Hermite-Hadamard type fractional integral inequalities for products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions

Artion Kashuri¹ orcid.org/0000-0003-0115-3079

Rozana Liko² orcid.org/0000-0003-2439-8538

University Ismail Qemali Vlora, Dept. of Mathematics, Vlora, Albania.

¹ artionkashuri@gmail.com

² rozanaliko86@gmail.com

Received: February 2019 | Accepted: July 2019

Abstract:

A new class of $MT_{(r;g,m,\varphi)}$ -preinvex functions is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions via Riemann-Liouville fractional integrals are established. These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities. At the end, some conclusions and future research are given.

Keywords: Hermite-Hadamard type inequality; Hölder's inequality; Minkowski's inequality; Cauchy's inequality; Power mean inequality; Riemann-Liouville fractional integral; s-convex function in the second sense; m-invex; P -function.

MSC (2010): 26A51; 26A33; 26D07; 26D10; 26D15.

Cite this article as (IEEE citation style):

A. Kashuri and R. Liko, "Hermite-Hadamard type fractional integral inequalities for products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions", *Proyecciones (Antofagasta, On line)*, vol. 39, no. 1, pp. 219-242, Feb. 2020, doi: 10.22199/issn.0717-6279-2020-01-0014. [Accessed dd-mm-yyyy].

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1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbf{R} = (-\infty, +\infty)$ and I° to denote the interior of I . The nonnegative real numbers are denoted by $\mathbf{R}_0 = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In [22], Tunç and Yildirim defined the following so-called MT -convex function:

Definition 1.2. *A function $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ is said to belong to the class of $MT(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the following inequality:*

$$(1.2) \quad f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y).$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, see [1]-[11],[19, 20, 24]. Fractional calculus, see [16], was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes, see [9]-[16].

Definition 1.4. [7] A nonnegative function $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}_o$ is said to be P -function or P -convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 1.5. [2] A set $K \subseteq \mathbf{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbf{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details, see [2, 23].

Definition 1.6. [18] The function f defined on the invex set $K \subseteq \mathbf{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$(1.3) \quad \int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^* |f|,$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^* |f|$, see [21].

Recently, Liu in [14] obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.4 of P -function. Also in [17], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of $MT_{(r;g,m,\varphi)}$ -preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type integral inequalities for products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions via Riemann-Liouville fractional integrals are given. At the end, a briefly conclusion is given.

2. New integral inequalities

Definition 2.1. [8] A set $K \subseteq \mathbf{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m -invex set degenerates an invex set on K .

We next give new definition, to be referred as $MT_{(r;g,m,\varphi)}$ -preinvex function.

Definition 2.3. Let $K \subseteq \mathbf{R}^n$ be an open m -invex set with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}^n$, $g : [0, 1] \longrightarrow (0, 1)$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0, \infty)$ is said to be $MT_{(r;g,m,\varphi)}$ -preinvex with respect to η , if

$$(2.1) \quad f(m\varphi(y) + g(t)\eta(\varphi(x), \varphi(y), m)) \leq M_r(mf(\varphi(x)), f(\varphi(y)); g(t))$$

holds for some fixed $m \in (0, 1]$ and for all $x, y \in I$, $t \in [0, 1]$, where

$$M_r(f(\varphi(x)), f(\varphi(y)), m; g(t))$$

$$= \begin{cases} \left[\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}} f^r(\varphi(x)) + \frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}} f^r(\varphi(y)) \right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \left[f(\varphi(x)) \right]^{\frac{m\sqrt{g(t)}}{2\sqrt{1-g(t)}}} \left[f(\varphi(y)) \right]^{\frac{m\sqrt{1-g(t)}}{2\sqrt{g(t)}}}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2.4. In Definition 2.3, it is worthwhile to note that the class $MT_{(r;g,m,\varphi)}(I)$ is a generalization of the class $MT(I)$ given in Definition 1.2 for $r = m = 1$ with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x, y \in I$, $g(t) = t$, $\forall t \in (0, 1)$.

Let give a nontrivial example for motivation of this new interesting class of $MT_{(r;g,m,\varphi)}$ -preinvex functions.

Example 2.5. $f_1, f_2 : (1, \infty) \longrightarrow (0, \infty)$, $f_1(x) = x^p$, $f_2(x) = (1+x)^p$, $p \in (0, \frac{1}{1000})$; $h : [1, 3/2] \longrightarrow (0, \infty)$, $h(x) = (1+x^2)^k$, $k \in (0, \frac{1}{100})$, are simple examples of the new class of $MT_{(1;t,m,x)}$ -preinvex functions with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $g(t) = t$, $r = 1$, for any fixed $m \in (0, 1]$, but they are not convex.

In this section, in order to prove our main results regarding some new integral inequalities involving products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions, we need the following lemma.

Lemma 2.6. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ is a differentiable function. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbf{R}$ are continuous functions on the interval of real numbers K° with respect to the same $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\begin{aligned}
&= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \\
&\times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \\
&\times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)].
\end{aligned}$$

Proof. It is easy to observe that

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b), \varphi(a), m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b), \varphi(a), m)-x)^q f(x) h(x) dx \\
&= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\
&\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - g(t)\eta(\varphi(b), \varphi(a), m))^q \\
&\times f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \\
&= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \\
&\times \int_0^1 g^p(t)(1-g(t))^q f(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \\
&\times h(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)].
\end{aligned}$$

The proof of Lemma 2.6 is completed.

The following definition will be used in the sequel.

