A system of nonlinear fractional BVPs with ϕ -Laplacian operators and nonlocal conditions

Bahia Temar¹ orcid.org/0000-0001-7493-5325

Ouiza Saifi²

Smaïl Djebali³ orcid.org/0000-0002-2318-1989

Ecole Normale Supérieure Cheikh Mohamed El-Bachir El-Ibrahimi, Laboratoire "Théorie du Point Fixe et Applications", Kouba, Algiers, Algeria.

¹ temarbahia@gmail.com; ² saifi_rayen@hotmail.com; ³ djebali@hotmail.com

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Abstract:

This work investigates the existence of multiple positive solutions for a system of two nonlinear higher-order fractional differential equations with ϕ -Laplacian operators and nonlocal conditions. A degenerate nonlinearity which obeys some general growth conditions is considered. The singularities are dealt with by approximating the fixed point operator. New existence results are presented by using the fixed point index theory. Examples of applications illustrate the theoretical results.

Keywords: Fixed point index; Positive solution; Fractional differential equation; Nonlocal boundary condition; ϕ -Laplacian.

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1. Introduction

Fractional differential equations are much more adequate and more realistic than integer-order differential equations to model several problems in science and engineering. These equations appear in many problems arising from physics, engineering, and dynamical systems but also from fluid flows, electrical networks, dynamics of earthquakes, ... because they seem to agree with experimental studies. This has motivated several researchers to use the fractional modelling in some recent research works in applied mathematics. In addition, several homogeneous or non-homogeneous boundary conditions (BCs for brevity) have been adopted in problem modeling, ranging from linear to nonlinear boundary conditions; actually they reveal information on the behaviour of the unknowns (temperature, displacement, velocity, ...) on boundaries of strings, rods, and plates (e.g., heat transfer across isolated surfaces according to Newton's law of cooling). By nonlocal BCs, it is meant here multi-point BCs, integral BCs, or more general differential and/or algebraic BCs.

In [8, 10, 14], we can find a detailed discussion of some fractional equations and systems of equations arising in physical phenomena. As for fractional calculus and main properties of fractional derivative operators, we refer the reader to [7, 11, 9].

From a mathematical point of view, a differential operator (with a fractional or an integer order) may be extended to p -Laplacian order operators. Notice that the latter is only linear for $p = 2$. The case of the ϕ -Laplacian is again a more general extension of the p -Laplacian. This is our main motivation to consider in this work such general fractional operator. More precisely, we consider the existence of positive solutions for the following system of two nonlinear higher-order fractional differential equations with ϕ -Laplacian operators and nonlocal BCs:

$$(1.1) \quad \begin{cases} -D_{0+}^{\alpha_1}(\phi_1(D_{0+}^{\beta_1}x(t))) = a_1(t)f(t, x(t), y(t)), & t \in (0, 1) \\ -D_{0+}^{\alpha_2}(\phi_2(D_{0+}^{\beta_2}y(t))) = a_2(t)g(t, x(t), y(t)), & t \in (0, 1) \\ x(0) = x(1) = x'(0) = x'(1) = 0, & D_{0+}^{\beta_1}x(0) = 0, D_{0+}^{\beta_1}x(1) = b_1D_{0+}^{\beta_1}x(\mu_1), \\ y(0) = y(1) = y'(0) = y'(1) = 0, & D_{0+}^{\beta_2}y(0) = 0, D_{0+}^{\beta_2}y(1) = b_2D_{0+}^{\beta_2}y(\mu_2), \end{cases}$$

where $\alpha_i \in (1, 2]$, $\beta_i \in (3, 4]$. Throughout this paper, D_{0+}^{α} will denote the standard Riemann-Liouville derivative of order α .

For $i = 1, 2$, the coefficient functions $b_i \in (0, \phi_i^{-1}(\mu_i^{1-\alpha_i}))$ and $a_i \in C([0, 1])$ while $\mu_i \in (0, 1)$. The nonlinearities $f = f(t, x, y)$, $g = g(t, x, y) :$

$[0, 1] \times \mathbf{R}_*^{+2} \longrightarrow \mathbf{R}^+$ are continuous and degenerate at $x = 0$ and $y = 0$. For $i = 1, 2$, the mappings $\phi_i : \mathbf{R} \longrightarrow \mathbf{R}$ are continuous homeomorphisms that are increasing and multiplicative, i.e., $(\phi_i(xy) = \phi_i(x)\phi_i(y), \forall x, y > 0)$, with $\phi_i(0) = 0$. Here and hereafter, we will use the notations $\mathbf{R}^+ = [0, +\infty)$ and $\mathbf{R}_*^+ = (0, +\infty)$. $C([0, 1])$ and $L([0, 1])$ will refer to the space of continuous functions and to the class of Lebesgue integrable functions, respectively. Indeed, we are here interested in classical solutions which is sufficient for our purpose for the nonlinearities do not depend on first derivatives.

The positivity of solutions is physically interpreted by the fact that some parameters and unknowns such as temperature, density, displacement, ... are positive data. The higher-order derivative operator (which encompasses second-order operators) used in each equation is motivated by the investigation here of BVPs (boundary value problems) for a coupling system instead of IVPs (initial value problems). To avoid heavy fixed point fractional operator, we have considered homogeneous BCs for solutions and derivatives and nonlocal BCs for higher fractional derivatives. Recall further that, in physics, the nonlinear functions f, g may stand for source terms and have behavior which range from polynomial to exponential growth. The nonlinearities considered in this work encompass some growth conditions adopted in recent papers. Next, we are going to describe briefly some of them.

In the particular case of the p -Laplacian, namely $\varphi_p(s) = |s|^{p-2}s$ ($p > 1$), the authors of [6] discussed the following system of nonlinear fractional differential equations subject to nonlocal BCs:

$$(1.2) \quad \begin{cases} -D_{0+}^{\alpha_1}(\varphi_{p_1}(D_{0+}^{\beta_1}u(t))) = \lambda f(t, u(t), v(t)), & 0 < t < 1 \\ -D_{0+}^{\alpha_2}(\varphi_{p_2}(D_{0+}^{\beta_2}v(t))) = \mu g(t, u(t), v(t)), & 0 < t < 1 \\ u(0) = u(1) = u'(0) = u'(1) = 0, & D_{0+}^{\beta_1}u(0) = 0, & D_{0+}^{\beta_1}u(1) = b_1 D_{0+}^{\beta_1}v(\eta_1), \\ v(0) = v(1) = v'(0) = v'(1) = 0, & D_{0+}^{\beta_2}v(0) = 0, & D_{0+}^{\beta_2}v(1) = b_2 D_{0+}^{\beta_2}v(\eta_2), \end{cases}$$

where $\alpha_i \in (1, 2]$, $\beta_i \in (3, 4]$, $\eta_i \in (0, 1)$, and $b_i \in (0, \eta_i^{\frac{1-\alpha_i}{p_i-1}})$, $i = 1, 2$. The nonlinearities $f, g \in C([0, 1] \times [0, +\infty)^2, [0, +\infty))$. According to the positive parameters λ, μ , the existence of positive solutions is obtained by making use of the Guo-Krasnosel'skii fixed point theorem. The authors have considered classes of superlinear and sublinear functions f, g .

In [12], the authors considered the following fractional three-point BVP:

$$(1.3) \quad \begin{cases} -D_{0+}^{\alpha_1}(\varphi_p(D_{0+}^{\beta_1}u(t))) = f(t, u(t)), 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, D_{0+}^{\beta_1}u(0) = 0, D_{0+}^{\beta_1}u(1) = bD_{0+}^{\beta_1}u(\eta), \end{cases}$$

where $\alpha \in (1, 2]$, $\beta \in (3, 4]$, $\eta \in (0, 1)$, $b \in (0, \eta^{\frac{1-\alpha}{p-1}})$, and $f \in C([0, 1] \times [0, +\infty), \mathbf{R}^+)$. Existence of positive solutions is established by the upper and lower solutions method combined with the Schauder fixed point theorem.

In papers [4, 5], the case of a single equation is discussed in the ϕ -Laplacian setting and existence of positive solutions were obtained, using fixed point arguments. More recently, the same method was employed by the authors of [13] to discuss the three-point BVP (1.3).

The main objective of this work is twofold: first, we extend bvp (1.2) to the case of the ϕ -Laplacian operator and then we allow the nonlinearities f, g to have space singularities (degenerate source terms). In addition, the latter functions obey some general condition that include classical superlinearities considered in above cited works. The existence of positive solutions is obtained here by introducing suitable positive cones in the space of continuous functions. The case where no singularity exists (regular case) is first discussed in Section 3. Then the singular cases are investigated in Sections 4 and 5, according to the number of singularities taken into account. For this purpose, a convenient approximating fixed point operator is introduced. It aims at approaching the singular problem by a sequence of regular ones and then to pass to the limit. Regarding the hypotheses, quite general growth conditions on the nonlinear functions f and g are supposed. We have adopted a fixed point method which turned out to be a powerful one to locate solutions since multiple positive solutions are obtained as well. Finally each existence theorem is illustrated by means of an example of application which exhibits some types of nonlinearities covered by our existence results.

2. Preliminaries

In this section, we collect some definitions and basic results used throughout.

2.1. Lemmas

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : \mathbf{R}_*^+ \rightarrow \mathbf{R}$ is defined by

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $\mathbf{R}_*^+$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : \mathbf{R}_*^+ \rightarrow \mathbf{R}$ is defined by

$$D_{0+}^\alpha u(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where n is the smallest integer greater than or equal to α , provided the right-hand side is pointwise defined on $\mathbf{R}_*^+$.

Lemma 2.1. Let $\alpha > 0$. Then the following equality holds when $u \in L(0, 1)$ and $D_{0+}^\alpha u(t) \in L(0, 1)$:

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

where $c_1, c_2, \dots, c_n \in \mathbf{R}$ and $n-1 < \alpha \leq n$.

Definition 2.3. A nonempty subset $\mathcal{P}$ of a Banach space E is called a cone if it is convex, closed, and satisfies the conditions:

- (i) $\alpha x \in \mathcal{P}$, for all $x \in \mathcal{P}$ and $\alpha \geq 0$,
- (ii) $x, -x \in \mathcal{P}$ implies that $x = 0$.

Lemma 2.2. Let Ω be a bounded open set in a real Banach space E , $\mathcal{P}$ be a cone of E , and $A : \overline{\Omega} \cap \mathcal{P} \rightarrow \mathcal{P}$ be a completely continuous map. Suppose that $\lambda Ax \neq x$, $\forall x \in \partial\Omega \cap \mathcal{P}$, $\lambda \in (0, 1]$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 1$.

Lemma 2.3. Let Ω be a bounded open set in a real Banach space E , $\mathcal{P}$ be a cone of E and $A : \overline{\Omega} \cap \mathcal{P} \rightarrow \mathcal{P}$ be a completely continuous map. Suppose that $Ax \not\leq x$, $\forall x \in \partial\Omega \cap \mathcal{P}$. Then $i(A, \Omega \cap \mathcal{P}, \mathcal{P}) = 0$.

For more details on the theory and the computation of the fixed point index, we refer, e.g., to [1, 2, 3].

