Nondifferentiable higher-order duality theorems for new type of dual model under generalized functions

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Abstract:

The motivation behind this article is to study a class of non-differentiable multiobjective fractional programming problem in which each component of objective functions contains a term including the support function of a compact convex set. For a differentiable function, we consider a class of higher order pseudo quasi/ strictly pseudo quasi/weak strictly pseudo quasi- (V, ρ, d) -type-I convex functions. Under these the higher-order pseudo quasi/ strictly pseudo quasi/weak strictly pseudo quasi- (V, ρ, d) -type-I convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems related to efficient solution. establish Jensen-type and Hermite-Hadamard-type inequalities.

Keywords: Fractional programming; Multiobjective; Support function; Efficient solutions.

MSC (2010): 90C26; 90C30; 90C32; 90C46.

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1. Introduction

The fractional optimization problem with multiple objective functions have been the subject of intense investigations in the past few years, which have produced a number of optimality and duality results for these problems. Higher-order duality in non-linear programming has been studied in last few years by many researchers. In various numerical algorithms, higher-order duality is considered over first-order as it gives more closer bounds.

Higher-order duality in nonlinear programs have been studied by some researchers. Mangasarian [8] formulated a class of higher-order dual problems for the nonlinear programming problem " $\min \{f(x) : g(x) \geq 0\}$ " by introducing twice differentiable function $h : R^n \times R^n \rightarrow R$ and $k : R^n \times R^n \rightarrow R^m$. The concept of higher-order convexity presented by Zhang [12] and derived duality results in multiobjective programming problem. Later on, Yang et al. [11] considered a unified higher-order dual model for nondifferentiable multiobjective programs and proved duality results under generalized assumptions. Suneja et al. [10] introduced a higher order (F, α, σ) -type I functions and formulated higher order dual programs for a nondifferentiable multiobjective fractional programming problem.

Motivated by several concepts of generalized convexity, Gulati and Agarwal [4] gave the concept of second-order V -type I functions for multiobjective programming problem which were recently extended to nondifferentiable case by Jayswal et al. [6]. Recently, Jayswal et al. [7] considered higher-Order duality for multiobjective programming problems and discussed duality theorems under $(F, \alpha, \rho, d) - V -$ type-I functions. Several researchers [[1], [2], [3], [9], [13]-[17]] have done their work in the related areas.

In this paper, we have generalized the definitions of higher-order pseudo quasi/ strictly pseudo quasi/weak strictly pseudo quasi- (V, ρ, d) -type-I functions for a nondifferentiable multiobjective higher-order fractional programming problem. We have formulated higher-order unified dual and established duality results under higher order pseudo quasi/ strictly pseudo quasi/weak strictly pseudo quasi- (V, ρ, d) -type-I assumptions.

2. Preliminaries and Definitions

Throughout this paper, we use the index sets $K = \{1, 2, \dots, k\}$ and $M = \{1, 2, \dots, m\}$.

Definition 2.1. Let $Q \subseteq R^n$ be a compact convex set. The support function of Q is defined by

$$s(y|Q) = \max\{y^T z : z \in Q\}.$$

Consider the following nondifferentiable multiobjective fractional programming problem:

$$\begin{aligned} \text{(MFP) Minimize } \Psi(x) &= \left(\frac{\phi_1(x) + s(x|C_1)}{\psi_1(x) - s(x|E_1)}, \frac{\phi_2(x) + s(x|C_2)}{\psi_2(x) - s(x|E_2)}, \dots, \frac{\phi_k(x) + s(x|C_k)}{\psi_k(x) - s(x|E_k)} \right)^T \\ \text{subject to } x &\in Y^0 = \{x \in Y : \pi_j(x) + s(x|D_j) \leq 0, j \in M\}, \end{aligned}$$

where $Y \subseteq R^n$ is an open set. The functions $\phi, \psi : Y \rightarrow R^k, \pi : Y \rightarrow R^m$ are differentiable on Y and C_i, E_i, D_j are compact convex sets in R^n for $i \in K$ and $j \in M$. Let $\phi_i(x) + s(x|C_i) \geq 0$ and $\psi_i(x) - s(x|E_i) > 0, i \in K$. Let $H = (H_1, H_2, \dots, H_k) : X \times R^n \rightarrow R^k$ and $K = (K_1, K_2, \dots, K_m) : X \times R^n \rightarrow R^m$ be differentiable functions, $d : X \times X \rightarrow R, z = (z_1, z_2, \dots, z_k), v = (v_1, v_2, \dots, v_k)$ and $w = (w_1, w_2, \dots, w_m)$, where $z_i \in C_i, v_i \in E_i$ and $w_j \in D_j$, for $i \in K$ and $j \in M$. Let $\rho = (\rho^1, \rho^2)$ such that $\rho^1 = (\rho_1, \rho_2, \dots, \rho_k) \in R^k, \rho^2 = (\rho_{k+1}, \rho_{k+2}, \dots, \rho_{k+m}) \in R^m$.

