R(N)$ is a graded prime ideal of R. Graded second submodules have been wonderfully studied by Çeken and Alkan in [14]. On the other hand, graded secondary modules have been introduced by Atani and Farzalipour in [13]. A nonzero graded R-module M is said to be graded secondary if for each $a \in h(R)$, the graded R-homomorphism $f : M \to M$ defined by $f(x) = ax$ is either surjective or nilpotent.

In [20], Naghani and Moghimi gave a generalization of graded prime ideals, called graded 2-absorbing ideals. A proper graded ideal P of R is said to be graded 2-absorbing if whenever $a, b, c \in h(R)$ such that $abc \in P$, then either $ab \in P$ or $ac \in P$ or $bc \in P$. Graded 2-absorbing ideals have been admirably studied in [6].

The authors in [5] extended graded 2-absorbing ideals to graded 2-absorbing submodules. A proper graded R-submodule N of M is said to be graded 2-absorbing if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in N$, then either $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Graded 2-absorbing submodules have been deeply studied in [7].

In [15], a proper $\mathbf{Z}$-graded R-submodule N of M is said to be graded completely irreducible if whenever $N = \bigcap_{k\in\Delta} N_k$ where $\{N_k\}_{k\in\Delta}$ is a family of $\mathbf{Z}$-graded R-submodules of M, then $N = N_k$ for some $k \in \Delta$. In

[19], the concept of graded completely irreducible submodules has been extended into G-graded case, for any group G. It has been proved that every graded R-submodule of M is an intersection of graded completely irreducible R-submodules of M. In many instances, we use the following basic fact without further discussion.

Remark 1.2. *Let N and L be two graded R-submodules of M. To prove that $N \subseteq L$, it is enough to prove that: If K is a graded completely irreducible R-submodule of M such that $L \subseteq K$, then $N \subseteq K$.*

The purpose of our article is to follow [11] in order to introduce and study the concept of graded 2-absorbing second submodules, that is a generalization of graded second submodules. A nonzero graded R-submodule N of M is said to be graded 2-absorbing second if whenever $x, y \in h(R)$ and K is a graded completely irreducible R-submodule of M such that $xyN \subseteq K$, then either $xN \subseteq K$ or $yN \subseteq K$ or $xy \in Ann_R(N)$. Also, we follow [11] to introduce another generalization, namely, graded strongly 2-absorbing second submodules. A nonzero graded R-submodule N of M is said to be graded strongly 2-absorbing second if whenever $x, y \in h(R)$ and K is a graded R-submodule of M such that $xyN \subseteq K$, then either $xN \subseteq K$ or $yN \subseteq K$ or $xy \in Ann_R(N)$.

In Corollary 3.4, we prove that if N is a graded strongly 2-absorbing second R-submodule of M, then $Ann_R(N)$ is a graded 2-absorbing ideal of R, and in Example 3.5, we show that the converse is not true in general. Motivated by this, we introduce and study a generalization for graded strongly 2-absorbing second submodules. A nonzero graded R-submodule N of M is said to be graded quasi 2-absorbing second if $Ann(N)$ is a graded 2-absorbing ideal of R. Related results have been obtained.

2. Graded 2-Absorbing Second Submodules

In this section, we introduce and study the concept of graded 2-absorbing second submodules.

Definition 2.1. *Let M be a graded R-module and N be a nonzero graded R-submodule of M. Then N is said to be a graded 2-absorbing second R-submodule of M if whenever $x, y \in h(R)$ and K is a graded completely irreducible R-submodule of M such that $xyN \subseteq K$, then either $xN \subseteq K$ or $yN \subseteq K$ or $xy \in Ann_R(N)$.*

Example 2.2. *Let $R = \mathbf{Z}$, $M = \mathbf{Z}_n[i]$ and $G = \mathbf{Z}_4$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = R_2 = R_3 = \{0\}$. Also, M is G-graded by $M_0 = \mathbf{Z}_n$, $M_2 = i\mathbf{Z}_n$ and $M_1 = M_3 = \{0\}$. Consider the graded R-submodule $N = \mathbf{Z}_n$ of M. If $n = p$ or $n = pq$ where p, q are primes, then N is a graded 2-absorbing second R-submodule of M.*

Example 2.3. *Let $R = \mathbf{Z}$, $M = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbf{Z}$ and $M_1 = i\mathbf{Z}$. Consider the graded R-submodule $N = n\mathbf{Z}$ of M. Obviously, $n = p_1^{r_1} p_2^{r_2} \ldots\ldots p_k^{r_k}$ where $p_i^{r_i}$ $(1 \leq i \leq k)$ are distinct primes. Now, $p_1 \in h(R)$ and $K = p_1^{r_1+2}\mathbf{Z}$ is a graded completely irreducible R-submodule of M such that $p_1 p_1 N \subseteq K$. But $p_1 N \subseteq K$ and $p_1 p_1 \notin Ann_R(N) = \{0\}$. Hence, N is not graded 2-absorbing second R-submodule of M.*

Remark 2.4. *Consider the $\mathbf{Z}$-module $\mathbf{Z}$ and assume it is G-graded by any group G. Since the only graded submodules are $n\mathbf{Z}$, then by Example 2.3, $\mathbf{Z}$ has no graded 2-absorbing second submodules.*

Let $\Omega(M)$ be the set of all graded completely irreducible R-submodules of M. Assume that P is a graded prime ideal of R and N is a graded R-submodule of M. Then we define

$$I_P^M(N) = \bigcap_{K \in \Omega(M)} \{K : rN \subseteq K \text{ for some } r \in h(R) - P\}$$

. The following lemma gives some characterizations for graded second R-submodules.

