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On the total irregularity strength of convex polytope graphs

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Abstract

A vertex (edge) irregular total k-labeling ϕ of a graph G is a labeling of the vertices and edges of G with labels from the set $\{1, 2, \ldots, k\}$ in such a way that any two different vertices (edges) have distinct weights. Here, the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x, whereas the weight of an edge is the sum of label of the edge and the vertices incident to that edge. The minimum k for which the graph G has a vertex (edge) irregular total k-labeling is called the total vertex (edge) irregularity strength of G.

In this paper, we are dealing with infinite classes of convex polytopes generated by prism graph and antiprism graph. We have determined the exact value of their total vertex irregularity strength and total edge irregularity strength.

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1. Introduction

The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of graph labeling, but also for the wide range of its application, for instance X-ray, crystallography, coding theory, radar, astronomy, circuit design, network design and communication design. Bloom and Golomb studied applications of graph labelings to other branches of science [5, 6].

Let us consider a simple (without loops and multiple edges) undirected graph $G = (V, E)$. A total labeling is defined as a labeling in which all the vertices and edges are labeled. For a graph G, we define a labeling $\phi : V \cup E \to \{1, 2, \ldots, k\}$ to be a vertex irregular total k-labeling of the graph G if for every two different vertices x and y of G, $wt(x) \neq wt(y)$ where the weight of a vertex x in the labeling ϕ is

$$wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy),$$

where $N(x)$ is the set of neighbors of x. On the other hand, ϕ is said to be edge irregular total labeling if for any two distinct edges e and f of G, $wt(e) \neq wt(f)$, where the weight of an edge e in the labeling ϕ is

$$wt(e) = \phi(e) + \phi(x) + \phi(y)$$

where x and y are vertices adjacent to edge e. In [2] Bača *et al.* defined a new graph invariant, called the total vertex (edge) irregularity strength of G, $tvs(G)$ $(tes(G))$, that is the minimum k for which the graph G has a vertex (edge) irregular total k-labeling.

The original motivation for the definition of the total vertex (edge) irregularity strength came from irregular assignments and the irregularity strength of graphs introduced in [9] by Chartrand *et al.* and studied by numerous in [7, 10, 11, 12, 14].

An *irregular assignment* is a k-labeling of the edges

$$f : E \to \{1, 2, \ldots, k\}$$

such that the vertex weights

$$w(x) = \sum_{y \in N(x)} f(xy)$$

are different for all vertices of G, and the smallest k for which there is an irregular assignment is the *irregularity strength*, $s(G)$.

The irregularity strength $s(G)$ can be interpreted as the smallest integer k for which G can be turned into a multigraph G' by replacing each edge by a set of at most k parallel edges, such that the degrees of the vertices in G' are all different.

It is easy to see that the irregularity strength $s(G)$ of a graph G is defined only for graphs containing at most one isolated vertex and no connected component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G. Thus, the total labeling ϕ is a vertex irregular total labeling and for graphs with no component of order ≤ 2, $tvs(G) \leq s(G)$. Nierhoff [16] proved that for every (p, q)-graph G (i.e. the graph with p vertices and q edges) with no component of order at most 2 and $G \neq K_3$, the irregularity strength $s(G) \leq p - 1$. From this result it follows that

$$tvs(G) \leq p - 1. \tag{1.1}$$

In [2], Bača *et al.* put forward the lower bounds for total vertex irregularity strength and total edge irregularity strength in terms of maximum degree Δ, minimum degree δ, $|V(G)| = p$ and $|E(G)| = q$, which may be stated as the theorems 1 and 2.

Theorem 1.1. *Let G be a (p, q)-graph having minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Then the lower bound for $tvs(G)$ is*

$$tvs(G) \geq \left\lceil \frac{p + \delta}{\Delta + 1} \right\rceil, \tag{1.2}$$

Theorem 1.2. *The lower bound for $tes(G)$ is*

$$tes(G) \geq max\{\left\lceil \frac{q + 2}{3} \right\rceil, \left\lceil \frac{\Delta + 1}{2} \right\rceil\} \tag{1.3}$$

For graphs with no component of order ≤ 2, Bača *et al.* in [2] proved the following inequality:

$$tvs(G) \leq p - 1 - \left\lceil \frac{p - 2}{\Delta + 1} \right\rceil. \tag{1.4}$$

These results were then improved by Przybylo in [17] for sparse graphs and for graphs with large minimum degree. In [18, 19], Wijaya *et al.*

found the exact values of the total vertex irregularity strength of wheels, fans, suns, friendship graphs and complete bipartite graphs. More recently, Bokhary *et al.* [8] determined an exact value of the total vertex irregularity strength for some families of cubic graphs.