Definition 2.2. A point $u \in Y^0$ is efficient solution of (MFP) if $\exists$ no $x \in Y^0$ such that $\Psi(x) \leq \Psi(u)$.

Definition 2.3. $\forall i \in K, j \in M, \left(\frac{\phi_i(.) + (.)^T z_i}{\psi_i(.) - (.)^T v_i}, \pi_j(.) + (.)^T w_j \right)$ is higher order pseudo quasi (V, ρ, d) -type I at u of (MFP) if, $\forall x \in Y^0$ and $p \in R^n$, such that

$$\begin{aligned} \frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} &< \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p) \\ \Rightarrow \eta^T(x, u) \left\{ \nabla \left(\frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_i^1 d^2(x, u) &< 0 \end{aligned}$$

and

$$\begin{aligned} -\pi_j(u) - u^T w_j &\leq K_j(u, p) - p^T \nabla_p K_j(u, p) \\ \Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_j^2 d^2(x, u) &\leq 0. \end{aligned}$$

Definition 2.4. $\forall i \in K, j \in M, \left(\frac{\phi_i(\cdot) + (\cdot)^T z_i}{\psi_i(\cdot) - (\cdot)^T v_i}, \pi_j(\cdot) + (\cdot)^T w_j \right)$ is higher order strictly pseudo quasi (V, ρ, d) -type I at u of (MFP) if, $\forall x \in Y^0$ and $p \in R^n$, such that

$$\begin{aligned} \frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} &\leq \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p) \\ \Rightarrow \eta^T(x, u) \left\{ \nabla \left(\frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_i^1 d^2(x, u) &< 0 \end{aligned}$$

and

$$\begin{aligned} -\pi_j(u) - u^T w_j &\leq K_j(u, p) - p^T \nabla_p K_j(u, p) \\ \Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_j^2 d^2(x, u) &\leq 0. \end{aligned}$$

Definition 2.5. $\forall i \in K, j \in M, \left(\frac{\phi_i(\cdot) + (\cdot)^T z_i}{\psi_i(\cdot) - (\cdot)^T v_i}, \pi_j(\cdot) + (\cdot)^T w_j \right)$ is higher order weak strictly pseudo quasi (V, ρ, d) -type I at u of (MFP) if, $\forall x \in Y^0$ and $p \in R^n$, such that

$$\frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} \leq \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p)$$

$$\Rightarrow \eta^T(x, u) \left\{ \nabla \left(\frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_i^1 d^2(x, u) < 0$$

and

$$\begin{aligned} -\pi_j(u) - u^T w_j &\leq K_j(u, p) - p^T \nabla_p K_j(u, p) \\ \Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_j^2 d^2(x, u) &\leq 0. \end{aligned}$$

Definition 2.6. $\forall i \in K, j \in M$, $\left(\frac{\phi_i(\cdot) + (\cdot)^T z_i}{\psi_i(\cdot) - (\cdot)^T v_i}, \pi_j(\cdot) + (\cdot)^T w_j \right)$ is higher order quasi strictly pseudo (V, ρ, d) -type I at u of (MFP) if, $\forall x \in Y^0$ and $p \in R^n$, such that

$$\begin{aligned} \frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} &\leq \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p) \\ \Rightarrow \eta^T(x, u) \left\{ \nabla \left(\frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_i^1 d^2(x, u) &\leq 0 \end{aligned}$$

and

$$\begin{aligned} -\pi_j(u) - u^T w_j &\leq K_j(u, p) - p^T \nabla_p K_j(u, p) \\ \Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_j^2 d^2(x, u) &< 0. \end{aligned}$$