Lemma 2.5. *Let N be a graded R-submodule of a graded R-module M. Then the following are equivalent.*

1. *If $N = \{0\}$, K is a graded completely irreducible R-submodule of M and $r \in h(R)$ such that $rN \subseteq K$, then either $rN = \{0\}$ or $N \subseteq K$.*
2. *N is a graded second R-submodule of M.*
3. *$P = Ann_R(N)$ is a graded prime ideal of R and $I_P^M(N) = N$.*

Proof. (1) $\Rightarrow$ (2): Suppose that $r \in h(R)$ and $rN = \{0\}$. If $rN \subseteq K$ for some graded completely irreducible R-submodule K of M, then by assumption, $N \subseteq K$. Hence, $N \subseteq rN$. (2) $\Rightarrow$ (3): By [9], $P = Ann_R(N)$ is a graded prime ideal of R. Now,let K be a graded completely irreducible

R-submodule of M and $r \in h(R) - P$ such that $rN \subseteq K$. Then $N \subseteq K$ by assumption. Therefore, $N \subseteq I_P^M(N)$. The reverse inclusion is clear. (3) $\Rightarrow$ (1): Since $Ann_R(N)$ is a graded prime ideal of R, $N = \{0\}$. Let K be a graded completely irreducible R-submodule of M and $r \in h(R)$ such that $rN \subseteq K$. Suppose that $rN = \{0\}$. Then $r \in h(R) - P$. Therefore, $I_P^M(N) \subseteq K$. But $I_P^M(N) = N$ by assumption. Hence, $N \subseteq K$, as desired. $\square$

Proposition 2.6. *Let M be a graded R-module. If either L is a graded second R-submodule of M or L is a sum of two graded second R-submodules of M, then L is a graded 2-absorbing second R-submodule of M.*

Proof. The first assertion is clear. Let N and L be two graded second R-submodules of M. We show that $N+L$ is a graded 2-absorbing second R-submodules of M. Let $x, y \in h(R)$ and K is a graded completely irreducible R-submodule of M such that $xy(N + L) \subseteq K$. Since N is graded second, either $xyN = \{0\}$ or $N \subseteq K$ by Lemma 2.5. Similarly, either $xyL = \{0\}$ or $L \subseteq K$. If $xyN = xyL = \{0\}$, then we are done. Also, if $N \subseteq K$ and $L \subseteq K$, then we are done. Assume that $xyN = \{0\}$ and $L \subseteq K$. Then $xN = \{0\}$ or $yN = \{0\}$ because $Ann_R(N)$ is a graded prime ideal of R. If $xN = \{0\}$, then $x(N + L) \subseteq xN + L \subseteq L \subseteq K$. Similarly, if $yN = \{0\}$, we have $y(N + L) \subseteq K$ as desired. $\square$

Proposition 2.7. *Let M be a graded R-module. If L is a graded secondary R-submodule of M and $R/Ann_R(L)$ has no nonzero nilpotent homogeneous element, then L is a graded 2-absorbing second R-submodule of M.*

Proof. Let $x, y \in h(R)$ and K be a graded completely irreducible R-submodule of M such that $xyL \subseteq K$. If $xL \subseteq K$ or $yL \subseteq K$, then we are done. Suppose that $xL \subseteq K$ and $yL \subseteq K$. Then $x, y \in R/Ann_R(L)$. Thus, $(xy)^r \in Ann_R(L)$ for some positive integer r. Therefore, $xy \in Ann_R(L)$ since $R/Ann_R(L)$ has no nonzero nilpotent homogeneous element. Hence, L is a graded 2-absorbing second R-submodule of M. $\square$

Proposition 2.8. *Let M be a G-graded R-module, I be a graded ideal of R and L be a graded 2-absorbing second R-submodule of M. If $x \in h(R)$ and K is a graded completely irreducible R-submodule of M such that $xIL \subseteq K$, then either $xL \subseteq K$ or $xI \subseteq Ann_R(L)$ or $I_gL \subseteq K$ for some $g \in G$.*

Proof. Suppose that $xL \subseteq K$ and $xI \subseteq Ann_R(L)$. Then there exists $y \in I$ such that $xyL = \{0\}$, and then there exists $g \in G$ such that $xy_gL = \{0\}$ where $y_g \in I$ since I is graded. Now, since L is graded 2-absorbing second and $xy_gL \subseteq K$, we have $y_gL \subseteq K$. We show that $I_gL \subseteq K$. Let $z_g \in I_g$. Then $(y_g + z_g)xL \subseteq K$. Hence, either $(y_g + z_g)L \subseteq L$ or $(y_g + z_g)x \in Ann_R(L)$. If $(y_g + z_g)L \subseteq K$, then since $y_gL \subseteq K$, we have $z_gL \subseteq K$. If $(y_g + z_g)x \in Ann_R(L)$, then $z_gx \notin Ann_R(L)$, but $z_gxL \subseteq K$. Thus, $z_gL \subseteq K$. Hence, we conclude that $I_gL \subseteq K$. □