In order to transform problem (1.1) into an equivalent integral equation, we make the change $\phi_1(D_{0+}^{\beta_1}x(t)) = u(t)$ and then consider the bvp:

$$(2.1) \quad \begin{cases} -D_{0+}^{\alpha_1}u(t) = v(t), t \in (0, 1), \\ u(0) = 0, u(1) = \phi_1(b_1)u(\mu_1) \end{cases}$$

whose solution is given by the follwong

Lemma 2.4. *If $v \in C([0, 1])$, then problem (2.1) has a unique solution*

$$u(t) = \int_0^1 H_1(t, s)v(s)ds,$$

where

$$H_1(t, s) = h_1(t, s) + \frac{\phi_1(b_1)t^{\alpha_1-1}}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}}h_1(\mu_1, s)$$

and

$$h_1(t, s) = \frac{1}{\Gamma(\alpha_1)} \begin{cases} (t(1-s))^{\alpha_1-1}, & 0 \leq t \leq s \leq 1, \\ (t(1-s))^{\alpha_1-1} - (t-s)^{\alpha_1-1}, & 0 \leq s \leq t \leq 1. \end{cases}$$

As a consequence

$$(2.2) \quad \begin{cases} -D_{0+}^{\alpha_1}(\phi_1(D_{0+}^{\beta_1}x(t))) = v(t), t \in (0, 1) \\ x(0) = x(1) = x'(0) = x'(1) = 0, \\ D_{0+}^{\beta_1}x(0) = 0, D_{0+}^{\beta_1}x(1) = b_1D_{0+}^{\beta_1}x(\mu_1) \end{cases}$$

reads

$$\begin{cases} D_{0+}^{\beta_1}x(t) = \phi_1^{-1}(\int_0^1 H_1(t, s)v(s)ds), & t \in (0, 1) \\ x(0) = x(1) = x'(0) = x'(1) = 0. \end{cases}$$

We have

Lemma 2.5. *If $v \in C[0, 1]$, then problem (2.2) has a unique solution given by*

$$x(t) = \int_0^1 G_1(t, s)\phi_1^{-1}\left(\int_0^1 H_1(s, \tau)v(\tau)d\tau\right)ds,$$

where

$$G_1(t, s) = \frac{1}{\Gamma(\beta_1)} \begin{cases} t^{\beta_1-2}(1-s)^{\beta_1-2}[(s-t) \\ +(\beta_1-2)(1-t)s], & 0 \leq t \leq s \leq 1, \\ t^{\beta_1-2}(1-s)^{\beta_1-2}[(s-t) \\ +(\beta_1-2)(1-t)s] + (t-s)^{\beta_1-1}, & 0 \leq s \leq t \leq 1. \end{cases}$$

Proof. By Lemma 2.1

$$-I_{0+}^{\alpha_1} D_{0+}^{\alpha_1} u(t) = I_{0+}^{\alpha_1} v(t).$$

Then

$$u(t) = -I_{0+}^{\alpha_1} v(t) + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2}, \quad c_1, c_2 \in \mathbf{R}.$$

The boundary conditions $u(0) = 0$ and $u(1) = \phi_1(b_1)u(\mu_1)$ entail

$$c_1 = -\frac{\phi_1(b_1)I_{0+}^{\alpha_1} v(\mu_1) - I_{0+}^{\alpha_1} v(1)}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}} \quad \text{and} \quad c_2 = 0.$$

Hence the solution $u = u(t)$ of problem (2.1) satisfies

$$\begin{aligned} u(t) &= -I_{0+}^{\alpha_1} v(t) + \frac{t^{\alpha_1-1}}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}} I_{0+}^{\alpha_1} v(1) - \frac{\phi_1(b_1)t^{\alpha_1-1}}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}} I_{0+}^{\alpha_1} v(\mu_1) \\ &= -I_{0+}^{\alpha_1} v(t) + \frac{t^{\alpha_1-1}}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}} I_{0+}^{\alpha_1} v(1) - \frac{\phi_1(b_1)t^{\alpha_1-1}}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}} I_{0+}^{\alpha_1} v(\mu_1) \\ &\quad + t^{\alpha_1-1} I_{0+}^{\alpha_1} v(1) - t^{\alpha_1-1} I_{0+}^{\alpha_1} v(1) \\ &= t^{\alpha_1-1} I_{0+}^{\alpha_1} v(1) - I_{0+}^{\alpha_1} v(t) + \frac{\phi_1(b_1)t^{\alpha_1-1}}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}} \left(\mu_1^{\alpha_1-1} I_{0+}^{\alpha_1} v(1) - I_{0+}^{\alpha_1} v(\mu_1) \right) \\ &= \int_0^1 h_1(t, s) v(s) ds + \frac{\phi_1(b_1)t^{\alpha_1-1}}{1 - \phi_1(b_1)\mu_1^{\alpha_1-1}} \int_0^1 h_1(\mu_1, s) v(s) ds. \end{aligned}$$

Similarly by Lemma 2.1, we have

$$-I_{0+}^{\beta_1} D_{0+}^{\beta_1} x(t) = I_{0+}^{\beta_1} \phi_1^{-1} \left(\int_0^1 H_1(t, s) v(s) ds \right).$$

Then

$$x(t) = -I_{0+}^{\beta_1} \phi_1^{-1} \left(\int_0^1 H_1(t, s) v(s) ds \right) + c_1 t^{\beta_1-1} + c_2 t^{\beta_1-2} + c_3 t^{\beta_1-3} + c_4 t^{\beta_1-4},$$

where $c_1, c_2, c_3, c_4 \in \mathbf{R}$. Applying the boundary conditions $x(0) = x(1) = x'(0) = x'(1) = 0$, we get

$$\begin{aligned} c_1 &= -\frac{1}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1-1} \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\ &\quad - \frac{\beta_1-1}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1-2} s \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\ c_2 &= \frac{\beta_1-1}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1-2} s \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds, \\ c_3 = c_4 &= 0. \end{aligned}$$

Therefore, the solution $x(t)$ of problem (2.2) is given by

$$\begin{aligned}
 x(t) &= \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1-1} \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &\quad - \frac{t^{\beta_1-1}}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1-1} \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &\quad - \frac{t^{\beta_1-1}(\beta_1-1)}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1-2} s \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &\quad + \frac{t^{\beta_1-2}(\beta_1-1)}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1-2} s \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &= \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1-1} \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &\quad + \frac{t^{\beta_1-2}(1-s)^{\beta_1-2}}{\Gamma(\beta_1)} \int_0^1 [(\beta_1-1)s(1-t) - t(1-s)] \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &= \frac{1}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1-1} \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &\quad + \frac{t^{\beta_1-2}(1-s)^{\beta_1-2}}{\Gamma(\beta_1)} \int_0^1 [(s-t) + (\beta_1-2)(1-t)s] \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds \\
 &= \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) v(\tau) d\tau \right) ds.
 \end{aligned}$$

□

Lemma 2.6. The functions $G_1(t, s), H_1(t, s)$ enjoy the following properties:

- (a) G_1, H_1 are continuous on $[0, 1] \times [0, 1]$,
- (b) $\frac{(\beta_1-2)K_1(t)L_1(s)}{\Gamma(\beta_1)} \leq G_1(t, s) \leq \frac{M_1}{\Gamma(\beta_1)}, \forall (t, s) \in (0, 1) \times (0, 1)$, where

$$K_1(t) = t^{\beta_1-2}(1-t)^2, L_1(s) = s^2(1-s)^{\beta_1-2}, M_1 = \max\{\beta_1-1, (\beta_1-2)^2\}$$

- (c) $G_1(t, s) \geq \frac{\beta_1-2}{M_1} t^2(1-t)^2 \sup_{t \in [0,1]} G_1(t, s), \forall (t, s) \in (0, 1) \times (0, 1)$,

- (d) $H_1(t, s) \leq \lambda_1(1-s)^{\alpha_1-1}, \forall (t, s) \in (0, 1) \times (0, 1)$, where

$$\lambda_1 = \frac{1}{\left[1 - \phi_1(b_1)\mu_1^{\alpha_1-1}\right] \Gamma(\alpha_1)}.$$

Proof. (b) In [6], the authors proved that

$$(\beta_1-2)K_1(t)L_1(s) \leq \Gamma(\beta_1)G_1(t, s) \leq M_1L_1(s), \forall (t, s) \in (0, 1) \times (0, 1).$$

(c) Let $t \in (0, 1)$, we have

$$\begin{aligned} G_1(t, s) &\geq \frac{(\beta_1-2)K_1(t)L_1(s)}{\Gamma(\beta_1)} \\ &\geq \frac{(\beta_1-2)K_1(t)M_1L_1(s)}{M_1\Gamma(\beta_1)} \\ &\geq \frac{(\beta_1-2)t^2(1-t)^2M_1L_1(s)}{M_1\Gamma(\beta_1)}. \end{aligned}$$

Furthermore,

$$\sup_{t \in [0,1]} G_1(t, s) \leq \frac{M_1L_1(s)}{\Gamma(\beta_1)},$$

whence comes the result.

(d) We have

$$h_1(t, s) \leq \frac{1}{\Gamma(\alpha_1)}[t(1-s)]^{\alpha_1-1} \leq \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}.$$

Then

$$\begin{aligned} H_1(t, s) &\leq \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} + \frac{\phi_1(b_1)}{1-\phi_1(b_1)\mu_1^{\alpha_1-1}} \frac{\mu_1^{\alpha_1-1}(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} \\ &\leq \frac{1}{1-\phi_1(b_1)\mu_1^{\alpha_1-1}} \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} \\ &\leq \lambda_1(1-s)^{\alpha_1-1}. \end{aligned}$$

□

Similar results are obtained for the following BVP, by replacing α_1 by α_2 and β_1 by β_2 :

$$(2.3) \quad \begin{cases} -D_{0+}^{\alpha_2}(\phi_2(D_{0+}^{\beta_2}y(t))) = v(t), t \in (0, 1) \\ y(0) = y(1) = y'(0) = y'(1) = 0, \\ D_{0+}^{\beta_2}y(0) = 0, D_{0+}^{\beta_2}y(1) = b_2D_{0+}^{\beta_2}y(\mu_2). \end{cases}$$

To complete the functional setting, we consider the Banach space $E = \{(x, y) \in C([0, 1] \times [0, 1])\}$ with the norm $\|(x, y)\| = \|x\| + \|y\|$, where $\|x\| = \sup_{t \in [0,1]} |x(t)|$. Finally, let the cone $\mathcal{P}$ be given by

$$\mathcal{P} = \{(x, y) \in E : x(t) \geq \frac{\beta_1-2}{M_1}\gamma(t)\|x\|, y(t) \geq \frac{\beta_2-2}{M_2}\gamma(t)\|y\|, \forall t \in [0, 1]\},$$

where $\gamma(t) = t^2(1-t)^2$, $t \in [0, 1]$.

3. Regular nonlinearities

In this section, we suppose that the functions $f, g : [0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ are continuous and satisfy

($\mathcal{H}_1$) There exist nonnegative real-valued functions $m_i \in L([0, 1])$ and $\psi_i \in C(\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}^+)$ such that ψ_i are nondecreasing functions in each argument (for $i \in \{0, 1\}$), with

$$(3.1) \quad f(t, x, y) \leq m_1(t)\psi_1(x, y), \quad \forall (t, x, y) \in [0, 1] \times (\mathbf{R}^+)^2$$

$$(3.2) \quad g(t, x, y) \leq m_2(t)\psi_2(x, y), \quad \forall (t, x, y) \in [0, 1] \times (\mathbf{R}^+)^2$$

with

$$\int_0^1 (1 - \tau)^{\alpha_i - 1} a_i(\tau) m_i(\tau) d\tau < +\infty, \quad i \in \{0, 1\}.$$

($\mathcal{H}_2$)

$$\sup_{c>0} \frac{c}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi_i^{-1} \left(\lambda_i \psi_i(c, c) \int_0^1 (1 - \tau)^{\alpha_i - 1} a_i(\tau) m_i(\tau) d\tau \right)} > 1.$$

($\mathcal{H}_3$)

There exists $\sigma \in (0, \frac{1}{2})$ such that

$$\lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{f(t, x, y)}{\phi_1(x+y)} \geq l_1 \quad \text{and} \quad \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{g(t, x, y)}{\phi_2(x+y)} \geq l_2,$$

where

$$l_i = 1 + \frac{\phi_i \left(\frac{\Gamma(\beta_i)}{2c_i(1-2\sigma)^2(\beta_i-2)} \right)}{\int_{\sigma}^{1-\sigma} a_i(\tau) H_i(d_i, \tau) d\tau}, \quad i \in \{1, 2\},$$

such that

$$c_i = \min_{(t,s) \in ([\sigma, 1-\sigma])^2} K_i(t) L_i(s), \quad H_i(d_i, \tau) = \min_{s \in [\sigma, 1-\sigma]} H_i(s, \tau), \quad \min_{t \in [\sigma, 1-\sigma]} a_i(\tau) \neq 0.$$

($\mathcal{H}_4$)

$$\lim_{x+y \rightarrow 0} \inf_{t \in [\sigma, 1-\sigma]} \frac{f(t, x, y)}{\phi_1(x+y)} \geq N_1 \quad \text{and} \quad \lim_{x+y \rightarrow 0} \inf_{t \in [\sigma, 1-\sigma]} \frac{g(t, x, y)}{\phi_2(x+y)} \geq N_2,$$

where

$$N_i = 1 + \frac{\phi_i(\frac{\Gamma(\beta_i)}{2Nc_i(1-2\sigma)^2(\beta_i-2)})}{\int_{\sigma}^{1-\sigma} a_i(\tau)\phi_i(\gamma(\tau))H_i(d_i, \tau)d\tau}, \quad i \in \{1, 2\}.$$

Let $N = \max\{\frac{\beta_1-2}{M_1}, \frac{\beta_2-2}{M_2}\}$.