Theorem 2.1 (K-K-T-type necessary condition)[5]. Let u be efficient solution of (MFP) at which the Kuhn-Tucker constraint qualification is satisfied on X . Then, $\exists 0 < \bar{\lambda} \in R^k, 0 \leq \bar{y}_j \in R^m, \bar{z}_i \in R^n, \bar{v}_i, \bar{w}_j \in R^n, i \in K, j \in M$ such that

$$(2.1) \quad \sum_{i=1}^k \bar{\lambda}_i \nabla \left(\frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \sum_{j=1}^m \bar{y}_j \nabla (\pi_j(u) + u^T \bar{w}_j) = 0,$$

$$(2.2) \quad \sum_{j=1}^m \bar{y}_j (\pi_j(u) + u^T \bar{w}_j) = 0,$$

$$(2.3) \quad u^T \bar{z}_i = S(u|C_i), \quad u^T \bar{v}_i = S(u|E_i), \quad u^T \bar{w}_j = S(u|D_j),$$

$$(2.4) \quad \bar{z}_i \in C_i, \quad \bar{v}_i \in D_i, \quad \bar{w}_j \in E_j, \quad i \in K, \quad j \in M.$$

In the following section, we consider the following mixed higher-order dual for (MFP) and derive duality theorems. The notation $\frac{\phi(\cdot) + (\cdot)^T z}{\psi(\cdot) - (\cdot)^T v} + \mu_{J_0}^T (\pi_j J_0 + (\cdot)^T w J_0) e$ denotes the vector whose components are $\frac{\phi_1(\cdot) + (\cdot)^T z_1}{\psi_1(\cdot) - (\cdot)^T v_1} + \sum_{j \in J_0} \mu_j (\pi_j + (\cdot)^T w_j),$
 $\frac{\phi_2(\cdot) + (\cdot)^T z_2}{\psi_2(\cdot) - (\cdot)^T v_2} + \sum_{j \in J_0} \mu_j (\pi_j + (\cdot)^T w_j), \dots, \frac{\phi_k(\cdot) + (\cdot)^T z_k}{\psi_k(\cdot) - (\cdot)^T v_k} + \sum_{j \in J_0} \mu_j (\pi_j + (\cdot)^T w_j)$
and $\{\pi + (\cdot)^T w\}_{J_\beta}^\mu$ denotes r -dimensional vector whose components are

$$\sum_{j \in J_1} \{\pi_j + (\cdot)^T w_j\}, \quad \sum_{j \in J_2} \{\pi_j + (\cdot)^T w_j\}, \dots, \quad \sum_{j \in J_r} \{\pi_j + (\cdot)^T w_j\}.$$

3. Unified higher-order duality model:

In this section, we formulate the following unified higher- order dual for (MFP) and establish duality theorems:

$$\begin{aligned} (\mathbf{HMDP}) : \text{Maximize} \quad & \left(\frac{\phi_1(y) + y^T z_1}{\psi_1(y) - y^T v_1} + H_1(y, p) - p^T \nabla_p H_1(y, p) + \sum_{j \in J_0} \mu_j \{ \pi_j(y) + \right. \\ & y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \}, \dots, \frac{\phi_k(y) + y^T z_k}{\psi_k(y) - y^T v_k} \\ & \left. + H_k(y, p) - p^T \nabla_p H_k(y, p) + \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \} \right) \end{aligned}$$

subject to $y \in Y,$

$$(3.1) \quad \sum_{i=1}^k \lambda_i \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\}$$

$$+ \sum_{j=1}^m \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} = 0,$$

$$(3.2) \quad \sum_{j \in J_\beta} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \} \geq 0, \quad \beta = 1, \dots, r,$$

$$(3.3) \quad \lambda_i \geq 0, \quad \sum_{i=1}^k \lambda_i = 1,$$

$$(3.4) \quad \mu_j \geq 0, \quad z_i \in C_i, \quad v_i \in E_i, \quad w_j \in D_j \quad \text{for } i \in K, \quad j \in M,$$

where $J_\delta \subseteq N$, $\delta = 0, 1, \dots, r$ with $\bigcup_{\delta=0}^r J_\delta = N$ and $J_{\delta_1} \cap J_{\delta_2} = \emptyset$ if $\delta_1 \neq \delta_2$. It may be noted that $J_0 = N$ and $J_\beta = \phi$ ($1 \leq \beta \leq r$), we obtain Wolfe type dual. If $J_0 = \phi$, $J_1 = N$ and $J_\beta = \phi$ ($2 \leq \beta \leq r$), then (HMDP) reduces to Mond-Weir Type dual.

Let W^0 be feasible solution of (HMDP).