Lemma 2.9. *Let M be a G-graded R-module and N a graded R-submodule of M. If $r \in h(R)$, then $(N :_M r) = \{m \in M : rm \in N\}$ is a graded R-submodule of M.*

Proof. Clearly, $(N :_M r)$ is a graded R-submodule of M. Let $m \in (N :_M r)$. Then $rm \in N$. Now, $m = \sum_{g\in G} m_g$ where $m_g \in M_g$ for all $g \in G$. Since $r \in h(R)$, $r \in R_h$ for some $h \in G$ and then $rm_g \in M_{hg} \subseteq h(M)$ for all $g \in G$ such that $\sum_{g\in G} rm_g = r\left(\sum_{g\in G} m_g\right) = rm \in N$. Since N is graded, $rm_g \in N$ for all $g \in G$ which implies that $m_g \in (N :_M r)$ for all $g \in G$. Hence, $(N :_M r)$ is a graded R-submodule of M. □

In [23], a graded R-module M is said to be graded cocyclic if the sum of all graded minimal R-submodules of M is a large and graded simple R-submodule of M.

Lemma 2.10. *A graded R-submodule K of M is a graded completely irreducible R-submodule of M if and only if M/K is a graded cocyclic R-module.*

Proof. It follows from ([18], Remark 1.1). □

Lemma 2.11. *Let K be a graded completely irreducible R-submodule of M. Then $(K :_M r)$ is a graded completely irreducible R-submodule of M for all $r \in h(R)$.*

Proof. This follows from Lemma 2.9, Lemma 2.10 and that $M/(K :_M r) \cong (rM + K)/K$. □

Proposition 2.12. *Let L be a graded 2-absorbing second R-submodule of M and K is a graded completely irreducible R-submodule of M such that $L \subseteq K$, Then $(K :_R L)$ is a graded 2-absorbing ideal of R.*

Proof. Since $L \subseteq K$, we have $(K :_R L) = R$. Let $x, y, z \in h(R)$ such that $xyz \in (K :_R L)$. Then $xyL \in (K :_M z)$. Thus $xL \subseteq (K :_M z)$ or $yL \subseteq (K :_M z)$ or $xyL = \{0\}$ since L is graded 2-absorbing second and $(K :_M z)$ is a graded completely irreducible R-submodule of M by Lemma 2.11. Therefore, $xz \in (K :_R L)$ or $yz \in (K :_R L)$ or $xy \in (K :_R L)$. Hence, $(K :_R L)$ is a graded 2-absorbing ideal of R. □

Corollary 2.13. *If M is a graded cocyclic R-module and L is a graded 2-absorbing second R-submodule of M, then $Ann_R(L)$ is a graded 2-absorbing ideal of R.*

Proof. Since M is graded cocyclic, $\{0\}$ is a graded completely irreducible R-submodule of M by Lemma 2.10. Thus the result follows from Proposition 2.12. □

Proposition 2.14. *Let L be a graded 2-absorbing second R-submodule of M. Then $x^n L = x^{n+1} L$ for all $x \in h(R)$ and $n \geq 2$.*

Proof. It is enough to prove that $x^2 L = x^3 L$. Let $x \in h(R)$. Then clearly, $x^3 L \subseteq x^2 L$. Let K be a graded completely irreducible R-submodule of M such that $x^3 L \subseteq K$. Then $x^2 L \subseteq (K :_M x)$. Thus $xL \subseteq (K :_M x)$ or $x^2 L = \{0\}$ since L is graded 2-absorbing second submodule of M and $(K :_M x)$ is a graded completely irreducible R-submodule of M by Lemma 2.11. Therefore, $x^2 L \subseteq K$. Hence, $x^2 L = x^3 L$. □

Proposition 2.15. *Let L be a graded 2-absorbing second R-submodule of M. If $Ann_R(L)$ is a graded prime ideal of R, then $(K :_R L)$ is a graded prime ideal of R for all graded completely irreducible R-submodule K of M with $L \subseteq K$.*

Proof. Let K be a graded completely irreducible R-submodule of M such that $L \subseteq K$. Assume that $x, y \in h(R)$ such that $xy \in (K :_R L)$. Then $xyL \subseteq K$, and then $xL \subseteq K$ or $yL \subseteq K$ or $xyL = \{0\}$. If $xyL = \{0\}$, then $xy \in Ann_R(L)$, and then $xL = \{0\}$ or $yL = \{0\}$. So, in all cases, we have $xL \subseteq K$ or $yL \subseteq K$, which implies that $x \in (K :_R L)$ or $y \in (K :_R L)$. Hence, $(K :_R L)$ is a graded prime ideal of R. □

Proposition 2.16. *Let L be a graded 2-absorbing second R-submodule of M. If $Grad(Ann_R(L)) = P$ for some graded prime ideal P of R and K is a graded completely irreducible R-submodule of M such that $L \subseteq K$, then $Grad((K :_R L))$ is a graded prime ideal of R containing P.*

Proof. Let $x, y \in h(R)$ such that $xy \in Grad((K :_R L))$. Then $x^r y^r L \subseteq K$ for some positive integer r, and then $x^r L \subseteq K$ or $y^r L \subseteq K$ or $x^r y^r L = \{0\}$. If $x^r L \subseteq K$ or $y^r L \subseteq K$, then we are done. Suppose that $x^r y^r L = \{0\}$. Then $xy \in Grad(Ann_R(L)) = P$. Thus $x \in P$ or $y \in P$. Clearly, $P = Grad(Ann_R(L)) \subseteq Grad((K :_R L))$. Therefore, $x \in Grad((K :_R L))$ or $y \in Grad((K :_R L))$. $\square$

3. Graded Strongly 2-Absorbing Second Submodules

In this section, we introduce and study the concept of graded strongly 2-absorbing second submodules.