For $(x, y) \in \mathcal{P}$, define operators $A, B : E \longrightarrow C([0, 1])$ by

$$A(x, y)(t) = \int_0^1 G_1(t, s)\phi_1^{-1} \left(\int_0^1 H_1(s, \tau)a_1(\tau)f(\tau, x(\tau), y(\tau))d\tau \right) ds,$$

$$B(x, y)(t) = \int_0^1 G_2(t, s)\phi_2^{-1} \left(\int_0^1 H_2(s, \tau)a_2(\tau)g(\tau, x(\tau), y(\tau))d\tau \right) ds,$$

and then

$$T(x, y) = (A(x, y), B(x, y)).$$

Proposition 3.1. Assume that condition $(\mathcal{H}_1)$ holds. Then $T : \mathcal{P} \longrightarrow \mathcal{P}$ is a completely continuous operator.

Proof. **Step 1:** We prove that for $(x, y) \in \mathcal{P}$, $T(x, y) \in \mathcal{P}$. We have $A(x, y)(t) \geq 0, B(x, y)(t) \geq 0; \forall t \in [0, 1]$, and by (c) of Lemma 2.6

$$\begin{aligned} A(x, y)(t) &= \int_0^1 G_1(t, s)\phi_1^{-1} \left(\int_0^1 H_1(s, \tau)a_1(\tau)f(\tau, x(\tau), y(\tau))d\tau \right) ds \\ &\geq \frac{\beta_1-2}{M_1}\gamma(t) \int_0^1 \sup_{t \in [0, 1]} G_1(t, s)\phi_1^{-1} \left(\int_0^1 H_1(s, \tau)a_1(\tau)f(\tau, x(\tau), y(\tau))d\tau \right) ds \\ &\geq \frac{\beta_1-2}{M_1}\gamma(t) \sup_{t \in [0, 1]} \int_0^1 G_1(t, s)\phi_1^{-1} \left(\int_0^1 H_1(s, \tau)a_1(\tau)f(\tau, x(\tau), y(\tau))d\tau \right) ds \\ &\geq \frac{\beta_1-2}{M_1}\gamma(t)\|A(x, y)\|. \end{aligned}$$

Similarly, by (c) of Lemma 2.6, we have

$$\begin{aligned} B(x, y)(t) &= \int_0^1 G_2(t, s)\phi_2^{-1} \left(\int_0^1 H_2(s, \tau)a_2(\tau)g(\tau, x(\tau), y(\tau))d\tau \right) ds \\ &\geq \frac{\beta_2-2}{M_2}\gamma(t) \int_0^1 \sup_{t \in [0, 1]} G_2(t, s)\phi_2^{-1} \left(\int_0^1 H_2(s, \tau)a_2(\tau)g(\tau, x(\tau), y(\tau))d\tau \right) ds \\ &\geq \frac{\beta_2-2}{M_2}\gamma(t) \sup_{t \in [0, 1]} \int_0^1 G_2(t, s)\phi_2^{-1} \left(\int_0^1 H_2(s, \tau)a_2(\tau)g(\tau, x(\tau), y(\tau))d\tau \right) ds \\ &\geq \frac{\beta_2-2}{M_2}\gamma(t)\|B(x, y)\|. \end{aligned}$$

Hence $T(\mathcal{P}) \subset \mathcal{P}$.

Step 2: Let D is a bounded set; then there exists $r > 0$ such that

$$\forall (x, y) \in D, \|(x, y)\| \leq r.$$

By $(\mathcal{H}_1)$ and the properties (b) and (d) of Lemma 2.6, we have

$$\begin{aligned} \|A(x, y)\| &= \left\| \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds \right\| \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \int_0^1 \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\int_0^1 \lambda_1 (1 - \tau)^{\alpha_1 - 1} a_1(\tau) m_1(\tau) \psi_1(x(\tau), y(\tau)) d\tau \right) \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \psi_1(r, r) \int_0^1 (1 - \tau)^{\alpha_1 - 1} a_1(\tau) m_1(\tau) d\tau \right) < +\infty. \end{aligned}$$

Similarly by $(\mathcal{H}_1)$ and the properties (b) and (d) of Lemma 2.6, we have

$$\|B(x, y)\| \leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \psi_2(r, r) \int_0^1 (1 - \tau)^{\alpha_2 - 1} a_2(\tau) m_2(\tau) d\tau \right) < +\infty.$$

Step 3: Given $t, t' \in [0, 1]$ ($t < t'$), we have

$$\begin{aligned} &|A(x, y)(t) - A(x, y)(t')| \\ &\leq \int_0^1 |G_1(t, s) - G_1(t', s)| \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f(\tau, x(\tau), y(\tau)) d\tau \right) ds. \end{aligned}$$

Similarly,

$$\begin{aligned} &|B(x, y)(t) - B(x, y)(t')| \\ &\leq \int_0^1 |G_2(t, s) - G_2(t', s)| \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g(\tau, x(\tau), y(\tau)) d\tau \right) ds. \end{aligned}$$

By the continuity of G_1, G_2 , and appealing to Arzela-Ascoli Lemma, we deduce that T is completely continuous. $\square$

Theorem 3.1. Assume that conditions $(\mathcal{H}_1) - (\mathcal{H}_4)$ hold. Then bvp (1.1) has at least two positive solutions.

Proof. From condition $(\mathcal{H}_2)$, there exists $R_2 > 0$, such that

$$(3.3) \quad \frac{R_2}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi_i^{-1} \left(\lambda_i \psi_i(R_2, R_2) \int_0^1 (1 - \tau)^{\alpha_i - 1} a_i(\tau) m_i(\tau) d\tau \right)} > 1.$$

And from condition $(\mathcal{H}_4)$ there exists $r > 0$ with $r < R_2$ such that $f(t, x, y) \geq N_1 \phi_1(x + y)$ with $x + y \leq r$, for all $t \in [\sigma, 1 - \sigma]$ and there exists $r' > 0$ such that $r' < R_2$ and $g(t, x, y) \geq N_2 \phi_2(x + y)$ with $x + y \leq r'$, for all $t \in [\sigma, 1 - \sigma]$.

Step 1. Define the open ball

$$\Omega_1 = \{(x, y) \in E : \|(x, y)\| \leq R_1\},$$

where $R_1 = \min\{r, r'\}$. We show that $T(x, y) \not\leq (x, y)$, for all $(x, y) \in \partial\Omega_1 \cap \mathcal{P}$. Suppose on the contrary that there exists $(x_0, y_0) \in \partial\Omega_1 \cap \mathcal{P}$ such that $T(x_0, y_0) \leq (x_0, y_0)$. By $(\mathcal{H}_4)$ and (b) of Lemma 2.6, we have the estimates

$$\begin{aligned} x_0(t) &\geq A(x_0(t), y_0(t)) \\ &\geq \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_{\sigma}^{1-\sigma} a_1(\tau) H_1(s, \tau) \phi_1(x_0(\tau) + y_0(\tau)) N_1 d\tau \right) ds \\ &\geq N R_1 \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_{\sigma}^{1-\sigma} a_1(\tau) \phi_1(\gamma(\tau)) H_1(d_1, \tau) N_1 d\tau \right) ds \\ &\geq N R_1 \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_{\sigma}^{1-\sigma} a_1(\tau) \phi_1(\gamma(\tau)) H_1(d_1, \tau) \right. \\ &\quad \left. + \phi_1 \left(\frac{\Gamma(\beta_1)}{2N_{c1}(1-2\sigma)^2(\beta_1-2)} \right) d\tau \right) ds \\ &> N R_1 \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_{\sigma}^{1-\sigma} \phi_1 \left(\frac{\Gamma(\beta_1)}{2N_{c1}(1-2\sigma)^2(\beta_1-2)} \right) d\tau \right) ds \\ &> \frac{R_1}{2}. \end{aligned}$$

Similarly, by $(\mathcal{H}_4)$ and (b) of Lemma 2.6, we get

$$\begin{aligned} y_0(t) &\geq B(x_0(t), y_0(t)) \\ &\geq \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} a_2(\tau) H_2(s, \tau) \phi_2(x_0(\tau) + y_0(\tau)) N_2 d\tau \right) ds \\ &\geq N R_1 \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} a_2(\tau) \phi_2(\gamma(\tau)) H_2(d_2, \tau) N_2 d\tau \right) ds \\ &\geq N R_1 \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} a_2(\tau) \phi_2(\gamma(\tau)) H_2(d_2, \tau) \right. \\ &\quad \left. + \phi_2 \left(\frac{\Gamma(\beta_2)}{2N_{c2}(1-2\sigma)^2(\beta_2-2)} \right) d\tau \right) ds \\ &> N R_1 \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} \phi_2 \left(\frac{\Gamma(\beta_2)}{2N_{c2}(1-2\sigma)^2(\beta_2-2)} \right) d\tau \right) ds \\ &> \frac{R_1}{2}. \end{aligned}$$

Hence

$$\|(x_0, y_0)\| > R_1,$$

contradicting $\|(x_0, y_0)\| = R_1$. Using Lemma 2.3, we conclude that

$$i(T, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 0.$$

Step 2. Let

$$\Omega_2 = \{(x, y) \in E : \|(x, y)\| < R_2\}.$$

We claim that $(x, y) \neq \lambda T(x, y)$ for all $(x, y) \in \partial\Omega_2 \cap \mathcal{P}$ and $\lambda \in (0, 1]$. On the contrary, suppose that there exist $(x_0, y_0) \in \partial\Omega_2 \cap \mathcal{P}$ and $\lambda_0 \in (0, 1]$ such that $(x_0, y_0) = \lambda_0 T(x_0, y_0)$. Then

$$x_0 = \lambda_0 A(x_0, y_0) \text{ and } y_0 = \lambda_0 B(x_0, y_0).$$

By $(\mathcal{H}_1)$ and the properties (b) and (d) of Lemma 2.6, we have

$$\begin{aligned}
 \|x_0\| &= \|\lambda_0 A(x_0, y_0)\| \\
 &= \lambda_0 \left\| \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f(\tau, x_0(\tau), y_0(\tau)) d\tau \right) ds \right\| \\
 &\leq \lambda_0 \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\int_0^1 a_1(\tau) m_1(\tau) \lambda_1 (1 - \tau)^{\alpha_1 - 1} \psi_1(x_0(\tau), y_0(\tau)) d\tau \right) \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1 - 1} \psi_1(R_2, R_2) d\tau \right) \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \psi_1(R_2, R_2) \int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1 - 1} d\tau \right)
 \end{aligned}$$

and

$$\begin{aligned}
 \|y_0\| &= \|\lambda_0 B(x_0, y_0)\| \\
 &= \lambda_0 \left\| \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 H_2(s, \tau) a_2(\tau) g(\tau, x_0(\tau), y_0(\tau)) d\tau \right) ds \right\| \\
 &\leq \lambda_0 \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\int_0^1 a_2(\tau) m_2(\tau) \lambda_2 (1 - \tau)^{\alpha_2 - 1} \psi_2(x_0(\tau), y_0(\tau)) d\tau \right) \\
 &\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \int_0^1 a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} \psi_2(R_2, R_2) d\tau \right) \\
 &\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \psi_2(R_2, R_2) \int_0^1 a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} d\tau \right).
 \end{aligned}$$

We conclude that

$$\begin{aligned}
 R_2 = \|(x_0, y_0)\| &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \psi_1(R_2, R_2) \int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1 - 1} d\tau \right) \\
 &\quad + \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \psi_2(R_2, R_2) \int_0^1 a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} d\tau \right),
 \end{aligned}$$

which is a contradiction to (3.3). Using Lemma 2.2, we deduce that

$$(3.4) \quad i(A, \Omega_2 \cap \mathcal{P}, \mathcal{P}) = 1.$$

Therefore

$$i(T, (\Omega_2 \setminus \overline{\Omega_1}) \cap \mathcal{P}, \mathcal{P}) = 1.$$

Then T has a least one fixed point (x_0, y_0) such that $R_1 < \|(x_0, y_0)\| < R_2$.