In [2], Baca et al. determined the total vertex irregularity strength of prisms. The prism and antiprism are the Archimedian convex polytopes defined e.g. in [15]. Ahmad determined the total vertex irregularity strength of some classes of convex polytopes in [1]. In this paper, we determined an exact value of total vertex irregularity strength as well as total edge irregularity strength for some infinite classes of convex polytopes.

2. The graph of convex polytope S_n

For $n \geq 3$, the graph of convex polytope S_n (Fig.1) consisting of $2n$ 3-sided faces, $2n$ 4-sided faces and a pair of n-sided faces, is obtained by the combination of the graph of convex polytope R_n [4] and the graph of a prism D_n. We have

$$V(S_n) = \{a_i, b_i, c_i, d_i | 1 \leq i \leq n\}$$

and

$\mathrm{E}(\mathrm{S}_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1} | 1 \leq i \leq n\}$
$\cup \{a_{i+1} b_i, a_i b_i, b_i c_i, c_i d_i | 1 \leq i \leq n\}$.

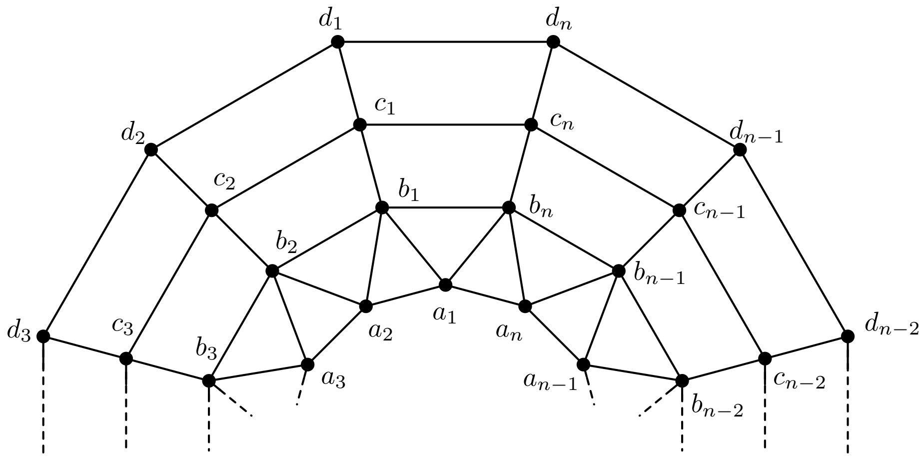

Figure 1: Graph S_n

The graph of convex polytope S_n can also be obtained from the graph of convex polytope Q_n defined in [4] by adding the edges $a_{i+1}b_i, c_i c_{i+1}$

and then deleting the edges $b_{i+1}c_i$. i.e., $V(S_n) = V(Q_n)$ and $E(S_n) = (E(Q_n) \cup \{a_{i+1}b_i, c_ic_{i+1} | 1 \leq i \leq n\}) \backslash \{b_{i+1}c_i | 1 \leq i \leq n\}$.

The total vertex irregularity strength of graph of convex polytope R_n and graph of a prism D_n have been studied in [1] and [2], respectively. In the next two theorems, we find the total vertex irregularity strength and total edge irregularity strength of the graph of convex polytope S_n.