Theorem 3.1 (Weak Duality Theorem). Let $x \in Y^0$ and $(y, \lambda, v, \mu, z, w, p) \in W^0$. Let

(i) $\left(\frac{\phi_i(\cdot) + (\cdot)^T z_i}{\psi_i(\cdot) - (\cdot)^T v_i} + \mu_{J_0}^T \left(\pi_j J_0 + (\cdot)^T w_j J_0 \right) e, \{ \pi_j(\cdot) + (\cdot)^T w_j \}_{J_\beta}^\mu \right)$ be higher-order weak strictly pseudo quasi (V, ρ, d) -type I at y ,

$$(ii) \quad \sum_{i=1}^k \lambda_i \rho_i^1 + \sum_{j=1}^r \mu_j \rho_j^2 \geq 0.$$

Then, the following cannot hold

$$\frac{\phi_i(x) + s(x|C_i)}{\psi_i(x) - s(x|E_i)} \leq \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} + H_i(y, p) - p^T \nabla_p H_i(y, p)$$

$$(3.5) \quad + \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \}, \quad \forall i \in K$$

and

$$\begin{aligned} \frac{\phi_j(x) + s(x|C_j)}{\psi_j(x) - s(x|E_j)} &< \frac{\phi_j(y) + y^T z_j}{\psi_j(x) - y^T v_j} + H_j(y, p) - p^T \nabla_p H_j(y, p) \\ &+ \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \}, \text{ for some } j \in K \end{aligned}$$

(3.6)

Proof: If possible, then suppose inequalities (3.5) and (3.6) hold. As $x^T z_i \leq s(x|C_i)$, $x^T v_i \leq s(x|E_i)$, $\forall i \in K$ and $\sum_{j \in J_0} \mu_j (\pi_j(x) + x^T w_j) \leq 0$, using the inequalities and the dual constraint (3.2), hypothesis (i) gives

$$\begin{aligned} \eta^T(x, y) \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) + \sum_{j \in J_0} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} e \right\} \\ < -\rho^1 d^2(x, y) \end{aligned}$$

and

$$\eta^T(x, y) \sum_{j \in J_\beta} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} + \rho_\beta^2 d^2(x, y) \leq 0, \quad \beta = 1, \dots, r.$$

Since $\lambda \geq 0$, $\lambda^T e = 1$, it follows that

$$\begin{aligned} \eta^T(x, y) \left(\sum_{i=1}^k \lambda_i \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right. \\ \left. + \sum_{j \in J_0} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \right) < - \sum_{i=1}^k \lambda_i \rho_i^1 d^2(x, y) \end{aligned}$$

and

$$\eta^T(x, y) \left(\sum_{j \in J_\beta} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \right) \leq -\rho_\beta^2 d^2(x, y), \quad \beta = 1, \dots, r.$$

Above inequalities follows that

$$\begin{aligned}
 & \eta^T(x, y) \left(\sum_{i=1}^k \lambda_i \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} + \sum_{j=1}^m \mu_j \{ \nabla \pi_j(y) + w_j \right. \\
 & \left. + \nabla_p K_j(y, p) \} \right) = \eta^T(x, y) \left(\sum_{i=1}^k \lambda_i \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right. \\
 & \left. + \sum_{j \in J_0} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} + \sum_{j \in J_1} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \right. \\
 & \left. + \dots + \sum_{j \in J_r} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \right) \\
 & \eta^T(x, y) \left(\sum_{i=1}^k \lambda_i \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} + \sum_{j=1}^m \mu_j \{ \nabla \pi_j(y) + w_j \right. \\
 & \left. + \nabla_p K_j(y, p) \} \right) \leq \eta^T(x, y) \left(\sum_{i=1}^k \lambda_i \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right. \\
 & \left. + \sum_{j \in J_0} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} + \sum_{j \in J_1} \mu_j \{ \nabla \pi_j(y) + w_j \right. \\
 & \left. + \nabla_p K_j(y, p) \} + \dots + \sum_{j \in J_r} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \right) \\
 & < - \left(\sum_{i=1}^k \lambda_i \rho_i^1 + \sum_{j=1}^r \mu_j \rho_j^2 \right) d^2(x, y).
 \end{aligned}$$

Further, using hypothesis (ii), we have

$$\begin{aligned}
 & \eta^T(x, y) \left(\sum_{i=1}^k \lambda_i \left\{ \nabla \left(\frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right. \\
 & \left. + \sum_{j=1}^m \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \right) < 0,
 \end{aligned}$$

which contradicts (3.1). Hence, completes the proof.