Definition 2.7. The Euler beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.8. Let $k > 1$, $r > 1$ and $r^{-1} + l^{-1} = 1$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ are continuous functions on K° with $\eta(\varphi(b), \varphi(a), m) > 0$. If $f^{\frac{k}{k-1}}, h^{\frac{k}{k-1}}$ are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function on an open m -invex set K° with respect to the same $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\begin{aligned}
 & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx \\
 (2.2) \quad & \leq \left(\frac{1}{2}\right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{1}{k}}(g(t); k, p, q) \\
 & \times \left[\left(\frac{m}{2}\right)^{\frac{2}{r}} \left[f^{\frac{rk}{k-1}}(\varphi(a)) A_1^{\frac{r}{2}}(g(t); r) + f^{\frac{rk}{k-1}}(\varphi(b)) A_2^{\frac{r}{2}}(g(t); r) \right]^{\frac{2}{r}} \right. \\
 & \left. + \left(\frac{m}{2}\right)^{\frac{2}{l}} \left[h^{\frac{lk}{k-1}}(\varphi(a)) A_1^{\frac{l}{2}}(g(t); l) + h^{\frac{lk}{k-1}}(\varphi(b)) A_2^{\frac{l}{2}}(g(t); l) \right]^{\frac{2}{l}} \right]^{\frac{k-1}{k}},
 \end{aligned}$$

where

$$B(g(t); k, p, q) := \int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)];$$

$$A_1(g(t); r) := \int_{g(0)}^{g(1)} \left(\frac{t}{1-t}\right)^{\frac{1}{r}} dt, \quad A_2(g(t); r) := \int_{g(0)}^{g(1)} \left(\frac{1-t}{t}\right)^{\frac{1}{r}} dt.$$

Proof. Since $f^{\frac{k}{k-1}}$ and $h^{\frac{k}{k-1}}$ are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function, combining with Lemma 2.6, Hölder inequality, Cauchy and Minkowski inequality, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\begin{aligned}
& \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\int_0^1 g^{kp}(t)(1-g(t))^{kq} d[g(t)] \right]^{\frac{1}{k}} \\
& \quad \times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) 44 \right. \\
& \quad \left. \times h^{\frac{k}{k-1}}(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \right]^{\frac{k-1}{k}} \\
& \leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{1}{k}}(g(t); k, p, q) \\
& \times \left[\int_0^1 \left(\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} f^{\frac{rk}{k-1}}(\varphi(a)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} f^{\frac{rk}{k-1}}(\varphi(b)) \right)^{\frac{1}{r}} \right. \\
& \times \left. \left(\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} h^{\frac{lk}{k-1}}(\varphi(a)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} h^{\frac{lk}{k-1}}(\varphi(b)) \right)^{\frac{1}{l}} d[g(t)] \right]^{\frac{k-1}{k}} \\
& \leq \left(\frac{1}{2} \right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{1}{k}}(g(t); k, p, q) \\
& \times \left[\int_0^1 \left(\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} f^{\frac{rk}{k-1}}(\varphi(a)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} f^{\frac{rk}{k-1}}(\varphi(b)) \right)^{\frac{2}{r}} d[g(t)] \right. \\
& + \int_0^1 \left(\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} h^{\frac{lk}{k-1}}(\varphi(a)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} h^{\frac{lk}{k-1}}(\varphi(b)) \right)^{\frac{2}{l}} d[g(t)] \left. \right]^{\frac{k-1}{k}} \\
& \leq \left(\frac{1}{2} \right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{1}{k}}(g(t); k, p, q) \\
& \times \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left\{ \left(\int_0^1 g^{\frac{1}{r}}(t)(1-g(t))^{-\frac{1}{r}} f^{\frac{2k}{k-1}}(\varphi(a)) d[g(t)] \right)^{\frac{r}{2}} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 g^{-\frac{1}{r}}(t)(1-g(t))^{\frac{1}{r}} f^{\frac{2k}{k-1}}(\varphi(b)) d[g(t)] \right)^{\frac{r}{2}} \Bigg\}^{\frac{2}{r}} \\
& + \left(\frac{m}{2} \right)^{\frac{2}{l}} \left\{ \left(\int_0^1 g^{\frac{1}{l}}(t)(1-g(t))^{-\frac{1}{l}} h^{\frac{2k}{k-1}}(\varphi(a)) d[g(t)] \right)^{\frac{l}{2}} \right. \\
& \left. + \left(\int_0^1 g^{-\frac{1}{l}}(t)(1-g(t))^{\frac{1}{l}} h^{\frac{2k}{k-1}}(\varphi(b)) d[g(t)] \right)^{\frac{l}{2}} \right\}^{\frac{2}{l}} \Bigg]^{\frac{k-1}{k}} \\
& = \left(\frac{1}{2} \right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{1}{k}}(g(t); k, p, q) \\
& \times \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left[f^{\frac{rk}{k-1}}(\varphi(a)) A_1^{\frac{r}{2}}(g(t); r) + f^{\frac{rk}{k-1}}(\varphi(b)) A_2^{\frac{r}{2}}(g(t); r) \right]^{\frac{2}{r}} \right. \\
& \left. + \left(\frac{m}{2} \right)^{\frac{2}{l}} \left[h^{\frac{lk}{k-1}}(\varphi(a)) A_1^{\frac{l}{2}}(g(t); l) + h^{\frac{lk}{k-1}}(\varphi(b)) A_2^{\frac{l}{2}}(g(t); l) \right]^{\frac{2}{l}} \right]^{\frac{k-1}{k}}.
\end{aligned}$$

The proof of Theorem 2.8 is completed. $\square$

Corollary 2.9. Under the same conditions as in Theorem 2.8 for $r = l = 2$ and $g(t) = t$, we get

$$\begin{aligned}
& \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx \\
(2.3) \quad & \leq \left(\frac{m\pi}{4} \right)^{\frac{k-1}{k}} \beta^{\frac{1}{k}}(kp+1, kq+1) \eta^{p+q+1}(\varphi(b), \varphi(a), m) \\
& \times \left[f^{\frac{2k}{k-1}}(\varphi(a)) + h^{\frac{2k}{k-1}}(\varphi(a)) + f^{\frac{2k}{k-1}}(\varphi(b)) + h^{\frac{2k}{k-1}}(\varphi(b)) \right]^{\frac{k-1}{k}}.
\end{aligned}$$