Definition 3.1. *Let M be a graded R-module. Then a nonzero graded R-submodule N of M is said to be graded strongly 2-absorbing second if whenever $x, y \in h(R)$ and K is a graded R-submodule of M such that $xyN \subseteq K$, then either $xN \subseteq K$ or $yN \subseteq K$ or $xy \in Ann_R(N)$.*

Clearly, every graded strongly 2-absorbing second submodule is a graded 2-absorbing second submodule. This motivates the following question.

Question 3.2. *Let M be a graded R-module. Is every graded 2-absorbing second R-submodule of M a graded strongly 2-absorbing second R-submodule of M?*

Proposition 3.3. *Let N be a graded R-submodule of M. Then N is a graded strongly 2-absorbing second R-submodule of M if and only if for every $x, y \in h(R)$, we have $xyN = xN$ or $xyN = yN$ or $xyN = \{0\}$.*

Proof. Suppose that N is a graded strongly 2-absorbing second R-submodule of M. Then $N = \{0\}$. Let $x, y \in h(R)$. Then $xyN \subseteq xyN$, which implies that $xN \subseteq xyN$ or $yN \subseteq xyN$ or $xyN = \{0\}$. Thus $xyN = xN$ or $xyN = yN$ or $xyN = \{0\}$. The converse is clear. $\square$

Corollary 3.4. *If N is a graded strongly 2-absorbing second R-submodule of M, then $Ann_R(N)$ is a graded 2-absorbing ideal of R.*

Proof. Let $x, y, z \in h(R)$ such that $xyz \in Ann_R(N)$. Then by Proposition 3.3, we have $xyN = xN$ or $xyN = yN$ or $xyN = \{0\}$. If $xyN = \{0\}$, then we are done. Suppose that $xyN = xN$. Then $zxN \subseteq zxyN = \{0\}$. Similarly, if $xyN = yN$. $\square$

The following example shows that the converse of Corollary 3.4 is not true in general.

Example 3.5. *In Example 2.3, $N = \langle p \rangle$ (where p is a prime number) is a graded R-submodule of $M = \mathbf{Z}[i]$ such that $Ann_R(N) = \{0\}$ is a graded 2-absorbing ideal of R, but N is not a graded strongly 2-absorbing second R-submodule of $\mathbf{Z}[i]$.*

Corollary 3.6. *Let N be a graded strongly 2-absorbing second R-submodule of M. If L is a graded R-submodule of M such that $N \subseteq L$, then $(L :_R N)$ is a graded 2-absorbing ideal of R.*

Proof. Let $x, y, z \in h(R)$ such that $xyz \in (L :_R N)$. Then $xyzN \subseteq L$, and then $xzN \subseteq L$ or $yzN \subseteq L$ or $xyzN = \{0\}$. If $xzN \subseteq L$ or $yzN \subseteq L$, then we are done. If $xyzN = \{0\}$, then the result follows by Corollary 3.4. □

A graded R-module M is said to be graded comultiplication if for every graded R-submodule N of M there exists a graded ideal J of R such that $N = (0 :_M J)$, equivalently, for every graded R-submodule N of M, we have $N = (0 :_M Ann_R(N))$. The concept of graded comultiplication modules was introduced by H. Ansari-Toroghy and F. Farshadifar in [10]. Some generalizations on graded comultiplication modules have been introduced in [3]. The next proposition shows that the converse of Corollary 3.4 is true if M is a graded comultiplication R-module.

Proposition 3.7. *Let M be a graded comultiplication R-module. If L is a graded R-submodule of M such that $Ann_R(L)$ is a graded 2-absorbing ideal of R, then L is a graded strongly 2-absorbing second R-submodule of M. In particular, L is a graded 2-absorbing second R-submodule of M.*

Proof. Let $x, y \in h(R)$ and K be a graded R-submodule of M such that $xyL \subseteq K$. Then $Ann_R(K)xyL = \{0\}$. So, $Ann_R(K)xL = \{0\}$ or $Ann_R(K)yL = \{0\}$ or $xyL = \{0\}$. If $xyL = \{0\}$, then we are done. If $Ann_R(K)xL = \{0\}$ or $Ann_R(K)yL = \{0\}$, then $Ann_R(K) \subseteq Ann_R(xL)$ or $Ann_R(K) \subseteq Ann_R(yL)$. Hence, $xL \subseteq K$ or $yL \subseteq K$ since M is graded comultiplication. □

Remark 3.8. *Example 3.5 Shows that the condition that M is a graded comultiplication R-module in Proposition 3.7 is necessary since M is not a graded comultiplication R-module.*

Corollary 3.9. *Let M be a graded comultiplication R-module. If M is graded cocyclic, then every graded 2-absorbing second R-submodule of M is graded strongly 2-absorbing second.*