Step 3. From $(\mathcal{H}_3)$, there exists $R > 0$ such that $R > R_2$ and $f(t, x, y) \geq l_1 \phi_1(x + y)$ with $x + y \geq R$, for all $t \in [\sigma, 1 - \sigma]$ and there exists $R' > 0$ such that $R' > R_2$ and $g(t, x, y) \geq l_2 \phi_2(x + y)$ with $x + y \geq R'$, for all $t \in [\sigma, 1 - \sigma]$. Define the open ball

$$\Omega_3 = \{(x, y) \in E : \|(x, y)\| \leq R_3\},$$

where $R_3 = \max\{R, R'\}$. To show that $T(x, y)(x, y)$ for all $(x, y) \in \partial\Omega_3 \cap \mathcal{P}$, suppose on the contrary that there exists $(x_1, y_1) \in \partial\Omega_3 \cap \mathcal{P}$ such that

$T(x_1, y_1) \leq (x_1, y_1)$. Using $(\mathcal{H}_3)$ and (b) of Lemma 2.6, we find

$$\begin{aligned} x_1(t) &\geq A(x_1(t), y_1(t)) \\ &\geq \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) \phi_1(x_1(\tau) + y_1(\tau)) l_1 d\tau \right) ds \\ &\geq R_3 \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_{\sigma}^{1-\sigma} a_1(\tau) H_1(d_1, \tau) l_1 d\tau \right) ds \\ &\geq R_3 \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_{\sigma}^{1-\sigma} a_1(\tau) H_1(d_1, \tau) \right. \\ &\quad \left. + \phi_1 \left(\frac{\Gamma(\beta_1)}{2c_1(1-2\sigma)^2(\beta_1-2)} \right) \right) d\tau ds \\ &> R_3 \frac{\beta_1-2}{\Gamma(\beta_1)} K_1(t) \int_{\sigma}^{1-\sigma} L_1(s) \phi_1^{-1} \left(\int_{\sigma}^{1-\sigma} \phi_1 \left(\frac{\Gamma(\beta_1)}{2c_1(1-2\sigma)^2(\beta_1-2)} \right) \right) d\tau ds \\ &> \frac{R_3}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} y_1(t) &\geq B(x_1(t), y_1(t)) \\ &\geq \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} a_2(\tau) H_2(s, \tau) \phi_2(x_0(\tau) + y_0(\tau)) l_2 d\tau \right) ds. \\ &\geq R_3 \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} a_2(\tau) H_2(d_2, \tau) l_2 d\tau \right) ds. \\ &\geq R_3 \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} a_2(\tau) H_2(d_2, \tau) \right. \\ &\quad \left. + \phi_2 \left(\frac{\Gamma(\beta_2)}{2c_2(1-2\sigma)^2(\beta_2-2)} \right) \right) d\tau ds \\ &> R_3 \frac{\beta_2-2}{\Gamma(\beta_2)} K_2(t) \int_{\sigma}^{1-\sigma} L_2(s) \phi_2^{-1} \left(\int_{\sigma}^{1-\sigma} \phi_2 \left(\frac{\Gamma(\beta_2)}{2c_2(1-2\sigma)^2(\beta_2-2)} \right) \right) d\tau ds \\ &> \frac{R_3}{2}. \end{aligned}$$

Consequently

$$\|(x_1, y_1)\| > R_3,$$

contradicting $\|(x_1, y_1)\| = R_3$. Making use of Lemma 2.3, we conclude that

$$i(T, \Omega_3 \cap \mathcal{P}, \mathcal{P}) = 0. \text{ and } i(T, (\Omega_3 \setminus \overline{\Omega_2}) \cap \mathcal{P}, \mathcal{P}) = -1.$$

Finally, operator T has a fixed point (x_1^*, y_1^*) such that $R_2 < \|(x_1^*, y_1^*)\| < R_3$. Since $R_1 < \|(x_0, y_0)\| < R_2 < \|(x_1^*, y_1^*)\| < R_3$, then T has two positive fixed points. $\square$

Example 3.1. Consider the bvp:

$$\begin{cases} -D_{0+}^{\frac{3}{2}} \left(\left(D_{0+}^{\frac{7}{2}} x(t) \right)^p \right) = \frac{12\delta^p}{10} \frac{6^p(1-2^{-p})\sqrt{\pi}}{5^p} (1-t)^{\frac{3}{2}} (t+1) (x^q + y^q + 1)^{\frac{1}{p}}, \\ -D_{0+}^{\frac{7}{4}} \left(\left(D_{0+}^{\frac{15}{4}} y(t) \right)^p \right) = \left(\frac{64\delta}{49} \right)^p \frac{3}{4} (1-2^{-2p}) \sqrt{\pi} (1-t)^{\frac{1}{4}} (t+1) ((x+y)^q + 1)^{\frac{1}{p}}, \\ x(0) = x(1) = x'(0) = x'(1) = 0, D_{0+}^{\frac{7}{2}} x(0) = 0, D_{0+}^{\frac{7}{2}} x(1) = 2^{\frac{1}{2p}-1} D_{0+}^{\frac{7}{2}} x\left(\frac{1}{2}\right), \\ y(0) = y(1) = y'(0) = y'(1) = 0, D_{0+}^{\frac{15}{4}} y(0) = 0, D_{0+}^{\frac{15}{4}} y(1) = 2^{\frac{3}{2p}-2} D_{0+}^{\frac{15}{4}} y\left(\frac{1}{4}\right), \end{cases} \quad (3.5)$$

where

$$f(t, x, y) = \frac{6^p(1-2^{-p})\sqrt{\pi}}{5^p}(t+1)(x^q + y^q + 1)^{\frac{1}{p}},$$

$$g(t, x, y) = \frac{3}{4}(1-2^{-2p})\sqrt{\pi}(t+1)((x+y)^q + 1)^{\frac{1}{p}},$$

and

$$a_1(t) = \frac{12\delta^p}{10}(1-t)^{\frac{3}{2}}, a_2(t) = \left(\frac{64\delta}{49}\right)^p(1-t)^{\frac{1}{4}} \quad (\delta > 0).$$

p is an odd number, $q > p^2$, and $\phi_1(t) = \phi_2(t) = t^p$ is the p -Laplacian. As a consequence, for $i \in \{1, 2\}$, ϕ_i are continuous, increasing, homeomorphism, multiplicative and $\phi_i(0) = 0$.

Let

$$\psi_1(x, y) = (x^q + y^q + 1)^{\frac{1}{p}}, \quad \psi_2(x, y) = ((x+y)^q + 1)^{\frac{1}{p}}$$

and

$$m_1(t) = \frac{6^p(1-2^{-p})\sqrt{\pi}}{5^p}(t+1), \quad m_2(t) = \frac{3}{4}(1-2^{-2p})\sqrt{\pi}(t+1).$$

Thus Hypothesis $(\mathcal{H}_1)$ is easily checked. Moreover, we can verify the rest of hypotheses:

$(\mathcal{H}_2)$

$$\begin{aligned} & \sup_{c>0} \frac{c}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi^{-1} \left(\lambda_i \psi_i(c, c) \int_0^1 (1-\tau)^{\alpha_i-1} a_i(\tau) m_i(\tau) d\tau \right)} \\ & > \sup_{c>0} \frac{c}{\delta \left((2c^q+1)^{\frac{1}{p^2}} + 2(2^q c^q+1)^{\frac{1}{p^2}} \right)}. \end{aligned}$$

$(\mathcal{H}_3)$

$$\min_{t \in [\sigma, 1-\sigma]} a_1(t) = \min_{t \in [\sigma, 1-\sigma]} \frac{12\delta^p}{10}(1-t)^{\frac{3}{2}} = \frac{12\delta^p}{10}\sigma^{\frac{3}{2}} \neq 0,$$

$$\begin{aligned} & \min_{t \in [\sigma, 1-\sigma]} a_2(t) \\ & = \min_{t \in [\sigma, 1-\sigma]} \left(\frac{64\delta}{49}\right)^p(1-t)^{\frac{1}{4}} = \left(\frac{64\delta}{49}\right)^p\sigma^{\frac{1}{4}} \neq 0, \end{aligned}$$

and

$$\begin{aligned} & \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{f(t, x, y)}{\phi_1(x+y)} \\ & = \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{\frac{6^p(1-2^{-p})\sqrt{\pi}}{5^p}(t+1)(x^q + y^q + 1)^{\frac{1}{p}}}{(x+y)^p} = +\infty, \end{aligned}$$

$$\lim_{x+y \rightarrow \infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{g(t, x, y)}{\phi_2(x+y)} \\ = \lim_{x+y \rightarrow \infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{\frac{3}{4}(1-2^{-2p})\sqrt{\pi}(t+1)((x+y)^q+1)^{\frac{1}{p}}}{(x+y)^p} = +\infty.$$

($\mathcal{H}_4$)

$$\lim_{x+y \rightarrow 0} \inf_{t \in [\sigma, 1-\sigma]} \frac{f(t, x, y)}{\phi_1(x+y)} \\ = \lim_{x+y \rightarrow 0} \inf_{t \in [\sigma, 1-\sigma]} \frac{\frac{6^p(1-2^{-p})\sqrt{\pi}}{5^p}(t+1)(x^q+y^q+1)^{\frac{1}{p}}}{(x+y)^p} = +\infty,$$

and

$$\lim_{x+y \rightarrow 0} \inf_{t \in [\sigma, 1-\sigma]} \frac{g(t, x, y)}{\phi_2(x+y)} \\ = \lim_{x+y \rightarrow 0} \inf_{t \in [\sigma, 1-\sigma]} \frac{\frac{3}{4}(1-2^{-2p})\sqrt{\pi}(t+1)((x+y)^q+1)^{\frac{1}{p}}}{(x+y)^p} = +\infty.$$

Finally we can choose δ such that $0 < \delta < \sup_{c>0} \frac{c}{\left((2c^q+1)^{\frac{1}{p^2}}+2(2^q c^q+1)^{\frac{1}{p^2}}\right)}$.

Then the hypotheses of Theorem 3.1 are satisfied and problem (3.5) has at least two positive solutions.

4. Nonlinearities with one singularity

In this section, we suppose that the nonlinear functions $f, g : [0, 1] \times \mathbf{R}_*^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ are continuous and satisfy

($\mathcal{A}_1$) For $i \in \{0, 1\}$, there exist $m_i \in C([0, 1], \mathbf{R}^+)$, $\psi_i \in C(\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}^+)$ increasing functions in each argument, and $q_i, p_i \in C(\mathbf{R}_*^+, \mathbf{R}^+)$ such that p_i are decreasing functions and $\frac{q_i}{p_i}$ are increasing functions with

$$(4.1) \quad \begin{aligned} f(t, x, y) &\leq m_1(t)q_1(x)\psi_1(x, y), & \forall (t, x, y) \in [0, 1] \times \mathbf{R}_*^+ \times \mathbf{R}^+, \\ g(t, x, y) &\leq m_2(t)q_2(x)\psi_2(x, y), & \forall (t, x, y) \in [0, 1] \times \mathbf{R}_*^+ \times \mathbf{R}^+, \end{aligned}$$

and for each $c > 0$,

$$(4.2) \quad \int_0^1 (1-\tau)^{\alpha_1-1} a_i(\tau) m_i(\tau) p_i(c\gamma(\tau)) d\tau < +\infty.$$

(A₂) For all $0 < \omega < 1$,

$$\sup_{c>0} \frac{c}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi_i^{-1} \left(\lambda_i \frac{q_i(c)}{p_i(c)} \psi_i(c, c) \int_0^1 (1-\tau)^{\alpha_i-1} a_i(\tau) m_i(\tau) p_i(\omega\gamma(\tau)) d\tau \right)} > 1.$$

(A₃) There exists $\sigma \in (0, \frac{1}{2})$ such that

$$\lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{f(t, x, y)}{\phi_1(x+y)} \geq l_1$$

and

$$\lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{g(t, x, y)}{\phi_2(x+y)} \geq l_2,$$

where

$$l_i = 1 + \frac{\phi_i\left(\frac{\Gamma(\beta_i)}{2c_i(1-2\sigma)^2(\beta_i-2)}\right)}{\int_{\sigma}^{1-\sigma} a_i(\tau) H_i(d_i, \tau) d\tau}, \quad i \in \{1, 2\},$$

$$c_i = \min_{(t,s) \in ([\sigma, 1-\sigma])^2} K_i(t) L_i(s), \quad \min_{t \in [\sigma, 1-\sigma]} a_i(\tau) \neq 0 \text{ and } H_i(d_i, \tau) = \min_{s \in [\sigma, 1-\sigma]} H_i(s, \tau).$$

(A₄) For every $c > 0$ and $\frac{1}{4} < \theta < \frac{1}{2}$, there exist $\rho_c, \rho_c^* \in C([0, 1], \mathbf{R}^+)$ such that $\rho_c(t) > 0, \rho_c^*(t) > 0$ on $(\theta - \frac{1}{4}, \theta)$ and

$$\begin{aligned} f(t, x, y) &\geq \rho_c(t), \quad \forall t \in [0, 1], \forall x, y \in (0, c], \\ g(t, x, y) &\geq \rho_c^*(t), \quad \forall t \in [0, 1], \forall x, y \in (0, c]. \end{aligned}$$

Define two sequences of approximating functions $\{f_n\}_{n \geq 1}$ and $\{g_n\}_{n \geq 1}$ by

$$\begin{aligned} f_n(t, x, y) &= f(t, \max\{\frac{1}{n}, x\}, y), \quad n \in \{1, 2, \dots\} \\ g_n(t, x, y) &= g(t, \max\{\frac{1}{n}, x\}, y), \quad n \in \{1, 2, \dots\} \end{aligned}$$

for $(t, x, y) \in (0, 1) \times \mathcal{P}$ and define the sequences of operators:

$$\begin{aligned} A_n(x, y)(t) &= \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f_n(\tau, x(\tau), y(\tau)) d\tau \right) ds, \\ B_n(x, y)(t) &= \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 H_2(s, \tau) a_2(\tau) g_n(\tau, x(\tau), y(\tau)) d\tau \right) ds, \\ T_n(x, y) &= (A_n(x, y), B_n(x, y)). \end{aligned}$$

We have,

Proposition 4.1. Assume that condition (A₁) holds. Then for each $n \geq 1$, the operator $T_n : P \longrightarrow P$ is completely continuous.