Theorem 2.10. Let $l \geq 1, r > 1$ and $r^{-1} + r_1^{-1} = 1$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ are continuous functions on the interval of real numbers K° with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. If f^l, h^l are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(r_1;g,m,\varphi)}$ -preinvex function on an open m -invex set K° with respect to the same $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\begin{aligned}
 & \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx \\
 (2.4) \quad & \leq \left(\frac{1}{2}\right)^{\frac{1}{t}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{l-1}{t}}(g(t); 1, p, q) \\
 & \times \left[\left(\frac{m}{2}\right)^{\frac{2}{r}} \left\{ f^{rl}(\varphi(a)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2\left(p + \frac{1}{2}\right), 2\left(q - \frac{1}{2}\right)\right) \right. \right. \\
 & \quad \left. \left. + f^{rl}(\varphi(b)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2\left(p - \frac{1}{2}\right), 2\left(q + \frac{1}{2}\right)\right) \right\}^{\frac{2}{r}} \right. \\
 & \quad \left. + \left(\frac{m}{2}\right)^{\frac{2}{r_1}} \left\{ h^{r_1 l}(\varphi(a)) B^{\frac{r_1}{2}} \left(g(t); \frac{1}{r_1}, 2\left(p + \frac{1}{2}\right), 2\left(q - \frac{1}{2}\right)\right) \right. \right. \\
 & \quad \left. \left. + h^{r_1 l}(\varphi(b)) B^{\frac{r_1}{2}} \left(g(t); \frac{1}{r_1}, 2\left(p - \frac{1}{2}\right), 2\left(q + \frac{1}{2}\right)\right) \right\}^{\frac{2}{r_1}} \right]^{\frac{1}{t}}.
 \end{aligned}$$

Proof. Since f^l and h^l are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(r_1;g,m,\varphi)}$ -preinvex function, combining with Lemma 2.6, power mean inequality, Cauchy and Minkowski inequality, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\begin{aligned}
 &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) \left[\int_0^1 g^p(t)(1-g(t))^q d[g(t)] \right]^{\frac{l-1}{l}} \\
 &\quad \times \left[\int_0^1 g^p(t)(1-g(t))^q f^l(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) \right. \\
 &\quad \left. \times h^l(m\varphi(a) + g(t)\eta(\varphi(b), \varphi(a), m)) d[g(t)] \right]^{\frac{1}{l}} \\
 &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{l-1}{l}}(g(t); 1, p, q) \\
 &\quad \times \left[\int_0^1 g^p(t)(1-g(t))^q \left(\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} f^{rl}(\varphi(a)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} f^{rl}(\varphi(b)) \right)^{\frac{1}{r}} \right. \\
 &\quad \left. \times \left(\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} h^{r_1 l}(\varphi(a)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} h^{r_1 l}(\varphi(b)) \right)^{\frac{1}{r_1}} d[g(t)] \right]^{\frac{1}{l}} \\
 &\leq \left(\frac{1}{2} \right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{1}{k}}(g(t); 1, p, q) \\
 &\quad \times \left[\int_0^1 \left(\frac{m}{2} g^{p+\frac{1}{2}}(t)(1-g(t))^{q-\frac{1}{2}} f^{rl}(\varphi(a)) + \frac{m}{2} g^{p-\frac{1}{2}}(t)(1-g(t))^{q+\frac{1}{2}} f^{rl}(\varphi(b)) \right)^{\frac{2}{r}} d[g(t)] \right. \\
 &\quad \left. + \int_0^1 \left(\frac{m}{2} g^{p+\frac{1}{2}}(t)(1-g(t))^{q-\frac{1}{2}} h^{r_1 l}(\varphi(a)) + \frac{m}{2} g^{p-\frac{1}{2}}(t)(1-g(t))^{q+\frac{1}{2}} h^{r_1 l}(\varphi(b)) \right)^{\frac{2}{r_1}} d[g(t)] \right]^{\frac{1}{l}} \\
 &\leq \left(\frac{1}{2} \right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{l-1}{l}}(g(t); 1, p, q) \\
 &\quad \times \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left\{ \left(\int_0^1 g^{\frac{2(p+\frac{1}{2})}{r}}(t)(1-g(t))^{\frac{2(q-\frac{1}{2})}{r}} f^{2l}(\varphi(a)) d[g(t)] \right)^{\frac{r}{2}} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 g^{\frac{2(p-\frac{1}{2})}{r}}(t)(1-g(t))^{\frac{2(q+\frac{1}{2})}{r}} f^{2l}(\varphi(b))d[g(t)] \right)^{\frac{r}{2}} \Bigg\}^{\frac{2}{r}} \\
& + \left(\frac{m}{2} \right)^{\frac{2}{r_1}} \left\{ \left(\int_0^1 g^{\frac{2(p+\frac{1}{2})}{r_1}}(t)(1-g(t))^{\frac{2(q-\frac{1}{2})}{r_1}} h^{2l}(\varphi(a))d[g(t)] \right)^{\frac{r_1}{2}} \right. \\
& + \left. \left(\int_0^1 g^{\frac{2(p-\frac{1}{2})}{r_1}}(t)(1-g(t))^{\frac{2(q+\frac{1}{2})}{r_1}} h^{2l}(\varphi(b))d[g(t)] \right)^{\frac{r_1}{2}} \Bigg\}^{\frac{2}{r_1}} \right]^{\frac{1}{l}} \\
& = \left(\frac{1}{2} \right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) B^{\frac{l-1}{l}}(g(t); 1, p, q) \\
& \times \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left\{ f^{rl}(\varphi(a)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2 \left(p + \frac{1}{2} \right), 2 \left(q - \frac{1}{2} \right) \right) \right. \right. \\
& \quad \left. \left. + f^{rl}(\varphi(b)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2 \left(p - \frac{1}{2} \right), 2 \left(q + \frac{1}{2} \right) \right) \right\}^{\frac{2}{r}} \right. \\
& + \left(\frac{m}{2} \right)^{\frac{2}{r_1}} \left\{ h^{r_1 l}(\varphi(a)) B^{\frac{r_1}{2}} \left(g(t); \frac{1}{r_1}, 2 \left(p + \frac{1}{2} \right), 2 \left(q - \frac{1}{2} \right) \right) \right. \\
& \quad \left. \left. + h^{r_1 l}(\varphi(b)) B^{\frac{r_1}{2}} \left(g(t); \frac{1}{r_1}, 2 \left(p - \frac{1}{2} \right), 2 \left(q + \frac{1}{2} \right) \right) \right\}^{\frac{2}{r_1}} \right]^{\frac{1}{l}}.
\end{aligned}$$

The proof of Theorem 2.10 is completed.