Proof. Let L be a graded 2-absorbing second R-submodule of M. Then by Corollary 2.13, $Ann_R(L)$ is a graded 2-absorbing ideal of R, and then the result follows by Proposition 3.7. □

Proposition 3.10. *Let N be a graded strongly 2-absorbing second R-submodule of M. If $(K \bigcap L :_R N)$ is a graded prime ideal of R for all graded completely irreducible R-submodules K and L of M with $N \subseteq K \bigcap L$, then $Ann_R(N)$ is a graded prime ideal of R.*

Proof. Suppose that $Ann_R(N)$ is not a graded prime ideal of R. Then there exist $x, y \in h(R)$ such that $xyN = \{0\}$, $xN = \{0\}$ and $yN = \{0\}$. So, there exist graded completely irreducible R-submodules K and L of M such that $xN \subseteq K$ and $yN \subseteq L$. Now, $xyN \subseteq K \bigcap L$, which implies that $xy \in (K \bigcap L :_R N)$, and then $xN \subseteq K \bigcap L$ or $yN \subseteq K \bigcap L$. In both cases, we have a contradiction. □

Lemma 3.11. *Suppose that $f : M \rightarrow S$ is a graded R-homomorphism of graded R-modules.*

1. *If f is a graded R-monomorphism and K is a graded R-submodule of $f(M)$, then $f^{-1}(K)$ is a graded R-submodule of M.*
2. *If L is a graded R-submodule of M with $Ker(f) \subseteq L$, then $f(L)$ is a graded R-submodule of $f(M)$.*

Proof.

1. Clearly, $f^{-1}(K)$ is an R-submodule of M. Let $x \in f^{-1}(K)$. Then $x \in M$ with $f(x) \in K$, and then $x = \sum\limits_{g \in G} x_g$ where $x_g \in M_g$ for all $g \in G$. So, for every $g \in G$, $f(x_g) \in f(M_g) \subseteq S_g$ such that $\sum\limits_{g \in G} f(x_g) = f\left(\sum\limits_{g \in G} x_g\right) = f(x) \in K$. Since K is graded, $f(x_g) \in K$ for all $g \in G$, i.e., $x_g \in f^{-1}(K)$ for all $g \in G$. Hence, $f^{-1}(K)$ is a graded R-submodule of M.
2. Clearly, $f(L)$ is an R-submodule of $f(M)$. Let $y \in f(L)$. Then $y \in f(M)$, and so there exists $x \in M$ such that $y = f(x)$, so $f(x) \in L$, which implies that $x \in L$ since $Ker(f) \subseteq L$, and hence $x_g \in L$ for all $g \in G$ since L is graded. Thus, $y_g = (f(x))_g = f(x_g) \in f(L)$ for all $g \in G$. Therefore, $f(L)$ is a graded R-submodule of S. □

Lemma 3.12. *Let $f : M \to S$ be a graded monomorphism of graded R-modules. If N is a graded completely irreducible R-submodule of M, then $f(N)$ is a graded completely irreducible R-submodule of $f(M)$.*

Proof. Let $\{L_i\}_{i\in\Delta}$ be a family of a graded R-submodules of $f(M)$ such that $f(N) = \bigcap\limits_{i\in\Delta} L_i$. Then $N = f^{-1}(f(N)) = f^{-1}\left(\bigcap\limits_{i\in\Delta} L_i\right) = \bigcap\limits_{i\in\Delta} f^{-1}(L_i)$. So, there exists $i \in \Delta$ such that $N = f^{-1}(L_i)$ since N is a graded completely irreducible R-submodule of M. Therefore, $f(N) = f(f^{-1}(L_i)) = f(M)\bigcap L_i = L_i$, as needed. $\square$

Similarly, one can prove the next lemma.

Lemma 3.13. *Let $f : M \to S$ be a graded monomorphism of graded R-modules. If L is a graded completely irreducible R-submodule of $f(M)$, then $f^{-1}(L)$ is a graded completely irreducible R-submodule of M.*

Proposition 3.14. *Let $f : M \to S$ be a graded monomorphism of graded R-modules. If N is a graded strongly 2-absorbing second R-submodule of M with $Ker(f) \subseteq N$, then $f(N)$ is a graded 2-absorbing second R-submodule of $f(M)$.*

Proof. Since $N = \{0\}$ and f is injective, we have $f(N) = \{0\}$. Let $x, y \in h(R)$ and K be a graded completely irreducible R-submodule of S such that $xyf(N) \subseteq K$. Then $xyN \subseteq f^{-1}(K)$, and then $xN \subseteq f^{-1}(K)$ or $yN \subseteq f^{-1}(K)$ or $xyN = \{0\}$. Therefore, $xf(N) \subseteq f(f^{-1}(K)) = f(M)\bigcap K \subseteq K$ or $yf(N) \subseteq K$ or $xyf(N) = \{0\}$, as needed. $\square$

Similarly, one can prove the next proposition.