Theorem 2.1. *For $n \geq 3$, the total vertex irregularity strength of S_n is $\lceil \frac{4n+3}{6} \rceil$.*

Proof. From inequality (1.2) it follows that $tvs(S_n) \geq \left\lceil \frac{4n+3}{6} \right\rceil$.
Let $k = \lceil \frac{4n+3}{6} \rceil$. It is sufficient to define an irregular total k-labeling as following. For $1 \leq i \leq n$, let

$$\begin{aligned}
\phi(d_i) = \phi(a_i) &= \lceil \frac{i}{2} \rceil \\
\phi(d_i d_{i+1}) = \phi(c_i) &= 1 \\
\phi(d_i c_i) &= \lceil \frac{i+1}{2} \rceil \\
\phi(c_i c_{i+1}) &= \lceil \frac{n+2}{3} \rceil \\
\phi(c_i b_i) &= \lceil \frac{i}{2} \rceil + \lceil \frac{n-3}{3} \rceil \\
\phi(b_i) = \phi(b_i a_{i+1}) = \phi(b_i b_{i+1}) &= k \\
\phi(b_i a_i) &= \lceil \frac{i+1}{2} \rceil + 1 \\
\phi(a_i a_{i+1}) &= \lceil \frac{4n+1}{6} \rceil.
\end{aligned}$$

One can easily check that different vertices in S_n have different weights and so $tvs(S_n) \leq k$. Hence, Inequality (1.2) implies that $tvs(S_n) = k$. □

Theorem 2.2. *For $n \geq 3$, the total edge irregularity strength of S_n is $\lceil \frac{8n+2}{3} \rceil$.*

Proof. From inequality (1.3) it follows that $tes(S_n) \geq \left\lceil \frac{8n+2}{3} \right\rceil$.
Let $\lceil \frac{8n+2}{3} \rceil = k$ and define an irregular total k-labeling in order to show that k is an upper bound for $tes(S_n)$.

For $1 \leq i \leq n$,
$\phi(d_i) = 1$,
$\phi(d_i d_{i+1}) = \phi(d_i c_i) = \phi(c_i c_{i+1}) = \phi(c_i b_i) = \phi(b_i b_{i+1}) = i$,
$\phi(c_i) = n + 1$,
$\phi(b_i) = 2n + 1$,
$\phi(a_i) = k$,
$\phi(b_i a_i) = 2i - 1 + \lceil \frac{n-1}{3} \rceil$,
$\phi(b_i a_{i+1}) = 2i + \lceil \frac{n-1}{3} \rceil$,
$\phi(a_i a_{i+1}) = n + i + 2\lceil \frac{n-1}{3} \rceil$.

One can easily check that distinct edges in S_n have different weights and so $tes(S_n) \leq k$. Hence, Inequality (1.3) implies that $tes(S_n) = k$. $\square$

3. The graph of convex polytope T_n

For $n \geq 3$, the graph of convex polytope T_n (Fig. 2) consists of $4n$ 3-sided faces, n 4-sided faces and a pair of n-sided faces, and is obtained by the combination of the graph of convex polytope R_n [4] and the graph of an antiprism A_n [3]. We have

$$V(T_n) = \{a_i, b_i, c_i, d_i | 1 \leq i \leq n\}$$

and

$\mathrm{E(T}_n) = \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1} | 1 \leq i \leq n\}$
$\cup \{a_{i+1} b_i, a_i b_i, b_i c_i, c_i d_i, c_{i+1} d_i | 1 \leq i \leq n\}$.