Proof. Step 1: We claim that for $(x, y) \in \mathcal{P}$ we have $T_n(x, y) \in \mathcal{P}$. We have $A_n(x, y)(t) \geq 0$ and $B_n(x, y)(t) \geq 0$; $\forall t \in [0, 1]$. By (c) of Lemma 2.6, we have

$$\begin{aligned} A_n(x, y)(t) &= \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f_n(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &\geq \frac{\beta_1-2}{M_1} \gamma(t) \int_0^1 \sup_{t \in [0, 1]} G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f_n(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &\geq \frac{\beta_1-2}{M_1} \gamma(t) \sup_{t \in [0, 1]} \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f_n(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &\geq \frac{\beta_1-2}{M_1} \gamma(t) \|A_n(x, y)\|. \end{aligned}$$

Similarly we have by (c) of Lemma 2.6

$$\begin{aligned} B_n(x, y)(t) &= \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 H_2(s, \tau) a_2(\tau) g_n(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &\geq \frac{\beta_2-2}{M_2} \gamma(t) \int_0^1 \sup_{t \in [0, 1]} G_2(t, s) \phi_2^{-1} \left(\int_0^1 H_2(s, \tau) a_2(\tau) g_n(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &\geq \frac{\beta_2-2}{M_2} \gamma(t) \sup_{t \in [0, 1]} \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 H_2(s, \tau) a_2(\tau) g_n(\tau, x(\tau), y(\tau)) d\tau \right) ds \\ &\geq \frac{\beta_2-2}{M_2} \gamma(t) \|B_n(x, y)\|. \end{aligned}$$

Hence $T_n(\mathcal{P}) \subset \mathcal{P}$.

Step 2: Let D is a bounded set, then there exists $r > 1$ such that

$$\forall (x, y) \in D, \|(x, y)\| \leq r.$$

Hypothesis $(\mathcal{A}_1)$ and the properties (b) and (d) of Lemma 2.6 lead to the following estimates:

$$\begin{aligned} \|A_n(x, y)\| &= \left\| \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f_n(\tau, x(\tau), y(\tau)) d\tau \right) ds \right\| \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\int_0^1 a_1(\tau) \lambda_1 (1 - \tau)^{\alpha_1-1} f(\tau, \max\{\frac{1}{n}, x(\tau)\}, y(\tau)) d\tau \right) \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1-1} m_1(\tau) q_1(\max\{\frac{1}{n}, x(\tau)\}) \right. \\ &\quad \left. \psi_1(\max\{\frac{1}{n}, x(\tau)\}, y(\tau)) d\tau \right) \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1-1} m_1(\tau) p_1(\max\{\frac{1}{n}, x(\tau)\}) \right. \\ &\quad \left. \frac{q_1(\max\{\frac{1}{n}, x(\tau)\})}{p_1(\max\{\frac{1}{n}, x(\tau)\})} \psi_1(\max\{\frac{1}{n}, x(\tau)\}, y(\tau)) d\tau \right) \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \frac{q_1(r)}{p_1(r)} \psi_1(r, r) \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1-1} m_1(\tau) p_1(\frac{1}{n}) d\tau \right) \\ &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \frac{q_1(r)}{p_1(r)} \psi_1(r, r) \int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1-1} p_1(\frac{1}{n} \gamma(\tau)) d\tau \right) \\ &< +\infty. \end{aligned}$$

Similarly,

$$\begin{aligned}\|B_n(x, y)\| &\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \frac{q_2(r)}{p_2(r)} \psi_2(r, r) \int_0^1 a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} p_2\left(\frac{1}{n} \gamma(\tau)\right) d\tau \right) \\ &< +\infty.\end{aligned}$$

Step 3: For $t, t' \in [0, 1]$ ($t < t'$), we have

$$\begin{aligned}&|A_n(x, y)(t) - A_n(x, y)(t')| \\ &\leq \int_0^1 |G_1(t, s) - G_1(t', s)| \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_n(\tau, x(\tau), y(\tau)) d\tau \right) ds.\end{aligned}$$

In the same way

$$\begin{aligned}&|B_n(x, y)(t) - B_n(x, y)(t')| \\ &\leq \int_0^1 |G_2(t, s) - G_2(t', s)| \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_n(\tau, x(\tau), y(\tau)) d\tau \right) ds.\end{aligned}$$

By the continuity of G_1, G_2 and Arzela-Ascoli Lemma, T_n is a completely continuous. $\square$

Theorem 4.1. Assume that Assumptions $(\mathcal{A}_1) - (\mathcal{A}_4)$ hold. Then problem (1.1) has at least two positive solutions.

Proof. **Step 1: An approximating solution.** From condition $(\mathcal{A}_2)$, there exists $R_1 > 0$ such that, for all $0 < \omega < 1$

$$\frac{R_1}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi^{-1} \left(\lambda_i \frac{q_i(R_1)}{p_i(R_1)} \psi_i(R_1, R_1) \int_0^1 a_i(\tau) m_i(\tau) (1 - \tau)^{\alpha_i - 1} p_i(\omega \gamma(\tau)) d\tau \right)} > 1. \quad (4.3)$$

Let

$$\Omega_1 = \{(x, y) \in E : \|(x, y)\| < R_1\}.$$

We claim that $(x, y) \neq \lambda T_n(x, y)$ for all $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0, 1]$ and $n \geq n_0 \geq \frac{1}{R_1}$. If not, suppose that there exist $n_1 \geq n_0, (x_1, y_1) \in \partial\Omega_1 \cap \mathcal{P}$ and $\lambda_1 \in (0, 1]$ such that $(x_1, y_1) = \lambda_1 T_{n_1}(x_1, y_1)$. As a consequence, we obtain

the estimates:

$$\begin{aligned}
 \|x_1\| &= \|\lambda_1 A_{n_1}(x_1, y_1)\| \\
 &\leq \|A_{n_1}(x_1, y_1)\| \\
 &\leq \left\| \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_{n_1}(\tau, x_1(\tau), y_1(\tau)) d\tau \right) ds \right\| \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\int_0^1 a_1(\tau) \lambda_1 (1 - \tau)^{\alpha_1 - 1} f(\tau, \max\{\frac{1}{n_1}, x_1(\tau)\}, y_1(\tau)) d\tau \right) \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1 - 1} m_1(\tau) q_1(\max\{\frac{1}{n_1}, x_1(\tau)\}) \right. \\
 &\quad \left. \psi_1(\max\{\frac{1}{n_1}, x_1(\tau)\}, y_1(\tau)) d\tau \right) \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1 - 1} m_1(\tau) p_1(\max\{\frac{1}{n_1}, x_1(\tau)\}) \right. \\
 &\quad \left. \frac{q_1(\max\{\frac{1}{n_1}, x_1(\tau)\})}{p_1(\max\{\frac{1}{n_1}, x_1(\tau)\})} \psi_1(\max\{\frac{1}{n_1}, x_1(\tau)\}, y_1(\tau)) d\tau \right) \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \frac{q_1(R_1)}{p_1(R_1)} \psi_1(R_1, R_1) \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1 - 1} m_1(\tau) p_1(\frac{1}{n_1}) d\tau \right) \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \frac{q_1(R_1)}{p_1(R_1)} \psi_1(R_1, R_1) \int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1 - 1} p_1(\frac{1}{n_1}) \gamma(\tau) d\tau \right).
 \end{aligned}$$

In a similar way

$$\begin{aligned}
 \|y_1\| &= \|\lambda_1 B_{n_1}(x_1, y_1)\| \\
 &\leq \|B_{n_1}(x_1, y_1)\| \\
 &\leq \left\| \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_{n_1}(\tau, x_1(\tau), y_1(\tau)) d\tau \right) ds \right\|, \\
 &\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \frac{q_2(R_1)}{p_2(R_1)} \psi_2(R_1, R_1) \int_0^1 a_2(\tau) (1 - \tau)^{\alpha_2 - 1} m_2(\tau) p_2(\frac{1}{n_1}) d\tau \right) \\
 &\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \frac{q_2(R_1)}{p_2(R_1)} \psi_2(R_1, R_1) \int_0^1 a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} p_2(\frac{1}{n_1}) \gamma(\tau) d\tau \right).
 \end{aligned}$$

Then,

$$\begin{aligned}
 R_1 &= \|(x_1, y_1)\| \\
 &\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \frac{q_1(R_1)}{p_1(R_1)} \psi_1(R_1, R_1) \int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1 - 1} p_1(\frac{1}{n_1}) \gamma(\tau) d\tau \right) \\
 &\quad + \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \frac{q_2(R_1)}{p_2(R_1)} \psi_2(R_1, R_1) \int_0^{1-\sigma} a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} p_2(\frac{1}{n_1}) \gamma(\tau) d\tau \right),
 \end{aligned}$$

which is a contradiction to (4.3). By Lemma 2.2, we infer that

$$(4.4) \quad i(T_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \text{ for all } n \in \{n_0, n_0 + 1, \dots\}.$$

Hence there exists $(x_n, y_n) \in \Omega_1 \cap \mathcal{P}$ such that $T_n(x_n, y_n) = (x_n, y_n)$ and $\|(x_n, y_n)\| < R_1$, for all $n \geq n_0$.

Step 2: a compactness argument.

- (a) Since $\|(x_n, y_n)\| < R_1$, by $(\mathcal{A}_4)$ there exist functions $\rho_{R_1}, \rho_{R_1}^* \in C([0, 1], \mathbf{R}^+)$ such that

$$f_n(t, x_n(t), y_n(t)) \geq \rho_{R_1}(t), \quad \forall t \in [0, 1]$$

and

$$g_n(t, x_n(t), y_n(t)) \geq \rho_{R_1}^*(t), \quad \forall t \in [0, 1],$$

Then

$$\begin{aligned} x_n(t) &= A_n(x_n, y_n)(t) \\ &= \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ &\geq \int_\theta^{1-\theta} \frac{(\beta_1-2)K_1(t)L_1(s)}{\Gamma(\beta_1)} \phi_1^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_1(\tau) H_1(s, \tau) \rho_{R_1}(\tau) d\tau \right) ds \\ &\geq \frac{\phi_1^{-1}(\frac{1}{\Gamma(\alpha_1)})(\beta_1-2)K_1(t)}{\Gamma(\beta_1)} \\ &\quad \int_\theta^{1-\theta} L_1(s) \phi_1^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_1(\tau) [s(1-\tau)]^{\alpha_1-1} - (s-\tau)^{\alpha_1-1} \right] \rho_{R_1}(\tau) d\tau ds \end{aligned}$$

and

$$\begin{aligned} y_n(t) &= B_n(x_n, y_n)(t) \\ &= \int_0^1 G_2(s, \tau) \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ &\geq \int_\theta^{1-\theta} \frac{(\beta_2-2)K_2(t)L_2(s)}{\Gamma(\beta_2)} \phi_2^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_2(\tau) H_2(s, \tau) g_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ &\geq \frac{\phi_2^{-1}(\frac{1}{\Gamma(\alpha_2)})(\beta_2-2)K_2(t)}{\Gamma(\beta_2)} \\ &\quad \int_\theta^{1-\theta} L_2(s) \phi_2^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_2(\tau) [s(1-\tau)]^{\alpha_2-1} - (s-\tau)^{\alpha_2-1} \right] \rho_{R_1}^*(\tau) d\tau ds. \end{aligned}$$

Let the positive numbers

$$c_1^* = \frac{\phi_1^{-1}(\frac{1}{\Gamma(\alpha_1)})(\beta_1-2)}{\Gamma(\beta_1)} \int_\theta^{1-\theta} L_1(s) \phi_1^{-1}$$

$$\left(\int_{\theta-\frac{1}{4}}^\theta a_1(\tau) [s(1-\tau)]^{\alpha_1-1} - (s-\tau)^{\alpha_1-1} \right] \rho_{R_1}(\tau) d\tau \Big) ds$$

and

$$c_2^* = \frac{\phi_2^{-1}(\frac{1}{\Gamma(\alpha_2)})(\beta_2-2)}{\Gamma(\beta_2)} \int_\theta^{1-\theta} L_2(s) \phi_2^{-1}$$

$$\left(\int_{\theta-\frac{1}{4}}^\theta a_2(\tau) [s(1-\tau)]^{\alpha_2-1} - (s-\tau)^{\alpha_2-1} \right] \rho_{R_1}^*(\tau) d\tau \Big) ds.$$