Proposition 3.15. *Let $f : M \to S$ be a graded monomorphism of graded R-modules. If N is a graded 2-absorbing second R-submodule of M with $Ker(f) \subseteq N$, then $f(N)$ is a graded 2-absorbing second R-submodule of $f(M)$.*

Proposition 3.16. *Let $f : M \to S$ be a graded monomorphism of graded R-modules. If L is a graded strongly 2-absorbing second R-submodule of $f(M)$, then $f^{-1}(L)$ is a graded 2-absorbing second R-submodule of M.*

Proof. If $f^{-1}(L) = \{0\}$, then $f(M) \bigcap L = f(f^{-1}(L)) = f(\{0\}) = \{0\}$. Thus $L = \{0\}$ which is a contradiction. Therefore, $f^{-1}(L) = \{0\}$. Let $x, y \in h(R)$ and K be a graded completely irreducible R-submodule of M such that $xyf^{-1}(L) \subseteq K$. Then $xyL \subseteq xy\,(f(M) \bigcap L) = xyf(f^{-1}(L)) \subseteq f(K)$. So, $xL \subseteq f(K)$ or $yL \subseteq f(K)$ or $xyL = \{0\}$. Hence, $xf^{-1}(L) \subseteq K$ or $yf^{-1}(L) \subseteq K$ or $xyf^{-1}(L) = \{0\}$, as needed. $\square$

Similarly, one can prove the next proposition.

Proposition 3.17. *Let $f : M \to S$ be a graded monomorphism of graded R-modules. If L is a graded 2-absorbing second R-submodule of $f(M)$, then $f^{-1}(L)$ is a graded 2-absorbing second R-submodule of M.*

4. Graded Quasi 2-Absorbing Second Submodules

In this section, we introduce and study the concept of graded quasi 2-absorbing second submodules.

Definition 4.1. *Let M be a graded ring and N be a nonzero graded R-submodule of M. Then N is said to be a graded quasi 2-absorbing second R-submodule of M if $Ann_R(N)$ is a graded 2-absorbing ideal of R.*

Remark 4.2. *By Corollary 3.4, every graded strongly 2-absorbing second R-submodule is a graded quasi 2-absorbing second R-submodule. But the converse is not true in general by Example 3.5.*

Proposition 4.3. *Let M be a graded comultiplication R-module. Then a graded R-submodule N of M is a graded strongly 2-absorbing second R-submodule of M if and only if it is a graded quasi 2-absorbing second R-submodule of M.*

Proof. It follows from Corollary 3.4 and Proposition 3.7. $\square$

Proposition 4.4. *Let M be a graded R-module and N be a graded quasi 2-absorbing second R-submodule of M. If J is a graded ideal of R such that $J \subseteq Ann_R(N)$, then JN is a graded quasi 2-absorbing second R-submodule of M.*

Proof. Since $J \subseteq Ann_R(N)$, we have $Ann_R(JN)$ is a proper graded ideal of R. Let $x, y, z \in h(R)$ such that $xyz \in Ann_R(JN)$. Then $xyzJN = \{0\}$, and then $xzN = \{0\}$ or $zyJN = \{0\}$ or $xyJN = \{0\}$. If $zyJN = \{0\}$ or $xyJN = \{0\}$, then we are done. Suppose that $xzN = \{0\}$. Then $xz \in Ann_R(N) \subseteq Ann_R(JN)$, so $xzJN = \{0\}$, as required. $\square$

Proposition 4.5. *Let M be a graded R-module and N be a graded quasi 2-absorbing second R-submodule of M. Then $Ann_R(J^nN) = Ann_R(J^{n+1}N)$ for all graded ideal J of R and for all $n \geq 2$.*

Proof. Let J be a graded ideal of R. It is enough to prove that $Ann_R(J^2N) = Ann_R(J^3N)$. Clearly, $Ann_R(J^2N) \subseteq Ann_R(J^3N)$. Since N is graded quasi 2-absorbing second, $Ann_R(J^3N)J^3N = \{0\}$ implies that $Ann_R(J^3N)J^2N = \{0\}$ or $J^2N = \{0\}$. If $Ann_R(J^3N)J^2N = \{0\}$, then $Ann_R(J^3N) \subseteq Ann_R(J^2N)$. If $J^2N = \{0\}$, then $Ann_R(J^2N) = R = Ann_R(J^3N)$. $\square$

For a graded R-submodule N of M, the graded second radical of N is defined as the sum of all graded second R-submodules of M contained in N, and its denoted by $GSec(N)$. If N does not contain any graded second R-submodule, then $GSec(N) = \{0\}$. The set of all graded second R-submodules of M is called the graded second spectrum of M, and is denoted by $GSpec^s(M)$. On the other hand, the set of all graded prime R-submodules of M is called the graded spectrum of M, and is denoted by $GSpec(M)$. The map $\phi : GSpec^s(M) \to GSpec(R/Ann_R(M))$ defined by $\phi(N) = Ann_R(N)/Ann_R(M)$ is called the natural map of $GSpec^s(M)$, see [9] and [14].

For a graded ideal I of R, the graded radical of I is defined to be the set of all $r \in R$ such that for each $g \in G$, there exists a positive integer n_g satisfies $r_g^{n_g} \in I$, and it is denoted by $Grad(I)$. One can see that if $r \in h(R)$, then $r \in Grad(I)$ if and only if $r^n \in I$ for some positive integer n, see [22].