Corollary 2.11. *Under the same conditions as in Theorem 2.10 for $r = r_1 = 2$ and $g(t) = t$, we get*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p (m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)h(x)dx$$

$$\begin{aligned}
 (2.5) \quad & \leq \left(\frac{m}{4}\right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b), \varphi(a), m) \beta^{\frac{l-1}{l}} (p+1, q+1) \\
 & \times \left[\beta \left(p + \frac{3}{2}, q + \frac{1}{2}\right) \left(f^{2l}(\varphi(a)) + h^{2l}(\varphi(a))\right) \right. \\
 & \left. + \beta \left(p + \frac{1}{2}, q + \frac{3}{2}\right) \left(f^{2l}(\varphi(b)) + h^{2l}(\varphi(b))\right) \right]^{\frac{1}{l}}.
 \end{aligned}$$

3. Hermite-Hadamard type fractional integral inequalities

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions via fractional integrals.

Theorem 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Suppose $K \subseteq \mathbf{R}$ be an open m -invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$ with $\eta(\varphi(y), \varphi(x), m) > 0$. Assume that $f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \longrightarrow (0, \infty)$ are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function on an open m -invex set K° . Then for $\alpha > 0$, $r > 1$ and $r^{-1} + l^{-1} = 1$, we have*

$$\begin{aligned}
 (3.1) \quad & \frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x)+g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{\alpha-1} f(t) h(t) dt \\
 & \leq \frac{1}{2} \left[\left(\frac{m}{2}\right)^{\frac{2}{r}} \left\{ f^r(\varphi(x)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2\left(\alpha - \frac{1}{2}\right), -1\right) \right. \right. \\
 & \quad \left. \left. + f^r(\varphi(y)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2\left(\alpha - \frac{3}{2}\right), 1\right) \right\}^{\frac{2}{r}} \right. \\
 & \quad \left. + \left(\frac{m}{2}\right)^{\frac{2}{l}} \left\{ h^l(\varphi(x)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2\left(\alpha - \frac{1}{2}\right), -1\right) \right. \right. \\
 & \quad \left. \left. + h^l(\varphi(y)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2\left(\alpha - \frac{3}{2}\right), 1\right) \right\}^{\frac{2}{l}} \right].
 \end{aligned}$$

Proof. Since f and h are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function, combining with Cauchy and Minkowski inequalities, we get

$$\begin{aligned}
& \frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x)+g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{\alpha-1} f(t) h(t) dt \\
&= \int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \\
&\quad \times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)] \\
&\leq \int_0^1 g^{(\alpha-1)(\frac{1}{r}+\frac{1}{l})}(t) \left[\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} f^r(\varphi(x)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} f^r(\varphi(y)) \right]^{\frac{1}{r}} \\
&\quad \times \left[\frac{m}{2} g^{\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} h^l(\varphi(x)) + \frac{m}{2} g^{-\frac{1}{2}}(t)(1-g(t))^{\frac{1}{2}} h^l(\varphi(y)) \right]^{\frac{1}{l}} d[g(t)] \\
&\leq \frac{1}{2} \left\{ \int_0^1 \left[\frac{m}{2} g^{\alpha-\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} f^r(\varphi(x)) + \frac{m}{2} g^{\alpha-\frac{3}{2}}(t)(1-g(t))^{\frac{1}{2}} f^r(\varphi(y)) \right]^{\frac{2}{r}} d[g(t)] \right. \\
&\quad \left. + \int_0^1 \left[\frac{m}{2} g^{\alpha-\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} h^l(\varphi(x)) + \frac{m}{2} g^{\alpha-\frac{3}{2}}(t)(1-g(t))^{\frac{1}{2}} h^l(\varphi(y)) \right]^{\frac{2}{l}} d[g(t)] \right\} \\
&\leq \frac{1}{2} \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left\{ \left(\int_0^1 g^{\frac{2(\alpha-\frac{1}{2})}{r}}(t)(1-g(t))^{-\frac{1}{r}} f^2(\varphi(x)) d[g(t)] \right)^{\frac{r}{2}} \right. \right. \\
&\quad \left. \left. + \left(\int_0^1 g^{\frac{2(\alpha-\frac{3}{2})}{r}}(t)(1-g(t))^{\frac{1}{r}} f^2(\varphi(y)) d[g(t)] \right)^{\frac{r}{2}} \right\} \right. \\
&\quad \left. + \left(\frac{m}{2} \right)^{\frac{2}{l}} \left\{ \left(\int_0^1 g^{\frac{2(\alpha-\frac{1}{2})}{l}}(t)(1-g(t))^{-\frac{1}{l}} h^2(\varphi(x)) d[g(t)] \right)^{\frac{l}{2}} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 g^{\frac{2(\alpha-\frac{3}{2})}{l}}(t)(1-g(t))^{\frac{1}{l}} h^2(\varphi(y)) d[g(t)] \right)^{\frac{l}{2}} \left\}^{\frac{2}{l}} \right] \\
& = \frac{1}{2} \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left\{ f^r(\varphi(x)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2 \left(\alpha - \frac{1}{2} \right), -1 \right) \right. \right. \\
& \quad \left. \left. + f^r(\varphi(y)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2 \left(\alpha - \frac{3}{2} \right), 1 \right) \right\}^{\frac{2}{r}} \right. \\
& \quad \left. + \left(\frac{m}{2} \right)^{\frac{2}{l}} \left\{ h^l(\varphi(x)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2 \left(\alpha - \frac{1}{2} \right), -1 \right) \right. \right. \\
& \quad \left. \left. + h^l(\varphi(y)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2 \left(\alpha - \frac{3}{2} \right), 1 \right) \right\}^{\frac{2}{l}} \right].
\end{aligned}$$

The proof of Theorem 3.1 is completed.