Then,

$$x_n(t) \geq c_1^* K_1(t) \geq c_1^* \gamma(t) \quad \text{and} \quad y_n(t) \geq c_2^* K_2(t) \geq c_2^* \gamma(t) \quad \forall t \in [0, 1], \forall n \geq n_0.$$

(b) For any $t, t' \in [0, 1]$ ($t > t'$), we have the estimates

$$\begin{aligned} & |x_n(t) - x_n(t')| \\ & \leq \int_0^1 \left| G_1(t, s) - G_1(t', s) \right| \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ & \leq \int_0^1 \left| G_1(t, s) - G_1(t', s) \right| \phi_1^{-1} \left(\int_0^1 a_1(\tau) \lambda_1 (1 - \tau)^{\alpha_1 - 1} m_1(\tau) p_1 \right. \\ & \quad \left. (c_1^* \gamma(\tau)) \frac{q_1(R_1)}{p_1(R_1)} \psi_1(R_1, R_1) d\tau \right) \\ & \leq \int_0^1 \left| G_1(t, s) - G_1(t', s) \right| \phi_1^{-1} \left(\lambda_1 \frac{q_1(R_1)}{p_1(R_1)} \psi_1(R_1, R_1) \right) \\ & \quad \phi_1^{-1} \left(\int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1 - 1} p_1 (c_1^* \gamma(\tau)) d\tau \right). \end{aligned}$$

Also

$$\begin{aligned} & |y_n(t) - y_n(t')| \\ & \leq \int_0^1 \left| G_2(t, s) - G_2(t', s) \right| \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ & \leq \int_0^1 \left| G_2(t, s) - G_2(t', s) \right| \phi_2^{-1} \left(\lambda_2 \frac{q_2(R_1)}{p_2(R_1)} \psi_2(R_1, R_1) \right) \\ & \quad \phi_2^{-1} \left(\int_0^1 a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} p_2 (c_2^* \gamma(\tau)) d\tau \right). \end{aligned}$$

Therefore $\{(x_n, y_n)\}_{n \geq n_0}$ is relatively compact in E . By the continuity of G_1, G_2 , there exists a subsequence $\{(x_{n_k}, y_{n_k})\}_{k \geq 1}$ such that $\lim_{k \rightarrow +\infty} (x_{n_k}, y_{n_k}) = (x_0, y_0)$. Hence $x_0(t) \geq c_1^* \gamma(t)$, $y_0(t) \geq c_2^* \gamma(t)$, $\forall t \in [0, 1]$. From (4.3), we have $\|(x_0, y_0)\| < R_1$. By the Lebesgue dominated convergence theorem and the continuity of f and g , we deduce that

$$\begin{aligned} x_0(t) &= \lim_{k \rightarrow +\infty} x_{n_k}(t) \\ &= \lim_{k \rightarrow +\infty} \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_{n_k}(\tau, x_{n_k}(\tau), y_{n_k}(\tau)) d\tau \right) ds \\ &= \lim_{k \rightarrow +\infty} \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f(s, \max\{\frac{1}{n_k}, x_{n_k}(s)\}, y_{n_k}(s)) d\tau \right) ds \\ &= \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f(\tau, x_0(\tau), y_0(\tau)) d\tau \right) ds \end{aligned}$$

and

$$\begin{aligned} y_0(t) &= \lim_{k \rightarrow +\infty} y_{n_k}(t) \\ &= \lim_{k \rightarrow +\infty} \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_{n_k}(\tau, x_{n_k}(\tau), y_{n_k}(\tau)) d\tau \right) ds \\ &= \lim_{k \rightarrow +\infty} \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g(s, \max\{\frac{1}{n_k}, x_{n_k}(s)\}, y_{n_k}(s)) d\tau \right) ds \\ &= \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g(\tau, x_0(\tau), y_0(\tau)) d\tau \right) ds. \end{aligned}$$

Then (x_0, y_0) is a positive solution of problem (1.1). Arguing as in the proof of Theorem 3.1 and using condition $(\mathcal{A}_3)$, we can choose a number R_2 and show that

$$i(T_n, \Omega_2 \cap \mathcal{P}, \mathcal{P}) = 0,$$

with

$$\Omega_2 = \{(x, y) \in \mathcal{P} : \|(x, y)\| \leq R_2\}$$

and also

$$i(T_n, (\Omega_2 \setminus \overline{\Omega_1}) \cap \mathcal{P}, \mathcal{P}) = -1, \text{ for all } n \in \{n_0, n_0 + 1, \dots\}$$

Therefore T_n has a second fixed point $(w_n, v_n) \in (\Omega_2 \setminus \overline{\Omega_1}) \cap \mathcal{P}$, $\forall n \geq n_0$, such that $R_1 < \|(w_n, v_n)\| < R_2$, $\forall n \geq n_0$. As above, we can show that $\{(w_n, v_n)\}_{n \geq n_0}$ has a subsequence $\{(w_{n_j}, v_{n_j})\}_{j \geq 1}$ such that $\lim_{j \rightarrow +\infty} (w_{n_j}, v_{n_j}) = (w_0, v_0)$ and (w_0, v_0) is a solution of problem (1.1). Since $0 < \|(x_0, y_0)\| < R_1 < \|(w_0, v_0)\| < R_2$ then (x_0, y_0) and (w_0, v_0) are two distinct positive solutions of problem (1.1). $\square$

Example 4.1. Consider the bvp:

$$(4.5) \quad \begin{cases} -D_{0+}^{\frac{5}{4}} \left(D_{0+}^{\frac{13}{4}} x(t) \right)^{\frac{1}{3}} = 4\sqrt{\pi} \left(\frac{8}{9} \delta_1 \right)^{\frac{1}{3}} t^2 (1-t)^{\frac{7}{4}} \frac{x^2 + \ln(x+2)}{x} (x+y+1), \\ -D_{0+}^{\frac{7}{4}} \left(D_{0+}^{\frac{7}{2}} y(t) \right)^{\frac{1}{5}} = (1-2^{\frac{-1}{5}}) \frac{\sqrt{\pi}}{2} \left(\frac{6}{5} \delta_2 \right)^{\frac{1}{5}} t^4 (1-t)^{\frac{13}{4}} \frac{x^3+x+1}{x^2} e^{(x+y)^{\frac{1}{5}}}, \\ x(0) = x(1) = x'(0) = x'(1) = 0, D_{0+}^{\frac{13}{4}} x(0) = 0, D_{0+}^{\frac{13}{4}} x(1) = \frac{1}{2^{\frac{1}{4}}} D_{0+}^{\frac{13}{4}} x\left(\frac{1}{2}\right), \\ y(0) = y(1) = y'(0) = y'(1) = 0, D_{0+}^{\frac{7}{2}} y(0) = 0, D_{0+}^{\frac{7}{4}} y(1) = 2^{\frac{11}{4}} D_{0+}^{\frac{7}{2}} y\left(\frac{1}{2}\right), \end{cases}$$

where

$$f(t, x, y) = (1-t)^{\frac{7}{4}} \frac{x^2 + \ln(x+2)}{x} (x+y+1), \quad g(t, x, y) = (1-t)^{\frac{13}{4}} \frac{x^3 + x + 1}{x^2} e^{(x+y)^{\frac{1}{5}}},$$

$$a_1(t) = 4\sqrt{\pi} \left(\frac{8}{9} \delta_1 \right)^{\frac{1}{3}} t^2, \quad a_2(t) = (1-2^{\frac{-1}{5}}) \frac{\sqrt{\pi}}{2} \left(\frac{6}{5} \delta_2 \right)^{\frac{1}{5}} t^4, \quad (\delta_1, \delta_2 > 0),$$

and $\phi_1(t) = t^{\frac{1}{3}}$, $\phi_2(t) = t^{\frac{1}{5}}$. Hence ϕ_i , $i \in \{1, 2\}$ are continuous homeomorphisms which are increasing and multiplicative with $\phi_i(0) = 0$. Let us check Assumptions $(\mathcal{A}_1)$ – $(\mathcal{A}_4)$:

$(\mathcal{A}_1)$ Let $m_1(t) = (1-t)^{\frac{7}{4}}$, $m_2(t) = (1-t)^{\frac{13}{4}}$, $q_1(x) = \frac{x^2 + \ln(x+2)}{x}$, $q_2(x) = \frac{x^3 + x + 1}{x^2}$, $\psi_1(x, y) = x + y + 1$, $\psi_2(x, y) = e^{(x+y)^{\frac{1}{5}}}$, $p_1(x) = \frac{1}{x}$, $p_2(x) = \frac{1}{x^2}$. Then

$$\begin{aligned} \int_0^1 (1-\tau)^{\alpha_1-1} a_1(\tau) m_1(\tau) p_1(c\gamma(\tau)) d\tau &= \frac{4\sqrt{\pi} \left(\frac{8}{9} \delta_1 \right)^{\frac{1}{3}}}{c^{\frac{1}{3}}} < +\infty, \\ \int_0^1 (1-\tau)^{\alpha_2-1} a_2(\tau) m_2(\tau) p_2(c\gamma(\tau)) d\tau &= \frac{(1-2^{\frac{-1}{5}}) \frac{\sqrt{\pi}}{2} \left(\frac{6}{5} \delta_2 \right)^{\frac{1}{5}}}{c^{\frac{1}{5}}} < +\infty. \end{aligned}$$

($\mathcal{A}_2$)

$$\begin{aligned} & \sup_{c>0} \frac{c}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi^{-1} \left(\lambda_i \frac{q_i(c)}{p_i(c)} \psi_i(c, c) \int_0^1 a_i(\tau) (1-\tau)^{\alpha_i-1} m_i(\tau) p_i(\omega\gamma(\tau)) d\tau \right)} \\ & \geq \sup_{c>0} \frac{c}{\frac{\delta_1}{\omega^3} [8(2c+1)(c^2 + \ln(c+2))]^3 + \frac{\delta_2}{\omega^{10}} (c^3 + c + 1)^5 e^{5(2c)} \frac{1}{5}} \\ & \geq \sup_{c>0} \frac{c}{\max\{\delta_1, \delta_2\} \left(\frac{1}{\omega^3} [8(2c+1)(c^2 + \ln(c+2))]^3 + \frac{1}{\omega^{10}} (c^3 + c + 1)^5 e^{5(2c)} \frac{1}{5} \right)}. \end{aligned}$$

($\mathcal{A}_3$)

$$\begin{aligned} \min_{t \in [\sigma, 1-\sigma]} a_1(t) &= \min_{t \in [\sigma, 1-\sigma]} 4\sqrt{\pi} \left(\frac{8}{9} \delta_1 \right)^{\frac{1}{3}} t^2 = 4\sqrt{\pi} \left(\frac{8}{9} \delta_1 \right)^{\frac{1}{3}} \sigma^2 \neq 0, \\ \min_{t \in [\sigma, 1-\sigma]} a_2(t) &= \min_{t \in [\sigma, 1-\sigma]} (1 - 2^{-\frac{1}{5}}) \frac{\sqrt{\pi}}{2} \left(\frac{6}{5} \delta_2 \right)^{\frac{1}{5}} t^4 = (1 - 2^{-\frac{1}{5}}) \frac{\sqrt{\pi}}{2} \left(\frac{6}{5} \delta_2 \right)^{\frac{1}{5}} \sigma^4 \neq 0, \\ \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{f(t, x, y)}{\phi_1(x+y)} &= \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{(1-t)^{\frac{7}{4}} \frac{x^2 + \ln(x+2)}{x} (x+y+1)}{(x+y)^{\frac{1}{3}}} \\ &= +\infty, \\ \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{g(t, x, y)}{\phi_2(x+y)} &= \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{(1-t)^{\frac{13}{4}} \frac{x^3 + x + 1}{x^2} e^{(x+y)} \frac{1}{5}}{(x+y)^{\frac{1}{5}}} \\ &= +\infty. \end{aligned}$$

($\mathcal{A}_4$) For any $c > 0$, we have

$$\begin{aligned} f(t, x, y) &\geq \frac{\ln(2)}{c} (1-t)^{\frac{7}{4}}, \quad \forall t \in [0, 1], \forall x, y \in (0, c], \\ g(t, x, y) &\geq \frac{1}{c^2} (1-t)^{\frac{13}{4}}, \quad \forall t \in [0, 1], \forall x, y \in (0, c]. \end{aligned}$$

If we choose δ_1, δ_2 such that

$$0 < \max\{\delta_1, \delta_2\} < \sup_{c>0} \frac{c}{\frac{1}{\omega^3} [8(2c+1)(c^2 + \ln(c+2))]^3 + \frac{1}{\omega^{10}} (c^3 + c + 1)^5 e^{5(2c)} \frac{1}{5}},$$

then all hypotheses of Theorem 4.1 are fulfilled. We conclude problem (4.5) has at least two positive solutions.