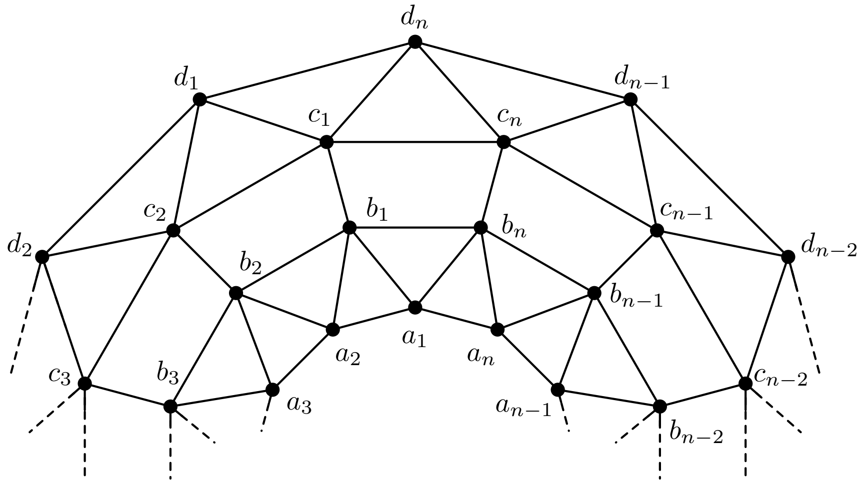

Figure 2: Graph T_n

The graph of convex polytope T_n can also be obtained from the graph of convex polytope S_n by adding the edges $c_{i+1}d_i$. i.e., $V(T_n) = V(S_n)$ and $E(T_n) = (E(S_n) \cup \{a_{i+1}b_i, c_ic_{i+1}, c_{i+1}d_i|1 \leq i \leq n\})\backslash\{b_{i+1}c_i|1 \leq i \leq n\}$. The total vertex irregularity strength of the graph of convex polytope R_n and graph of an antiprism A_n have been studied in [1]. In the next two theorems we find the total vertex irregularity strength and total edge irregularity strength of the graph of convex polytope T_sn.

Theorem 3.1. *For $n \geq 6$, the total vertex irregularity strength of T_n is $\lceil \frac{4n+4}{6} \rceil$.*

Proof. Let $k = \lceil \frac{4n+4}{6} \rceil$. From inequality (1.2) it is clear that $tvs(T_n) \geq k$. Now we will define an irregular total k-labeling in order to show that k is an upper bound for $tvs(T_n)$.
For $1 \leq i \leq n$,

$\phi(d_i) = \lceil \frac{i}{2} \rceil$,
$\phi(d_id_{i+1}) = \phi(d_ic_{i+1}) = \phi(a_ia_{i+1}) = 1$,
$\phi(d_ic_i) = \lceil \frac{i+1}{2} \rceil$,
$\phi(c_i) = \lceil \frac{n+1}{2} \rceil$,
$\phi(c_ib_i) = \lceil \frac{i}{2} \rceil + \lfloor \frac{n+1}{6} \rfloor$,
$\phi(c_ic_{i+1}) = \phi(b_i) = \phi(b_ia_{i+1}) = \phi(b_ib_{i+1}) = k$,
$\phi(b_ia_i) = \lceil \frac{i+1}{2} \rceil + \lfloor \frac{n+4}{6} \rfloor$,
$\phi(a_i) = \lceil \frac{i}{2} \rceil + \lfloor \frac{n+1}{6} \rfloor$.

One can easily check that distinct vertices in T_n have different weights and so $tvs(T_n) \leq k$. Hence, Inequality (1.2) implies that $tvs(T_n) = k$. □

Theorem 3.2. *For $n \geq 3$, the total edge irregularity strength of T_n is $\lceil \frac{9n+2}{3} \rceil$.*

Proof. Let $k = \lceil \frac{9n+2}{3} \rceil$. From inequality (1.3), we have $tvs(T_n) \geq k$. To prove the equality, we define an irregular total k-labeling as follows:

For $1 \leq i \leq n$,
$\phi(d_i) = 1$,
$\phi(d_id_{i+1}) = i + 1$,
$\phi(c_i) = n + 1$,
$\phi(d_ic_i) = 2i - 1$,
$\phi(d_ic_{i+1}) = 2i$,

$\phi(c_ic_{i+1}) = \phi(c_ib_i) = \phi(b_ib_{i+1}) = \phi(a_ia_{i+1}) = n+i,$
$\phi(b_i) = 2n+1,$
$\phi(a_i) = k,$
$\phi(b_ia_i) = 2n+2i-1,$
$\phi(b_ia_{i+1}) = 2n+2i.$

One can easily check that distinct edges in T_n have different weights and so $tes(T_n) \leq k$. Hence, Inequality (1.3) implies that $tes(T_n) = k$. □

4. The graph of convex polytope U_n

For $n \geq 3$, the graph of convex polytope U_n (Fig. 3) consists of n 4-sided faces, $2n$ 5-sided faces and a pair of n-sided faces, and is obtained as a combination of the graph of convex polytope D_n [3] and graph of a prism D_n. We have

$$V(U_n) = \{a_i, b_i, c_i, d_i, e_i | 1 \leq i \leq n\}$$

and

$\mathrm{E(U}_n) = \{a_ia_{i+1}, b_ib_{i+1}, e_ie_{i+1} | 1 \leq i \leq n\}$
$\cup \{a_ib_i, b_ic_i, c_id_i, d_ie_i, c_{i+1}d_i | 1 \leq i \leq n\}.$