Corollary 3.2. Under the same conditions as in Theorem 3.1 for $r = l = 2$ and $g(t) = t$, we get

$$\begin{aligned}
& \frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(y), \varphi(x), m)} J_{(m\varphi(x)+\eta(\varphi(y), \varphi(x), m))}^\alpha - f(m\varphi(x))h(m\varphi(x)) \\
(3.2) \quad & \leq \frac{m}{4} \left[\beta \left(\alpha + \frac{1}{2}, \frac{1}{2} \right) \left(f^2(\varphi(x)) + h^2(\varphi(x)) \right) \right. \\
& \quad \left. + \beta \left(\alpha - \frac{1}{2}, \frac{3}{2} \right) \left(f^2(\varphi(y)) + h^2(\varphi(y)) \right) \right].
\end{aligned}$$

Theorem 3.3. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Assume that $f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \longrightarrow (0, \infty)$ are continuous functions on K° with $\eta(\varphi(y), \varphi(x), m) > 0$. Let $0 < r, l \leq 1, q > 1$ and $p^{-1} + q^{-1} = 1$. If f^p, h^q are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function on an open m -invex set K° with respect to the same $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$, then for $\alpha > 0$, we have

$$\begin{aligned}
& \frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x)+g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt \\
(3.3) \quad & \leq \left(\frac{m}{2}\right)^{\frac{1}{rp} + \frac{1}{lq}} \left[f^{rp}(\varphi(x))B^r\left(g(t); \frac{1}{r}, \frac{1}{2} + rp(\alpha - 1), -\frac{1}{2}\right) \right. \\
& \quad \left. + f^{rp}(\varphi(y))B^r\left(g(t); \frac{1}{r}, rp(\alpha - 1) - \frac{1}{2}, \frac{1}{2}\right) \right]^{\frac{1}{rp}} \\
& \quad \times \left[h^{lq}(\varphi(x))B^l\left(g(t); \frac{1}{l}, \frac{1}{2}, -\frac{1}{2}\right) + h^{lq}(\varphi(y))B^l\left(g(t); \frac{1}{l}, -\frac{1}{2}, \frac{1}{2}\right) \right]^{\frac{1}{lq}}.
\end{aligned}$$

Proof. Since f^p, h^q are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function, combining with Hölder and Minkowski inequalities, we get

$$\begin{aligned}
& \frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x)+g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt \\
& = \int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \\
& \quad \times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)] \\
& \leq \left(\int_0^1 g^{p(\alpha-1)}(t) f^p(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)] \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_0^1 h^q(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)] \right)^{\frac{1}{q}} \\
& \leq \left(\int_0^1 g^{p(\alpha-1)}(t) \left[\frac{m}{2} g^{\frac{1}{2}}(t) (1 - g(t))^{-\frac{1}{2}} f^{rp}(\varphi(x)) + \frac{m}{2} g^{-\frac{1}{2}}(t) (1 - g(t))^{\frac{1}{2}} f^{rp}(\varphi(y)) \right]^{\frac{1}{r}} d[g(t)] \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\left[\frac{m}{2} g^{\frac{1}{2}}(t) (1 - g(t))^{-\frac{1}{2}} h^{lq}(\varphi(x)) + \frac{m}{2} g^{-\frac{1}{2}}(t) (1 - g(t))^{\frac{1}{2}} h^{lq}(\varphi(y)) \right]^{\frac{1}{l}} d[g(t)] \right)^{\frac{1}{q}} \\
& \leq \left(\frac{m}{2} \right)^{\frac{1}{rp}} \left\{ \left(\int_0^1 g^{p(\alpha-1)+\frac{1}{2r}}(t) (1 - g(t))^{-\frac{1}{2r}} f^p(\varphi(x)) d[g(t)] \right)^r \right. \\
& \quad \left. + \left(\int_0^1 g^{p(\alpha-1)-\frac{1}{2r}}(t) (1 - g(t))^{\frac{1}{2r}} f^p(\varphi(y)) d[g(t)] \right)^r \right\}^{\frac{1}{rp}} \\
& \quad \times \left(\frac{m}{2} \right)^{\frac{1}{lq}} \left\{ \left(\int_0^1 g^{\frac{1}{2l}}(t) (1 - g(t))^{-\frac{1}{2l}} h^q(\varphi(x)) d[g(t)] \right)^l \right. \\
& \quad \left. + \left(\int_0^1 g^{-\frac{1}{2l}}(t) (1 - g(t))^{\frac{1}{2l}} h^q(\varphi(y)) d[g(t)] \right)^l \right\}^{\frac{1}{lq}} \\
& = \left(\frac{m}{2} \right)^{\frac{1}{rp} + \frac{1}{lq}} \left[f^{rp}(\varphi(x)) B^r \left(g(t); \frac{1}{r}, \frac{1}{2} + rp(\alpha - 1), -\frac{1}{2} \right) \right. \\
& \quad \left. + f^{rp}(\varphi(y)) B^r \left(g(t); \frac{1}{r}, rp(\alpha - 1) - \frac{1}{2}, \frac{1}{2} \right) \right]^{\frac{1}{rp}} \\
& \quad \times \left[h^{lq}(\varphi(x)) B^l \left(g(t); \frac{1}{l}, \frac{1}{2}, -\frac{1}{2} \right) + h^{lq}(\varphi(y)) B^l \left(g(t); \frac{1}{l}, -\frac{1}{2}, \frac{1}{2} \right) \right]^{\frac{1}{lq}}.
\end{aligned}$$

The proof of Theorem 3.3 is completed.