Lemma 4.6. *Let N and K be two graded R-submodules of M. Then*

1. *$GSec(N) \subseteq N$.*
2. *If $N \subseteq K$, then $GSec(N) \subseteq GSec(K)$.*
3. *$GSec((0 :_M I)) = GSec((0 :_M Gr(I))$ for all graded ideal I of R.*
4. *$GSec(N) \subseteq (0 :_M Gr(Ann_R(N)))$.*

Proof. (1) and (2) are straightforward. (3) Let I be a graded ideal of R. If $GSec((0 :_M I)) = \{0\}$, then we are done. Suppose that there exists a graded second R-submodule L of M such that $L \subseteq (0 :_M I)$. Then $I \subseteq Ann_R(L)$. Since $Ann_R(L)$ is a graded prime ideal of R, we have $Gr(I) \subseteq Ann_R(L)$. Hence, $L \subseteq (0 :_M Ann_R(L)) \subseteq (0 :_M Gr(I))$. Consequently, $GSec((0 :_M I) \subseteq GSec((0 :_M Gr(I))$. The reverse inclusion follows by part (2). (4) Since $N \subseteq (0 :_M Ann_R(N))$, the result follows by parts (1), (2) and (3). □

Lemma 4.7. *Let M be a graded R-module and N be a graded R-submodule of M. If the natural map ϕ of $GSpec^s(N)$ is surjective, then $Ann_R(GSec(N)) = Gr(Ann_R(N))$.*

Proof. If $N = \{0\}$, then we are done. Suppose that $N = \{0\}$. Then by Lemma 4.6 part (4), we have $Gr(Ann_R(N)) \subseteq Ann_R(GSec(N))$. Assume that $Gr(Ann_R(N)) = \bigcap_i P_i$ where P_i is a graded prime ideal of R containing $Ann_R(N)$. Since ϕ is surjective, for every P_i, there exists $L_i \in GSpec^s(N)$ such that $Ann_R(L_i) = P_i$. Hence, $\sum_i L_i \subseteq GSec(N)$. Therefore, $Ann_R(GSec(N)) \subseteq Ann_R\left(\sum_i L_i\right) = \bigcap_i P_i = Gr(Ann_R(N))$, as needed. □

Proposition 4.8. *Let N be a graded quasi 2-absorbing second R-submodule of M. If the natural map ϕ of $GSpec^s(N)$ is surjective, then $GSec(N)$ is a graded quasi 2-absorbing second R-submodule of M.*

Proof. By Lemma 4.7, $Ann_R(GSec(N)) = Gr(Ann_R(N))$, and then the result follows from the fact that $Gr(Ann_R(N))$ is a graded 2-absorbing ideal of R by ([6], Lemma 2.5). □

Proposition 4.9. *Let M be a graded comultiplication R-module, $N \subseteq L$ be two graded R-submodules of M and L be a graded quasi 2-absorbing second R-submodule of M. Then L/N is a graded quasi 2-absorbing second R-submodule of M/N.*

Proof. Let $x, y, z \in h(R)$ such that $xyz(L/N) = \{0\}$. Then $xyzL \subseteq N$, and then $Ann_R(N)xyzL = \{0\}$. Thus, $Ann_R(N)xyL = \{0\}$ or $Ann_R(N)xzL = \{0\}$ or $yzL = \{0\}$. If $yzL = \{0\}$, then $yz(L/N) = \{0\}$, and then we are done. If $Ann_R(N)xyL = \{0\}$ or $Ann_R(N)xzL = \{0\}$, then $xyL \subseteq (0 :_M Ann_R(N))$ or $xzL \subseteq (0 :_M Ann_R(N))$. Since M is graded comultiplication, we have $N = (0 :_M Ann_R(N))$, and the result follows obviously. □

The next example shows that the condition M is a graded comultiplication R-module is necessary in Proposition 4.9.

Example 4.10. *Let $R = \mathbf{Z}$, $M = \mathbf{Z}[i]$ and $G = \mathbf{Z}_2$. Then R is G-graded by $R_0 = \mathbf{Z}$ and $R_1 = \{0\}$. Also, M is G-graded by $M_0 = \mathbf{Z}$ and $M_1 = i\mathbf{Z}$. Clearly, $\mathbf{Z}$ is a graded quasi 2-absorbing second R-submodule of M. On the other hand, $12\mathbf{Z}$ is a graded R-submodule of M such that $12\mathbf{Z} \subseteq \mathbf{Z}$ and $\mathbf{Z}/12\mathbf{Z}$ is not graded quasi 2-absorbing second. Note that, M is not graded comultiplication R-module.*

Proposition 4.11. *Let $f : M \to S$ be a graded monomorphism of graded R-modules. Then N is a graded quasi 2-absorbing second R-submodule of M if and only if $f(N)$ is a graded quasi 2-absorbing second R-submodule of S.*

Proof. It follows from the fact that $Ann_R(N) = Ann_R(f(N))$. □

References

[1] R. Abu-Dawwas and K. Al-Zoubi, "On graded weakly classical prime submodules", *Iranian journal of mathematical sciences and informatics*, vol. 12, no. 1, pp. 153-161, 2017, doi: 10.7508/ijmsi.2017.01.012

[2] R. Abu-Dawwas, K. Al-Zoubi and M. Bataineh, "Prime submodules of graded modules", *Proyecciones (Antofagasta. On line)*, vol. 31, no. 4, pp. 355-361, Dec. 2012, doi: 10.4067/S0716-09172012000400004