Theorem 3.2 (Weak Duality Theorem). Let $x \in Y^0$ and $(y, \lambda, v, \mu, z, w, p) \in W^0$. Let

- (i) $\left(\frac{\phi_i(\cdot) + (\cdot)^T z_i}{\psi_i(\cdot) + (\cdot)^T v_i} + \mu_{J_0}^T \left(\pi_j J_0 + (\cdot)^T w_j J_0 \right) e, \{ \pi_j(\cdot) + (\cdot)^T w_j \}_{J_\beta}^\mu \right)$ be higher-order pseudo quasi (V, ρ, d) -type I at y ,
- (ii) $\sum_{i=1}^k \lambda_i \rho_i^1 + \sum_{j=1}^r \mu_j \rho_j^2 \geq 0$.

Then, the following cannot hold

$$\begin{aligned} \frac{\phi_i(x) + s(x|C_i)}{\psi_i(x) - s(x|E_i)} &\leq \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} + H_i(y, p) - p^T \nabla_p H_i(y, p) \\ (3.7) \quad &+ \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \}, \quad \forall i \in K \end{aligned}$$

and

$$\begin{aligned} \frac{\phi_j(x) + s(x|C_j)}{\psi_j(x) - s(x|E_j)} &< \frac{\phi_j(y) + y^T z_j}{\psi_j(y) - y^T v_j} + H_j(y, p) - p^T \nabla_p H_j(y, p) \\ &+ \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \}, \text{ for some } j \in K. \\ (3.8) \end{aligned}$$

Proof The proof follows on the lines of Theorem 3.1.

Theorem 3.3 (Strong Duality Theorem). If $\bar{u}$ is an efficient solution of (MFP) and let the Kuhn-Tucker constraint qualification be satisfied. Then, $\exists \bar{\lambda} \in R^k, \bar{y} \in R^m, \bar{z}_i \in R^n, \bar{v}_i \in R^n$ and $\bar{w}_j \in R^n, i \in K, j \in M$, such that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p}) \in W^0$ and the (MFP) and (HMDP) have equal values. Also, if

$$H(\bar{u}, 0) = 0, \nabla_p H(\bar{u}, 0) = 0, K(\bar{u}, 0) = 0, \nabla_p K(\bar{u}, 0) = 0.$$

Furthermore, if the assumptions of Theorem 3.1 or 3.2 hold for Y^0 and W^0 , then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is an efficient solution of (HMDP).

Proof. Since $\bar{u}$ is an efficient solution for (MFP) and the Slaters constraint qualification is satisfied, from Theorem 2.1, there exist $0 < \bar{\lambda} \in R^k$, $0 \leq \bar{y}_j \in R^m$, $\bar{z}_i \in R^n$, $\bar{v}_i, \bar{w}_j \in R^n$, $i \in K$, $j \in M$ such that

$$(3.9) \quad \sum_{i=1}^k \bar{\lambda}_i \nabla \left(\frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \sum_{j=1}^m \bar{y}_j \nabla (\pi_j(u) + u^T \bar{w}_j) = 0,$$

$$(3.10) \quad \sum_{j=1}^m \bar{y}_j (\pi_j(u) + u^T \bar{w}_j) = 0,$$

$$(3.11) \quad u^T \bar{z}_i = S(u|C_i), \quad u^T \bar{v}_i = S(u|E_i), \quad u^T \bar{w}_j = S(u|D_j),$$

$$(3.12) \quad \bar{z}_i \in C_i, \quad \bar{v}_i \in D_i, \quad \bar{w}_j \in E_j, \quad i \in K, \quad j \in M.$$

Using the assumption $H(\bar{u}, 0) = 0$, $\nabla_p H(\bar{u}, 0) = 0$, $K(\bar{u}, 0) = 0$, $\nabla_p K(\bar{u}, 0) = 0$, we find that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0) \in W^0$ and the two objective values are same. With the help of contradiction, we can prove efficiency results. Hence, the results. $\square$

Theorem 3.4 (Strict Converse Duality Theorem). Let $u \in Y^0$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in W^0$ such that

$$(i) \quad \sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right\} \leq \sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} + \nabla \rho H_i(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p H_i(\bar{y}, \bar{p}) \right. \\ \left. + \sum_{j \in J_0} \bar{\mu}_j \{ \pi_j(\bar{y}) + \bar{y}^T \bar{w}_j + K_j(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p K_j(\bar{y}, \bar{p}) \} \right\},$$

$$(ii) \quad \rho_i^1 + \sum_{j=1}^r \rho_j^2 \geq 0, \quad \forall i, j,$$

$$(iii) \quad \left(\sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(\cdot) + (\cdot)^T \bar{z}_i}{\psi_i(\cdot) - (\cdot)^T \bar{v}_i} \right\} + \sum_{j \in J_0} \bar{\mu}_j \{ \pi_j + (\cdot)^T \bar{w}_j \}, \{ \pi_j(\cdot) + (\cdot)^T \bar{w}_j \}_{J_\beta}^{\bar{\mu}} \right)$$

is higher order strictly pseudoquasi $(V, \rho, d) - type I$ at $\bar{y}$.