Corollary 3.4. Under the same conditions as in Theorem 3.3 for $p = q = 2$ and $g(t) = t$, we get

$$\frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(y), \varphi(x), m)} J_{(m\varphi(x)+\eta(\varphi(y), \varphi(x), m))}^\alpha - f(m\varphi(x)) h(m\varphi(x))$$

$$\begin{aligned}
(3.4) \quad &\leq \left(\frac{m}{2}\right)^{\frac{1}{2r}+\frac{1}{2l}} \beta^{\frac{1}{2}} \left(\frac{1}{2l} + 1, 1 - \frac{1}{2l}\right) \left[h^{2l}(\varphi(x)) + h^{2l}(\varphi(y))\right]^{\frac{1}{2l}} \\
&\quad \times \left[f^{2r}(\varphi(x))\beta^r \left(\frac{1}{2r} + 2(\alpha - 1) + 1, 1 - \frac{1}{2r}\right) \right. \\
&\quad \left. + f^{2r}(\varphi(y))\beta^r \left(2(\alpha - 1) + 1 - \frac{1}{2r}, 1 + \frac{1}{2r}\right)\right]^{\frac{1}{2r}}.
\end{aligned}$$

Theorem 3.5. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow (0, 1)$ is a differentiable function. Assume that $f, h : K = [m\varphi(x), m\varphi(x) + \eta(\varphi(y), \varphi(x), m)] \longrightarrow (0, \infty)$ are continuous functions on K° with $\eta(\varphi(y), \varphi(x), m) > 0$. Let $q \geq 1, r > 1$ and $r^{-1} + l^{-1} = 1$. If f, h^q are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function on an open m -invex set K° with respect to the same $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$, then for $\alpha > 0$, we have

$$\begin{aligned}
&\frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x)+g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{\alpha-1} f(t)h(t)dt \\
(3.5) \quad &\leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(\frac{m}{2}\right)^{\frac{q-1}{rq}} \left[f^r(\varphi(x))B^r \left(g(t); \frac{1}{r}, \frac{1}{2} + r(\alpha - 1), -\frac{1}{2}\right) \right. \\
&\quad \left. + f^r(\varphi(y))B^r \left(g(t); \frac{1}{r}, r(\alpha - 1) - \frac{1}{2}, \frac{1}{2}\right) \right]^{\frac{q-1}{rq}} \\
&\quad \times \left[\left(\frac{m}{2}\right)^{\frac{2}{r}} \left\{ f^r(\varphi(x))B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha - \frac{1}{2}), -1\right) \right. \right. \\
&\quad \left. \left. + f^r(\varphi(y))B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha - \frac{3}{2}), 1\right) \right\}^{\frac{2}{r}} \right. \\
&\quad \left. + \left(\frac{m}{2}\right)^{\frac{2}{l}} \left\{ h^{lq}(\varphi(x))B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha - \frac{1}{2}), -1\right) \right. \right. \\
&\quad \left. \left. + h^{lq}(\varphi(y))B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha - \frac{3}{2}), 1\right) \right\}^{\frac{2}{l}} \right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Since f and h^q are respectively $MT_{(r;g,m,\varphi)}$ -preinvex function and $MT_{(l;g,m,\varphi)}$ -preinvex function, combining with power mean inequality, Cauchy and Minkowski inequalities, we get