[3] R. Abu-Dawwas, M. Bataineh and A. Da'keek, "Graded weak comultiplication modules", *Hokkaido mathematical journal*, vol. 48, no. 2, pp. 253-261, 2019, doi: 10.14492/HOKMJ/1562810507

[4] K. Al-Zoubi and R. Abu-Dawwas, "On graded quasi-prime submodules", *Kyungpook mathematical journal*, vol. 55, no. 2 pp. 259-266, 2015, doi: 10.5666/KMJ.2015.55.2.259

[5] K. Al-Zoubi and R. Abu-Dawwas, "On graded 2-absorbing and weakly graded 2-absorbing submodules", *Journal of mathematical sciences: advances and applications*, vol. 28, pp. 45-60, 2014. [On line]. Available: https://bit.ly/34KEQ1x

[6] K. Al-Zoubi, R. Abu-Dawwas, and S. Ceken, "On graded 2-absorbing and graded weakly 2-absorbing ideals", *Hacettepe journal of mathematics and statistics*, vol. 48, no. 3, pp. 724-731, 2019, doi: 10.15672/hjms.2018.543

[7] K. Al-Zoubi and M. Al-Azaizeh, "Some properties of graded 2-absorbing and graded weakly 2-absorbing submodules", *Journal of nonlinear sciences and applications*, vol. 12, no. 8, pp. 503-508, Mar. 2019, doi: 10.22436/jnsa.012.08.01

[8] K. Al-Zoubi, M. Jaradat and R. Abu-Dawwas, "On graded classical prime and graded prime submodules", *Bulletin of the iranian mathematical society*, vol. 41, no. 1, pp. 205-2013, 2015. [On line]. Available: https://bit.ly/383ka6O

[9] H. Ansari-Toroghy and F. Farshadifar, "On graded second modules", *Tamkang journal of mathematics*, vol. 43, no. 4, pp. 499-505, 2012, doi: 10.5556/j.tkjm.43.2012.1319

[10] H. Ansari-Toroghy and F. Farshadifar, "Graded comultiplication modules", *Chiang mai journal of science*, vol. 38, no. 3, pp. 311-320, Jul. 2011. [On line]. Available: https://bit.ly/34Oe48w

[11] H. Ansari-Toroghy and F. Farshadifar, "Some generalizations of second submodules", *Palestine journal of mathematics*, vol. 8, no. 2, pp. 159-168, 2019. [On line]. Available: https://bit.ly/2JmQKGu

[12] S. E. Atani, "On graded prime submodules", *Chiang mai journal of science*, vol. 33, no. 1, pp. 3-7, 2006. [On line]. Available: https://bit.ly/2Jz1nGB

[13] S. E. Atani and F. Farzalipour, "On graded secondary modules", *Turkish journal of mathematics*, vol. 31, no. 4, 2007, pp. 371-378. [On line], Available: https://bit.ly/3jMSnd3

[14] S. Çeken and M. Alkan, "On graded second and coprimary modules and graded second representations", *Bulletin of Malaysian Mathematical Society*, vol. 38, no. 4, Oct. 2015, pp. 1317-1330, doi: 10.1007/s40840-014-0097-6

[15] J. Chen and Y. Kim, "Graded irreducible modules are irreducible", *Communications in algebra*, vol. 45, no. 5, pp. 1907-1913, Oct. 2017, doi: 10.1080/00927872.2016.1226864

[16] F. Farzalipour and P. Ghiasvand, "On the union of graded prime submodules", *Thai journal of mathematics*, vol. 9, no. 1, pp. 49-55, 2011. [On line]. Available: https://bit.ly/38423xD

[17] F. Farzalipour and P. Ghiasvand, "On graded weak multiplication modules", *Tamkang journal of mathematics*, vol. 43, no. 2, pp. 171-177, 2012, doi: 10.5556/j.tkjm.43.2012.712

[18] L. Fuchs, W. Heinzer, and B. Olberding, "Commutative ideal theory without finiteness conditions: irreducibility in the quotient field," in *Abelian groups, rings, modules, and homological algebra*, P. Goeters and O. M. G. Jenda , Eds. Boca Raton, CA: Chapman and Hall/CRC, 2006, pp. 121–145.

[19] C. Meng, "G-graded irreducibility and the index of reducibility", *Communications in algebra*, vol. 48, no. 2, pp. 826–832, Oct. 2019. doi: 10.1080/00927872.2019.1662914

[20] S. R. Naghani and H. F. Moghimi, "On graded 2-absorbing and graded weakly 2-absorbing ideals of a commutative ring", *Çankaya University journal of science and engineering*, vol. 13, no. 2, pp. 11-17, 2016, doi: 10.15672/hjms.2018.543

[21] C. Năstăsescu and F. V. Oystaeyen, *Methods of graded rings*. Berlin: Springer, 2004, doi: 10.1007/b94904

[22] M. Refai and K. Al-Zoubi, "On graded primary ideals", *Turkish journal of mathematics*, vol. 28, no. 3, pp. 217-229, 2004. [On line]. Available: https://bit.ly/34QIO91

[23] J. Zhang, "A 'natural' graded Hopf algebra and its graded Hopf-cyclic cohomology", *Journal of noncommutative geometry*, vol. 6, no. 2, pp. 389- 405, 2012. [On line]. Available: https://bit.ly/3jQsTLW