5. Nonlinearities with two singularities

In this section, we suppose that the nonlinearities $f, g : [0, 1] \times \mathbf{R}_*^+ \times \mathbf{R}_*^+ \longrightarrow \mathbf{R}^+$ are continuous and satisfy

($\mathcal{B}_1$) There exist $m_i \in C([0, 1], \mathbf{R}^+)$, $\psi_i \in C(\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}^+)$ increasing functions for each argument and $k_i, p_i, q_i, l_i \in C(\mathbf{R}_*^+, \mathbf{R}^+)$ such that p_i, l_i

are decreasing functions and $\frac{q_i}{p_i 0}, \frac{k_i}{l_i}$ are increasing functions ($i \in \{1, 2\}$) with

$$(5.1) \quad \begin{aligned} f(t, x, y) &\leq m_1(t)q_1(x)k_1(y)\psi_1(x, y), \quad \forall t \in [0, 1] \times \mathbf{R}_*^+ \times \mathbf{R}_*^+, \\ g(t, x, y) &\leq m_2(t)q_2(x)k_2(y)\psi_2(x, y), \quad \forall t \in [0, 1] \times \mathbf{R}_*^+ \times \mathbf{R}_*^+ \end{aligned}$$

and for each $c, c' > 0$

$$(5.2) \quad \int_0^1 (1 - \tau)^{\alpha_2 - 1} a_i(\tau) m_i(\tau) p_i(c\gamma(\tau)) l_i(c'\gamma(\tau)) d\tau < +\infty.$$

(B₂) For all $0 < \omega < 1$,

$$\sup_{c>0} \frac{c}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi_i^{-1} \left(\lambda_{i,c} \int_0^1 a_i(\tau) (1 - \tau)^{\alpha_i - 1} m_i(\tau) p_i(\omega\gamma(\tau)) l_i(\omega\gamma(\tau)) d\tau \right)} > 1,$$

with

$$\lambda_{i,c} = \lambda_i \frac{q_i(c)}{p_i(c)} \frac{k_i(c)}{l_i(c)} \psi_i(c, c).$$

(B₃) There exist $\sigma \in (0, \frac{1}{2})$ and $F_1, F_2 \in C((\mathbf{R}_*^+)^2, \mathbf{R}^+)$, such that

$$f(t, x, y) \geq F_1(x, y), \quad g(t, x, y) \geq F_2(x, y), \quad \forall (t, x, y) \in [0, 1] \times (\mathbf{R}_*^+)^2,$$

where

$$\lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{F_1(x, y)}{\phi_1(x+y)} \geq l_1, \quad \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{F_2(x, y)}{\phi_2(x+y)} \geq l_2,$$

and

$$l_i = 1 + \frac{\phi_i\left(\frac{\Gamma(\beta_i)}{2c_i(1-2\sigma)^2(\beta_i-2)}\right)}{\int_{\sigma}^{1-\sigma} a_i(\tau) H_i(d_i, \tau) d\tau}, \quad i \in \{1, 2\} \quad \text{with} \quad \min_{t \in [\sigma, 1-\sigma]} a_i(\tau) \neq 0$$

and $c_i = \min_{(t,s) \in ([\sigma, 1-\sigma])^2} k_i(t) L_i(s)$, $H_i(d_i, \tau) = \min_{s \in [\sigma, 1-\sigma]} H_i(s, \tau)$.

(B₄) For any $c > 0$ and $\frac{1}{4} < \theta < \frac{1}{2}$, there exists $\rho_c, \rho_c^* \in C([0, 1], \mathbf{R}^+)$ such that $\rho_c(t) > 0, \rho_c^*(t) > 0$ on $(\theta - \frac{1}{4}, \theta)$ and

$$\begin{aligned} f(t, x, y) &\geq \rho_c(t), \quad \forall t \in [0, 1], \quad \forall x, y \in (0, c] \\ g(t, x, y) &\geq \rho_c^*(t), \quad \forall t \in [0, 1], \quad \forall x, y \in (0, c] \end{aligned}$$

Define two sequences of approximating functions $\{f_n\}_{n \geq 1}$ and $\{g_n\}_{n \geq 1}$ by

$$\begin{aligned} f_n(t, x, y) &= f(t, \max\{\frac{1}{n}, x\}, \max\{\frac{1}{n}, y\}), \quad n \in \{1, 2, \dots\}, \\ g_n(t, x, y) &= g(t, \max\{\frac{1}{n}, x\}, \max\{\frac{1}{n}, y\}), \quad n \in \{1, 2, \dots\}. \end{aligned}$$

For $(x, y) \in \mathcal{P}$, define the sequences of operators

$$A_n(x, y)(t) = \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 H_1(s, \tau) a_1(\tau) f_n(\tau, x(\tau, y(\tau))) d\tau \right) ds,$$

$$B_n(x, y)(t) = \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 H_2(s, \tau) a_2(\tau) g_n(\tau, x(\tau, y(\tau))) d\tau \right) ds,$$

and

$$T_n(x, y) = (A_n(x, y), B_n(x, y)).$$

As for Proposition 3.1, we can prove

Proposition 5.1. *Assume that condition $(\mathcal{B}_1)$ holds. Then for each $n \geq 1$, the operator $T_n : P \longrightarrow P$ is completely continuous.*

Our main result in this section concerns the existence of two positive solutions.

Theorem 5.1. *Assume that Assumptions $(\mathcal{B}_1) - (\mathcal{B}_4)$ hold. Then problem (1.1) has at least two positive solutions.*

Proof. Step 1: An approximating solution. From condition $(\mathcal{B}_2)$, there exists $R_1 > 0$ such that, for all $0 < \omega < 1$

$$\frac{R_1}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi_i^{-1} \left(\lambda_{i,R_1} \int_0^1 a_i(\tau) (1-\tau)^{\alpha_i-1} m_i(\tau) p_i(\omega \gamma(\tau)) l_i(\omega \gamma(\tau)) d\tau \right)} > 1. \quad (5.3)$$

Let

$$\Omega_1 = \{(x, y) \in E : \|(x, y)\| < R_1\}.$$

We claim that $(x, y) \neq \lambda T_n(x, y)$ for any $x \in \partial\Omega_1 \cap \mathcal{P}, \lambda \in (0, 1]$ and $n \geq n_0 \geq \frac{1}{R_1}$. On the contrary, suppose that there exists $n_1 \geq n_0$, $(x_1, y_1) \in \partial\Omega_1 \cap \mathcal{P}$, and $\lambda_1 \in (0, 1]$ such that $(x_1, y_1) = \lambda_1 T_{n_1}(x_1, y_1)$.

As a consequence, we have the estimates:

$$\begin{aligned}
\|x_1\| &= \|\lambda_1 A_{n_1}(x_1, y_1)\| \\
&\leq \|A_{n_1}(x_1, y_1)\| \\
&\leq \left\| \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_{n_1}(\tau, x_1(\tau), y_1(\tau)) d\tau \right) ds \right\| \\
&\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\int_0^1 a_1(\tau) \lambda_1 (1 - \tau)^{\alpha_1 - 1} f(\tau, \max\{\frac{1}{n_1}, x_1(\tau)\}, \right. \\
&\quad \left. \max\{\frac{1}{n_1}, y_1(\tau)\}) d\tau \right) \\
&\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1 - 1} m_1(\tau) q_1(\max\{\frac{1}{n_1}, x_1(\tau)\}) \right. \\
&\quad \left. k_1(\max\{\frac{1}{n_1}, y_1(\tau)\}) \psi_1(\max\{\frac{1}{n_1}, x_1(\tau)\}, \max\{\frac{1}{n_1}, y_1(\tau)\}) d\tau \right) \\
&\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1 - 1} m_1(\tau) p_1(\max\{\frac{1}{n_1}, x_1(\tau)\}) \right. \\
&\quad \left. l_1(\max\{\frac{1}{n_1}, y_1(\tau)\}) \frac{q_1(\max\{\frac{1}{n_1}, x_1(\tau)\}) k_1(\max\{\frac{1}{n_1}, y_1(\tau)\})}{p_1(\max\{\frac{1}{n_1}, x_1(\tau)\}) l_1(\max\{\frac{1}{n_1}, y_1(\tau)\})} \right. \\
&\quad \left. \psi_1(\max\{\frac{1}{n_1}, x_1(\tau)\}, \max\{\frac{1}{n_1}, y_1(\tau)\}) d\tau \right) \\
&\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \frac{q_1(R_1)}{p_1(R_1)} \frac{k_1(R_1)}{l_1(R_1)} \psi_1(R_1, R_1) \right) \\
&\quad \phi_1^{-1} \left(\int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1 - 1} m_1(\tau) p_1\left(\frac{1}{n_1}\right) l_1\left(\frac{1}{n_1}\right) d\tau \right) \\
&\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} \left(\lambda_1 \frac{q_1(R_1)}{p_1(R_1)} \frac{k_1(R_1)}{l_1(R_1)} \psi_1(R_1, R_1) \right) \\
&\quad \phi_1^{-1} \left(\int_0^1 a_1(\tau) (1 - \tau)^{\alpha_1 - 1} m_1(\tau) p_1\left(\frac{1}{n_1}\right) l_1\left(\frac{1}{n_1}\right) d\tau \right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|y_1\| &= \|\lambda_1 B_{n_1}(x_1, y_1)\| \\
&\leq \|B_{n_1}(x_1, y_1)\| \\
&\leq \left\| \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_{n_1}(\tau, x_1(\tau), y_1(\tau)) d\tau \right) ds \right\|, \\
&\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\int_0^1 a_2(\tau) \lambda_2 (1 - \tau)^{\alpha_2 - 1} g(\tau, \max\{\frac{1}{n_1}, x_1(\tau)\}, \right. \\
&\quad \left. \max\{\frac{1}{n_1}, y_1(\tau)\}) d\tau \right) \\
&\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \int_0^1 a_2(\tau) (1 - \tau)^{\alpha_2 - 1} m_2(\tau) q_2(\max\{\frac{1}{n_1}, x_1(\tau)\}) \right. \\
&\quad \left. k_2(\max\{\frac{1}{n_1}, y_1(\tau)\}) \psi_2(\max\{\frac{1}{n_1}, x_1(\tau)\}, \max\{\frac{1}{n_1}, y_1(\tau)\}) d\tau \right) \\
&\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \frac{q_2(R_1)}{p_2(R_1)} \frac{k_2(R_1)}{l_2(R_1)} \psi_2(R_1, R_1) \right) \\
&\quad \phi_2^{-1} \left(\int_0^1 a_2(\tau) (1 - \tau)^{\alpha_2 - 1} m_2(\tau) p_2\left(\frac{1}{n_1}\right) l_2\left(\frac{1}{n_1}\right) d\tau \right) \\
&\leq \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} \left(\lambda_2 \frac{q_2(R_1)}{p_2(R_1)} \frac{k_2(R_1)}{l_2(R_1)} \psi_2(R_1, R_1) \right) \\
&\quad \phi_2^{-1} \left(\int_0^1 a_2(\tau) (1 - \tau)^{\alpha_2 - 1} m_2(\tau) p_2\left(\frac{1}{n_1}\right) l_2\left(\frac{1}{n_1}\right) d\tau \right).
\end{aligned}$$

Then

$$\begin{aligned}
&\|(x_1, y_1)\| \\
&\leq \frac{M_1}{\Gamma(\beta_1)} \phi_1^{-1} (\lambda_{1, R_1} \int_0^1 a_1(\tau) m_1(\tau) (1 - \tau)^{\alpha_1 - 1} p_1\left(\frac{1}{n_1}\right) l_1\left(\frac{1}{n_1}\right) d\tau) \\
&+ \frac{M_2}{\Gamma(\beta_2)} \phi_2^{-1} (\lambda_{2, R_1} \int_0^1 a_2(\tau) m_2(\tau) (1 - \tau)^{\alpha_2 - 1} p_2\left(\frac{1}{n_1}\right) l_2\left(\frac{1}{n_1}\right) d\tau),
\end{aligned}$$

which is a contradiction to (5.3). By Lemma 2.2, we deduce that

$$(5.4) \quad i(T_n, \Omega_1 \cap \mathcal{P}, \mathcal{P}) = 1, \text{ for all } n \in \{n_0, n_0 + 1, \dots\}.$$

Consequently there exists a sequence of fixed points $(x_n, y_n) \in \Omega_1 \cap \mathcal{P}$ such that $T_n(x_n, y_n) = (x_n, y_n)$, $\forall n \geq n_0$ and $\|(x_n, y_n)\| < R_1$.