$$\begin{aligned}
 & \frac{1}{\eta^\alpha(\varphi(y), \varphi(x), m)} \int_{m\varphi(x)+g(0)\eta(\varphi(y), \varphi(x), m)}^{m\varphi(x)+g(1)\eta(\varphi(y), \varphi(x), m)} (t - m\varphi(x))^{\alpha-1} f(t) h(t) dt \\
 &= \int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \\
 & \quad \times h(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)] \\
 &\leq \left(\int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)] \right)^{1-\frac{1}{q}} \\
 & \quad \times \left[\int_0^1 g^{\alpha-1}(t) f(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) \right. \\
 & \quad \left. \times h^q(m\varphi(x) + g(t)\eta(\varphi(y), \varphi(x), m)) d[g(t)] \right]^{\frac{1}{q}} \\
 &\leq \left(\int_0^1 g^{\alpha-1}(t) \left[\frac{m}{2} g^{\frac{1}{2}}(t) (1-g(t))^{-\frac{1}{2}} f^r(\varphi(x)) + \frac{m}{2} g^{-\frac{1}{2}}(t) (1-g(t))^{\frac{1}{2}} f^r(\varphi(y)) \right]^{\frac{1}{r}} d[g(t)] \right)^{1-\frac{1}{q}} \\
 & \quad \times \left\{ \int_0^1 g^{\alpha-1}(t) \left[\frac{m}{2} g^{\frac{1}{2}}(t) (1-g(t))^{-\frac{1}{2}} f^r(\varphi(x)) + \frac{m}{2} g^{-\frac{1}{2}}(t) (1-g(t))^{\frac{1}{2}} f^r(\varphi(y)) \right]^{\frac{1}{r}} \right. \\
 & \quad \left. \times \left[\frac{m}{2} g^{\frac{1}{2}}(t) (1-g(t))^{-\frac{1}{2}} h^{lq}(\varphi(x)) + \frac{m}{2} g^{-\frac{1}{2}}(t) (1-g(t))^{\frac{1}{2}} h^{lq}(\varphi(y)) \right]^{\frac{1}{l}} d[g(t)] \right\}^{\frac{1}{q}} \\
 &\leq \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\frac{m}{2} \right)^{\frac{q-1}{rq}} \left\{ \left(\int_0^1 g^{\alpha-1+\frac{1}{2r}}(t) (1-g(t))^{-\frac{1}{2r}} f(\varphi(x)) d[g(t)] \right)^r \right.
 \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 g^{\alpha-1-\frac{1}{2r}}(t)(1-g(t))^{\frac{1}{2r}} f(\varphi(y)) d[g(t)] \right)^r \Bigg\}^{\frac{q-1}{rq}} \\
& \times \left\{ \int_0^1 \left[\frac{m}{2} g^{\alpha-\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} f^r(\varphi(x)) + \frac{m}{2} g^{\alpha-\frac{3}{2}}(t)(1-g(t))^{\frac{1}{2}} f^r(\varphi(y)) \right]^{\frac{2}{r}} d[g(t)] \right. \\
& \left. + \int_0^1 \left[\frac{m}{2} g^{\alpha-\frac{1}{2}}(t)(1-g(t))^{-\frac{1}{2}} h^{lq}(\varphi(x)) + \frac{m}{2} g^{\alpha-\frac{3}{2}}(t)(1-g(t))^{\frac{1}{2}} h^{lq}(\varphi(y)) \right]^{\frac{2}{l}} d[g(t)] \right\}^{\frac{1}{q}} \\
& \leq \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\frac{m}{2} \right)^{\frac{q-1}{rq}} \left\{ \left(\int_0^1 g^{\alpha-1+\frac{1}{2r}}(t)(1-g(t))^{-\frac{1}{2r}} f(\varphi(x)) d[g(t)] \right)^r \right. \\
& \quad \left. + \left(\int_0^1 g^{\alpha-1-\frac{1}{2r}}(t)(1-g(t))^{\frac{1}{2r}} f(\varphi(y)) d[g(t)] \right)^r \right\}^{\frac{q-1}{rq}} \\
& \quad \times \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left\{ \left(\int_0^1 g^{\frac{2(\alpha-\frac{1}{2})}{r}}(t)(1-g(t))^{-\frac{1}{r}} f^2(\varphi(x)) d[g(t)] \right)^{\frac{r}{2}} \right. \right. \\
& \quad \left. \left. + \left(\int_0^1 g^{\frac{2(\alpha-\frac{3}{2})}{r}}(t)(1-g(t))^{\frac{1}{r}} f^2(\varphi(y)) d[g(t)] \right)^{\frac{r}{2}} \right\}^{\frac{2}{r}} \right. \\
& \quad \left. + \left(\frac{m}{2} \right)^{\frac{2}{l}} \left\{ \left(\int_0^1 g^{\frac{2(\alpha-\frac{1}{2})}{l}}(t)(1-g(t))^{-\frac{1}{l}} h^{2q}(\varphi(x)) d[g(t)] \right)^{\frac{l}{2}} \right. \right. \\
& \quad \left. \left. + \left(\int_0^1 g^{\frac{2(\alpha-\frac{3}{2})}{l}}(t)(1-g(t))^{\frac{1}{l}} h^{2q}(\varphi(y)) d[g(t)] \right)^{\frac{l}{2}} \right\}^{\frac{2}{l}} \right]^{\frac{1}{q}} \\
& = \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\frac{m}{2} \right)^{\frac{q-1}{rq}} \left[f^r(\varphi(x)) B^r \left(g(t); \frac{1}{r}, \frac{1}{2} + r(\alpha-1), -\frac{1}{2} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + f^r(\varphi(y)) B^r \left(g(t); \frac{1}{r}, r(\alpha - 1) - \frac{1}{2}, \frac{1}{2} \right) \Bigg]^{\frac{q-1}{rq}} \\
& \times \left[\left(\frac{m}{2} \right)^{\frac{2}{r}} \left\{ f^r(\varphi(x)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha - \frac{1}{2}), -1 \right) \right. \right. \\
& \quad \left. \left. + f^r(\varphi(y)) B^{\frac{r}{2}} \left(g(t); \frac{1}{r}, 2(\alpha - \frac{3}{2}), 1 \right) \right\}^{\frac{2}{r}} \right. \\
& \quad \left. + \left(\frac{m}{2} \right)^{\frac{2}{l}} \left\{ h^{lq}(\varphi(x)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha - \frac{1}{2}), -1 \right) \right. \right. \\
& \quad \left. \left. + h^{lq}(\varphi(y)) B^{\frac{l}{2}} \left(g(t); \frac{1}{l}, 2(\alpha - \frac{3}{2}), 1 \right) \right\}^{\frac{2}{l}} \right]^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 3.5 is completed.

Corollary 3.6. *Under the same conditions as in Theorem 3.5 for $q = 1$, we get Theorem 3.1.*

Corollary 3.7. *Under the same conditions as in Theorem 3.5 for $r = l = 2$ and $g(t) = t$, we get*

$$\begin{aligned}
& \frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(y), \varphi(x), m)} J_{(m\varphi(x)+\eta(\varphi(y), \varphi(x), m))}^\alpha - f(m\varphi(x)) h(m\varphi(x)) \\
& \leq \left(\frac{m}{4} \right)^{\frac{1}{q}} \left(\frac{m}{2} \right)^{\frac{q-1}{2q}} \left[f^2(\varphi(x)) \beta^2 \left(\alpha + \frac{1}{4}, \frac{3}{4} \right) + f^2(\varphi(y)) \beta^2 \left(\alpha - \frac{1}{4}, \frac{5}{4} \right) \right]^{\frac{q-1}{2q}} \\
& (3.6) \\
& \quad \times \left[\beta \left(\alpha + \frac{1}{2}, \frac{1}{2} \right) \left(f^2(\varphi(x)) + h^{2q}(\varphi(x)) \right) \right. \\
& \quad \left. + \beta \left(\alpha - \frac{1}{2}, \frac{3}{2} \right) \left(f^2(\varphi(y)) + h^{2q}(\varphi(y)) \right) \right]^{\frac{1}{q}}.
\end{aligned}$$

Remark 3.8. For $\alpha > 0$, for different choices of positive values $r, l = \frac{1}{2}, \frac{1}{3}, 2$, etc., for any fixed $m \in (0, 1]$, for a particular choices of a differentiable function $g(t) = e^{-(t+1)}, \ln(t+2), \sin\left(\frac{\pi(t+1)}{3}\right), \cos\left(\frac{\pi(t+1)}{3}\right)$, etc., and a particular choices of a continuous function $\varphi(x) = e^x$ for all $x \in \mathbf{R}$, x^n for all $x > 0$ and for all $n \in \mathbf{N}$, etc., by Theorems 3.1, 3.3 and Theorem 3.5 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities for products of two $MT_{(r;g,m,\varphi)}$ -preinvex functions.

Conclusion

Interested reader can obtain in a similar way new results for different operators such as the k -Riemann-Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc., and this results can be applied in convex analysis, optimization and also different areas of pure and applied sciences.