Then, $u = \bar{y}$.

Proof. Suppose $u \neq \bar{y}$. The dual constraint (3.2) and the hypothesis (iii), for $\beta = 1, \dots, r$ yield

$$(3.13) \quad \eta^T(u, \bar{y}) \sum_{j \in J_\beta} \bar{\mu}_j \{ \nabla \pi_j(\bar{y}) + \nabla \rho K_j(\bar{y}, p) + \bar{w}_j \} \leq -\rho_\beta^2 d^2(u, \bar{y})$$

By the dual constraint (3.1), we have

$$\begin{aligned} \eta^T(u, \bar{y}) & \left(\sum_{i=1}^k \bar{\lambda}_i \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right\} \right. \\ & \left. + \sum_{j=1}^m \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \right) = 0, \end{aligned}$$

above inequalities with (3.13) give

$$\begin{aligned} & \eta^T(u, \bar{y}) \left(\sum_{i=1}^k \bar{\lambda}_i \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right\} + \sum_{j \in J_0} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) \right. \right. \\ & \left. \left. + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \right) \geq -\eta^T(u, \bar{y}) \left(\sum_{j \in J_1} \bar{\mu}_j \{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w}_j \} \right) \\ & \quad - \dots - \eta^T(u, \bar{y}) \left(\sum_{j \in J_r} \bar{\mu}_j \{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w}_j \} \right) \end{aligned}$$

or

$$\begin{aligned} & \eta^T(u, \bar{y}) \left(\sum_{i=1}^k \bar{\lambda}_i \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right\} \right) \\ & + \sum_{j \in J_0} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \geq \sum_{j=1}^r \rho_j^2 d^2(u, \bar{y}), \end{aligned}$$

by hypothesis (ii),

$$\begin{aligned} & \eta^T(u, \bar{y}) \left(\sum_{i=1}^k \bar{\lambda}_i \left\{ \nabla \left(\frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right\} \right) \\ & + \sum_{j \in J_0} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \geq -\rho_i^1 d^2(u, \bar{y}). \end{aligned}$$

Therefore, hypothesis (iii) in view of $\sum_{j \in J_0} \mu_j \{\pi_j(u) + u^T \bar{w}_j\} \leq 0$ yields

$$\begin{aligned} & \sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right\} > \sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} + \nabla_p H_i(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p H_i(\bar{y}, \bar{p}) \right\} \\ & + \sum_{j \in J_0} \bar{\mu}_j \{\pi_j(\bar{y}) + \bar{y}^T \bar{w}_j + K_j(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p K_j(\bar{y}, \bar{p})\}, \end{aligned}$$

which contradicts hypothesis (i). Hence, the result.

Theorem 3.5 (Strict Converse Duality Theorem). Let $u \in Y^0$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in W^0$ such that

$$\begin{aligned} (i) \quad & \sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right\} \leq \sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} + \nabla_p H_i(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p H_i(\bar{y}, \bar{p}) \right\} \\ & + \sum_{j \in J_0} \bar{\mu}_j \{\pi_j(\bar{y}) + \bar{y}^T \bar{w}_j + K_j(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p K_j(\bar{y}, \bar{p})\}, \\ (ii) \quad & \rho_i^1 + \sum_{j=1}^r \rho_j^2 \geq 0, \quad \forall i, j, \end{aligned}$$

$$(iii) \quad \left(\sum_{i=1}^k \bar{\lambda}_i \left\{ \frac{\phi_i(\cdot) + (\cdot)^T \bar{z}_i}{\psi_i(\cdot) - (\cdot)^T \bar{v}_i} \right\} + \sum_{j \in J_0} \bar{\mu}_j \{ \pi_j + (\cdot)^T \bar{w}_j \}, \{ \pi_j(\cdot) + (\cdot)^T \bar{w}_j \}_{J_\beta}^{\bar{\mu}} \right)$$

is higher order quasistrictly pseudo (V, ρ, d) -type I at $\bar{y}$.

Then, $u = \bar{y}$.

Proof The proof follows on the lines of theorem 3.4.

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