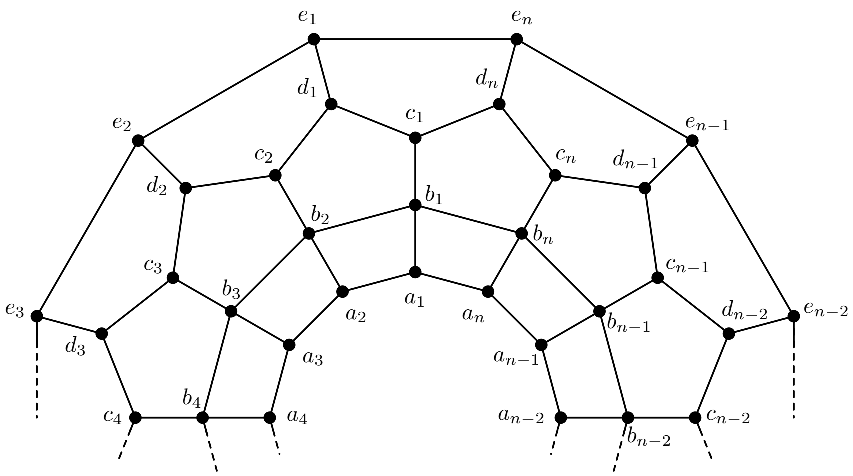

Figure 3: Graph U_n

The graph of convex polytopes D_n and graph of a prism D_n have been studied in [1] and [2]. In the next two theorems, we find the The total vertex and edge irregularity strength of the graph of convex polytope U_n.

Theorem 4.1. *For $n \geq 5$, the total vertex irregularity strength of U_n is* $\lceil \frac{5n+3}{5} \rceil$.

Proof. Let $k = \lceil \frac{5n+3}{5} \rceil$. To show that $tvs(U_n) = k$, we define the following k-labelling:
For $1 \leq i \leq n$,
$\phi(e_i) = \phi(e_i e_{i+1}) = \phi(d_i) = 1$,
$\phi(e_i d_i) = \phi(b_i) = \phi(c_i) = i$,
$\phi(d_i c_{i+1}) = \phi(d_i c_i) = \lceil \frac{n+2}{2} \rceil$,
$\phi(c_i b_i) = \phi(b_i a_i) = \phi(b_i b_{i+1}) = \phi(a_i a_{i+1}) = k$,
$\phi(a_i) = i + 1$.

From the above labeling, it is clear that every vertex has distinct weight. Thus, ϕ is a vertex irregular k-labelling and therefore $tvs(U_n) \leq k$. Hence, Inequality (1.2) implies that $tvs(U_n) = k$. □

Theorem 4.2. *For $n \geq 3$, the total edge irregularity strength of U_n is $\lceil \frac{8n+2}{3} \rceil$.*

Proof. Let $k = \lceil \frac{8n+2}{3} \rceil$. Define an irregular total k-labeling as follows.
For $1 \leq i \leq n$,
$\phi(e_i) = 1$,
$\phi(e_i e_{i+1}) = \phi(e_i d_i) = i$,
$\phi(d_i) = \phi(c_i) = n + 1$,
$\phi(d_i c_i) = 2i - 1$,
$\phi(d_i c_{i+1}) = 2i$,
$\phi(b_i) = \lceil \frac{7n+2}{3} \rceil$,
$\phi(c_i b_i) = \lceil \frac{2n-1}{3} \rceil + i$,

$$\phi(b_i b_{i+1}) = \begin{cases} \lceil \frac{n-2}{3} \rceil + i & \text{for } n \equiv 2 (mod 3) \\ \lceil \frac{n}{3} \rceil + i & \text{for } n \equiv 0, 1 (mod 3) \end{cases}$$

$\phi(a_i) = k$,
$\phi(b_i a_i) = n + i$,
$\phi(a_i a_{i+1}) = 2\lceil \frac{n-1}{3} \rceil + n + i$.

From the above labeling, it is clear that every edge has distinct weight, so $tes(U_n) \leq k$. Hence, Inequality (1.3) implies that $tes(U_n) = k$. □

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