References

- [1] A. Akkurt and H. Yildirim, "On some fractional integral inequalities of Hermite-Hadamard type for r -preinvex functions", *Khayyam journal of mathematics*, vol. 2, no. 2, pp. 120-127, 2016, doi: 10.22034/kjm.2016.40640.
- [2] T. Antczak, "Mean value in invexity analysis", *Nonlinear analysis: theory, methods & applications*, vol. 60, no. 8, pp. 1473-1484, Mar. 2005, doi: 10.1016/j.na.2004.11.005.
- [3] F. Chen, "A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals", *Italian journal of pure applied mathematics*, no. 33, pp. 299-306, 2014. [On line]. Available: <https://bit.ly/2vJP0QR>
- [4] F. X. Chen and S. H. Wu, "Several complementary inequalities to inequalities of Hermite-Hadamard type for s -convex functions", *Journal of nonlinear sciences and applications*, vol. 9, no. 2, pp. 705-716, 2016, doi: 10.22436/jnsa.009.02.32.
- [5] Y. M. Chu, M. A. Khan, T. U. Khan, and T. Ali, "Generalizations of Hermite-Hadamard type inequalities for MT -convex functions", *Journal of nonlinear sciences and applications*, vol. 9, no. 6, pp. 4305-4316, 2016, doi: 10.22436/jnsa.009.06.72.

- [6] Y. M. Chu, G. D. Wang, and X. H. Zhang, "Schur convexity and Hadamard's inequality", *Mathematical inequalities & applications*, vol. 13, no. 4, pp. 725-731, 2010, doi: 10.7153/mia-13-51.
- [7] S. S. Dragomir, J. Pečarić, and L. E. Persson, "Some inequalities of Hadamard type", *Soochow journal of mathematics*, vol. 21, no. 3, pp. 335-341, Jul. 1995. [On line]. Available: <https://bit.ly/37UvW09>
- [8] T. S. Du, J. G. Liao, and Y. J. Li, "Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions", *Journal of nonlinear sciences and applications*, vol. 9, no. 5, pp. 3112-3126, 2016, doi: 10.22436/jnsa.009.05.102.
- [9] H. Hudzik and L. Maligranda, "Some remarks on s-convex functions", *Aequationes mathematicae*, vol. 48, no. 1, pp. 100-111, Aug. 1994, doi: 10.1007/BF01837981.
- [10] W. D. Jiang, D. W. Niu, and F. Qi, "Some inequalities of Hermite-Hadamard type for r - ϕ -preinvex functions", *Tamkang journal of mathematics*, vol. 45, no. 1, pp. 31-38, 2014, doi: 10.5556/j.tkjm.45.2014.1261.
- [11] H. Kavurmaci, M. Avci. and M. E. Özdemir, "New inequalities of Hermite-Hadamard type for convex functions with applications", Jun. 2010, *arXiv:1006.1593v1*
- [12] M. A. Khan, Y. Khurshid, T. Ali, and N. Rehman, "Inequalities for three times differentiable functions", *The Punjab university journal of mathematics*, vol. 48, no. 2, pp. 35-48, 2016. [On line]. Available: <https://bit.ly/2vNZjU7>
- [13] M. A. Khan, Y. Khurshid, and T. Ali, "Hermite-Hadamard inequality for fractional integrals via η -convex functions", *Acta mathematica universitatis comenianae*, vol. 86, no. 1, pp. 153—164, 2017. [On line]. Available: <https://bit.ly/2v3TEJm>
- [14] W. Liu, "New integral inequalities involving beta function via P -convexity", *Miskolc mathematical notes (Online)*, vol. 15, no. 2, pp. 585-591, 2014, doi: 10.18514/mmn.2014.660.
- [15] W. Liu, W. Wen and J. Park, "Ostrowski type fractional integral inequalities for MT -convex functions", *Miskolc mathematical notes (Online)*, vol. 16, no. 1, pp. 249-256, 2015, doi: 10.18514/MMN.2015.1131.
- [16] W. Liu, W. Wen and J. Park, "Hermite-Hadamard type inequalities for MT -convex functions via classical integrals and fractional integrals", *Journal of nonlinear sciences and applications*, vol. 9, no. 3, pp. 766-777, 2016, doi: 10.22436/jnsa.009.03.05.
- [17] M. E. Özdemir, E. Set, and M. Alomari, "Integral inequalities via several kinds of convexity", *Creative mathematics and informatics*, vol. 20, no. 1, pp. 62-73, 2011. [On line]. Available: <https://bit.ly/31rqeRw>

- [18] R. Pini, "Invexity and generalized convexity", *Optimization*, vol. 22, no. 4. pp. 513— 525, Jun. 1991, doi: 10.1080/02331939108843693.
- [19] F. Qi and B. Y. Xi, "Some integral inequalities of Simpson type for GA- ϵ -convex functions", *Georgian mathematical journal*, vol. 20, no. 4, pp. 775-788, Oct. 2013, doi: 10.1515/gmj-2013-0043.
- [20] H. N. Shi, "Two Schur-convex functions related to Hadamard-type integral inequalities", *Publicationes mathematicae Debrecen*, vol. 78, no. 2, pp. 393-403, 2011. [On line]. Available: <https://bit.ly/2v2X9PW>
- [21] D. D. Stancu, G. Coman, O. Agratini, and Trimbitea Radu, *Analiza numerica si teoria aproximarii*, vol. 2. Cluj-Napoca: Presa Universitara Clujeana, 2001.
- [22] M. Tunç and H. Yildirim, "On MT -convexity", May 2012. *arXiv:1205.5453v1*
- [23] X. M. Yang, X. Q. Yang, and K. L. Teo, "Generalized invexity and generalized invariant monotonicity", *Journal of optimization theory and applications*, vol. 117, no. 3, pp. 607-625, Jun. 2003, doi: 10.1023/A:1023953823177.
- [24] X. M. Zhang, Y. M. Chu and X. H. Zhang, "The Hermite-Hadamard type inequality of GA-convex functions and its applications", *Journal of inequalities and applications*, Art. ID. 507560, Dec. 2010, doi: 10.1155/2010/507560.