Step 2: a compactness argument.

(a) Since $\|(x_n, y_n)\| < R_1$, by $(\mathcal{B}_4)$ there exist functions $\rho_{R_1}, \rho_{R_1}^* \in C([0, 1], \mathbf{R}^+)$ such that

$$f_n(t, x(t), y(t)) \geq \rho_{R_1}(t), \quad \forall t \in [0, 1],$$

$$g_n(t, x(t), y(t)) \geq \rho_{R_1}^*(t), \quad \forall t \in [0, 1].$$

Then

$$\begin{aligned} x_n(t) &= A_n(x_n, y_n)(t) \\ &\geq \int_0^1 G_1(t, s) \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ &\geq \int_\theta^{1-\theta} \frac{(\beta_1-2)K_1(t)L_1(s)}{\Gamma(\beta_1)} \phi_1^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_1(\tau) H_1(s, \tau) \rho_{R_1}(\tau) d\tau \right) ds \\ &\geq \frac{\phi_1^{-1}(\frac{1}{\Gamma(\alpha_1)})(\beta_1-2)K_1(t)}{\Gamma(\beta_1)} \\ &\quad \int_\theta^{1-\theta} L_1(s) \phi_1^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_1(\tau) [s(1-\tau)]^{\alpha_1-1} - (s-\tau)^{\alpha_1-1} \right) \rho_{R_1}(\tau) d\tau ds \end{aligned}$$

$$\begin{aligned} y_n(t) &= B_n(x_n, y_n)(t) \\ &\geq \int_0^1 G_2(t, s) \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ &\geq \int_\theta^{1-\theta} \frac{(\beta_2-2)K_2(t)L_2(s)}{\Gamma(\beta_2)} \phi_2^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_2(\tau) H_2(s, \tau) g_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ &\geq \frac{\phi_2^{-1}(\frac{1}{\Gamma(\alpha_2)})(\beta_2-2)K_2(t)}{\Gamma(\beta_2)} \\ &\quad \int_\theta^{1-\theta} L_2(s) \phi_2^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_2(\tau) [s(1-\tau)]^{\alpha_2-1} - (s-\tau)^{\alpha_2-1} \right) \rho_{R_1}^*(\tau) d\tau ds. \end{aligned}$$

Let the positive numbers

$$c_1^* = \frac{\phi_1^{-1}(\frac{1}{\Gamma(\alpha_1)})(\beta_1-2)}{\Gamma(\beta_1)}$$

$$\int_\theta^{1-\theta} L_1(s) \phi_1^{-1} \left(\int_{\theta-\frac{1}{4}}^\theta a_1(\tau) [s(1-\tau)]^{\alpha_1-1} - (s-\tau)^{\alpha_1-1} \right) \rho_{R_1}(\tau) d\tau ds$$

and

$$c_2^* = \frac{\phi_2^{-1}(\frac{1}{\Gamma(\alpha_2)})(\beta_2-2)}{\Gamma(\beta_2)}$$

$$\int_{\theta}^{1-\theta} L_2(s) \phi_2^{-1} \left(\int_{\theta-\frac{1}{4}}^{\theta} a_2(\tau) [[s(1-\tau)]^{\alpha_2-1} - (s-\tau)^{\alpha_2-1}] \rho_{R_1}^*(\tau) d\tau \right) ds.$$

Then, $x_n(t) \geq c_1^* \gamma(t)$, $y_n(t) \geq c_2^* \gamma(t)$, $\forall t \in [0, 1], \forall n \geq n_0$.

(b) For any $t, t' \in [0, 1]$ ($t > t'$), we have the following estimates:

$$\begin{aligned} & |x_n(t) - x_n(t')| \\ & \leq \int_0^1 \left| G_1(t, s) - G_1(t', s) \right| \phi_1^{-1} \left(\int_0^1 a_1(\tau) H_1(s, \tau) f_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ & \leq \int_0^1 \left| G_1(t, s) - G_1(t', s) \right| \phi_1^{-1} \left(\lambda_{1, R_1} \int_0^1 a_1(\tau) m_1(\tau) (1-\tau)^{\alpha_1-1} p_1 \right. \\ & \quad \left. (c_1^* \gamma(\tau)) l_1(c_2^* \gamma(\tau)) d\tau \right) ds. \end{aligned}$$

Similarly,

$$\begin{aligned} & |y_n(t) - y_n(t')| \\ & \leq \int_0^1 \left| G_2(t, s) - G_2(t', s) \right| \phi_2^{-1} \left(\int_0^1 a_2(\tau) H_2(s, \tau) g_n(\tau, x_n(\tau), y_n(\tau)) d\tau \right) ds \\ & \leq \int_0^1 \left| G_2(t, s) - G_2(t', s) \right| \phi_2^{-1} \left(\lambda_{2, R_1} \int_0^1 a_2(\tau) m_2(\tau) (1-\tau)^{\alpha_2-1} p_2 \right. \\ & \quad \left. (c_1^* \gamma(\tau)) l_2(c_2^* \gamma(\tau)) d\tau \right) ds. \end{aligned}$$

We conclude that $\{(x_n, y_n)\}_{n \geq n_0}$ has a convergent subsequence $\{(x_{n_{k'}}, y_{n_{k'}})\}_{k' \geq 1}$ such that $\lim_{k' \rightarrow +\infty} (x_{n_{k'}}, y_{n_{k'}}) = (x^*, y^*)$ such that $x^* \geq c_1^* \gamma(t)$ and $y^* \geq c_2^* \gamma(t)$. Consequently (x^*, y^*) is a positive solution of problem (1.1). Arguing as in the proof of Theorem 4.1, we can show by condition $(\mathcal{B}_3)$ that problem (1.1) admits a second positive solution. $\square$

Example 5.1. Consider the bvp

$$\begin{cases} -D_{0+}^{\frac{5}{4}} \left(\phi_1 \left(D_{0+}^{\frac{13}{4}} x(t) \right) \right) = \frac{9\sqrt{\pi}}{32} t^4 (t^2 + 1) (1-t)^{\frac{15}{4}} \frac{(x^2+1)(y^2+1)}{xy} ((x+y)^3 + 1), \\ -D_{0+}^{\frac{7}{4}} \left(\phi_2 \left(D_{0+}^{\frac{7}{2}} y(t) \right) \right) = \sqrt{\pi} t^8 (t+1) (1-t)^{\frac{29}{4}} \frac{(x^3+1)(y^3+1)((x+y)e^{|x|+|y|}+1)}{x^2 y^2}, \\ x(0) = x(1) = x'(0) = x'(1) = 0, D_{0+}^{\frac{13}{4}} x(0) = 0, D_{0+}^{\frac{13}{4}} x(1) = \frac{2^{-\frac{7}{8}}}{\sqrt{\delta}} D_{0+}^{\frac{13}{4}} x(\frac{1}{2}), \\ y(0) = y(1) = y'(0) = y'(1) = 0, D_{0+}^{\frac{7}{2}} y(0) = 0, D_{0+}^{\frac{7}{2}} y(1) = \frac{2^{-\frac{5}{8}}}{\sqrt{\delta}} D_{0+}^{\frac{7}{2}} y(\frac{1}{2}), \end{cases} \quad (5.5)$$

where

$$f(t, x, y) = \frac{(t^2 + 1)(x^2 + 1)(y^2 + 1)}{xy} ((x + y)^3 + 1),$$

$$g(t, x, y) = \frac{(t+1)(x^3+1)(y^3+1)((x+y)e^{|x|+|y|}+1)}{x^2y^2},$$

$$a_1(t) = \frac{9\sqrt{\pi}}{32}t^4(1-t)^{\frac{15}{4}}, \quad a_2(t) = \sqrt{\pi}t^8(1-t)^{\frac{29}{4}},$$

and $\phi_1(t) = \phi_2(t) = \delta t|t|, \delta > 0$. Hence for $i \in \{1, 2\}$, the functions ϕ_i are continuous homeomorphisms which are increasing, multiplicative with $\phi_i(0) = 0$ and $\phi_i^{-1}(t) = \frac{1}{\sqrt{\delta}}t|t|^{\frac{1}{2}}$. Next, we check Assumptions $(\mathcal{B}_1)$ and $(\mathcal{B}_4)$:

$(\mathcal{B}_1)$ Let

$$\begin{aligned} m_1(t) &= t^2 + 1, \quad m_2(t) = t + 1, \quad q_1(x) = \frac{x^2+1}{x}, \quad q_2(x) = \frac{x^3+1}{x^2} \\ k_1(y) &= \frac{y^2+1}{y}, \quad k_2(y) = \frac{y^3+1}{y^2}, \quad \psi_1(x, y) = (x+y)^3 + 1, \\ \psi_2(x, y) &= (x+y)e^{|x|+|y|} + 1, \quad p_1(x) = \frac{1}{x}, \quad p_2(x) = \frac{1}{x^2}, \\ l_1(y) &= \frac{1}{y}, \quad l_2(y) = \frac{1}{y^2}. \end{aligned}$$

Then we have

$$\begin{aligned} \int_0^1 (1-\tau)^{\alpha_1-1} a_1(\tau) m_1(\tau) p_1(c\gamma(\tau)) l_1(c'\gamma(\tau)) d\tau &= \frac{3\sqrt{\pi}}{8cc'} < +\infty, \\ \int_0^1 (1-\tau)^{\alpha_2-1} a_2(\tau) m_2(\tau) p_2(c\gamma(\tau)) l_2(c'\gamma(\tau)) d\tau &= \frac{3\sqrt{\pi}}{2c^2c'^2} < +\infty. \end{aligned}$$

$(\mathcal{B}_2)$

$$\begin{aligned} &\sup_{c>0} \frac{c}{\sum_{i=1}^2 \frac{M_i}{\Gamma(\beta_i)} \phi_i^{-1} \left(\lambda_{i,c} \int_0^1 a_i(\tau) m_i(\tau) (1-\tau)^{\alpha_i-1} p_i(\omega\gamma(\tau)) l_i(\omega\gamma(\tau)) d\tau \right)} \\ &\geq \sup_{c>0} \frac{\sqrt{\delta} \min\{\omega, \omega^2\} c}{2\sqrt{(2c)^3+1(c^2+1)+2\sqrt{(c^3+1)^2(2ce^{2c}+1)}}}. \end{aligned}$$

$(\mathcal{B}_3)$

$$\begin{aligned} \min_{t \in [\sigma, 1-\sigma]} a_1(t) &= \min_{t \in [\sigma, 1-\sigma]} \frac{9\sqrt{\pi}}{32} t^4 (1-t)^{\frac{15}{4}} \\ &\geq \frac{9\sqrt{\pi}}{32} \sigma^{\frac{31}{4}} \neq 0, \\ \min_{t \in [\sigma, 1-\sigma]} a_2(t) &= \min_{t \in [\sigma, 1-\sigma]} \sqrt{\pi} t^8 (1-t)^{\frac{29}{4}} \geq \sqrt{\pi} \sigma^{\frac{65}{4}} \neq 0, \\ \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{F_1(x, y)}{\phi_1(x+y)} &= \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{\frac{(x^2+1)(y^2+1)((x+y)^3+1)}{xy}}{\delta(x+y)|x+y|} \\ &= +\infty \end{aligned}$$

$$\begin{aligned} \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{F_2(x,y)}{\phi_2(x+y)} &= \lim_{x+y \rightarrow +\infty} \inf_{t \in [\sigma, 1-\sigma]} \frac{(x^3+1)(y^3+1)((x+y)e^{|x|+|y|}+1)}{\delta(xy)^2(x+y)|x+y|} \\ &= +\infty. \end{aligned}$$

(B₄)

$$\begin{aligned} f(t, x, y) &\geq \frac{t^2+1}{c^2}, & \forall t \in [0, 1], \forall x, y \in (0, c] \\ g(t, x, y) &\geq \frac{t+1}{c^4}, & \forall t \in [0, 1], \forall x, y \in (0, c] \end{aligned}$$

Let us pick some δ such that

$$\delta > \left(\sup_{c>0} \frac{\min\{\omega, \omega^2\}c}{2\sqrt{(2c)^3+1}(c^2+1)+2(c^3+1)\sqrt{2ce^{2c}+1}} \right)^{-2}.$$

Then the hypotheses of Theorem 5.1 are satisfied and problem (5.5) has at least two positive solutions.

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