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# Horadam polynomials and their applications to new family of bi-univalent functions with respect to symmetric conjugate points

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## Abstract:

*In the current paper, by making use of the Horadam polynomials, we introduce and investigate a new family of holomorphic and biunivalent functions with respect to symmetric conjugate points defined in the open unit disk* $\mathbf{D}$. *We derive upper bounds for the second and third coefficients and solve Fekete-Szegö problem of functions belongs to this family.*

**Keywords:** Bi-univalent function; Horadam polynomials; Upper bounds; Symmetric conjugate; Fekete-Szegö problem; Subordination.

**MSC (2020):** 30C45, 30C50.

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## 1. Introduction

Denote by $\mathcal{A}$ the collection of holomorphic functions in the open unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, let $S$ indicate the sub-collection of $\mathcal{A}$ consisting of functions in $\mathbf{D}$ satisfying (1.1) which are univalent in $\mathbf{D}$.

Also, let $S_{sc}^{*}$ be the subclass of $S$ consisting of functions given by (1.1) satisfying

$$Re\left\{\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right\} > 0, \quad z \in \mathbf{D}.$$

These functions are called starlike with respect to symmetric conjugate points and were introduced by El-Ashwah and Thomas [6]. The class can be extended to other class in $\mathbf{D}$, namely convex functions with respect to symmetric conjugate points. Let $C_{sc}$ denote the class of convex functions with respect to symmetric conjugate points and satisfy the conditions

$$Re\left\{\frac{(zf'(z))'}{\left(f(z) - \overline{f(-\overline{z})}\right)'}\right\} > 0, \quad z \in \mathbf{D}.$$

According to the Koebe One-Quarter Theorem [5] "every function $f \in S$ has an inverse $f^{-1}$ defined by $f^{-1}(f(z)) = z$, $(z \in \mathbf{D})$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$", where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbf{D}$ if both $f$ and $f^{-1}$ are univalent in $\mathbf{D}$. Let $\Sigma$ stands for the class of bi-univalent functions in $\mathbf{D}$ given by (1.1). In fact, Srivastava et al. [16] has apparently revived the study of holomorphic and bi-univalent functions in recent years, it was followed by such works as those by Bulut [4], Altınkaya and Yalçın [2, 3], Adegani et al. [1] and others (see, for example [13, 14, 15, 17, 18, 19]). We notice that the class $\Sigma$ is not empty. For example, the functions $z$, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of $\Sigma$. However, the Koebe function is not a member of $\Sigma$. Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$, $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

"With a view to recalling the principal of subordination between holomorphic functions, let the functions $f$ and $g$ be holomorphic in $\mathbf{D}$. We say that the function $f$ is said to be subordinate to $g$, if there exists a Schwarz function $w$ holomorphic in $\mathbf{D}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbf{D})$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbf{D})$. It is well known that (see [12]), if the function $g$ is univalent in $\mathbf{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbf{D}) \subset g(\mathbf{D})$".

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [8]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbf{R}, n \in \mathbf{N} = \{1, 2, 3, \cdots\}), \tag{1.3}$$

with $h_1(r) = a \quad and \quad h_2(r) = br$, for some real constant $a, b, p$ and $q$. The characteristic equation of repetition relation (1.3) is $t^2 - prt - q = 0$. This equation has two real roots $x = \frac{pr+\sqrt{p^2r^2+4q}}{2}$ and $y = \frac{pr-\sqrt{p^2r^2+4q}}{2}$.

**Remark 1.1.** *By selecting the particular values of $a$, $b$, $p$ and $q$, the Horadam polynomial $h_n(r)$ reduces to several polynomials. Some of them are illustrated below:*

1. *Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(r)$.*

2. *Taking $a = 2$ and $b = p = q = 1$, we attain the Lucas polynomials $L_n(r)$.*

3. *Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(r)$.*

4. *Taking $a = b = p = 2$ and $q = 1$, we get the Pell-Lucas polynomials $Q_n(r)$.*

5. *Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind.*

6. *Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(r)$ of the second kind.*

These polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in many of sciences, specially in the mathematics, statistics and physics. For more information associated with these polynomials see [7, 8, 10, 11].

The generating function of the Horadam polynomials $h_n(r)$ (see [9]) is given by

$$\Pi(r,z)=\sum_{n=1}^{\infty}h_n(r)z^{n-1}=\frac{a+(b-ap)rz}{1-prz-qz^2}. \tag{1.4}$$

## 2. Main Results

We begin this section by defining the family $\mathcal{G}_\Sigma(\lambda,\eta,r)$ as follows:

**Definition 2.1.** *For $0\le\eta\le\lambda\le1$ and $r\in\mathbf{R}$, a function $f\in\Sigma$ with $a_n\in\mathbf{R}$ is said to be in the class $\mathcal{G}_\Sigma(\lambda,\eta,r)$ if it fulfills the subordinations:*

$$\frac{2\left[\lambda\eta z^3f'''(z)+(\lambda+\eta(2\lambda-1))\,z^2f''(z)+zf'(z)\right]}{\lambda\eta z^2\left(f(z)-\overline{f(-\overline{z})}\right)''+(\lambda-\eta)z\left(f(z)-\overline{f(-\overline{z})}\right)'+(1-\lambda+\eta)\left(f(z)-\overline{f(-\overline{z})}\right)}$$
$$\prec\Pi(r,z)+1-a$$

*and*

$$\frac{2\left[\lambda\eta w^3g'''(w)+(\lambda+\eta(2\lambda-1))\,w^2g''(w)+wg'(w)\right]}{\lambda\eta w^2\left(g(w)-\overline{g(-\overline{w})}\right)''+(\lambda-\eta)w\left(g(w)-\overline{g(-\overline{w})}\right)'+(1-\lambda+\eta)\left(g(w)-\overline{g(-\overline{w})}\right)}$$
$$\prec\Pi(r,w)+1-a,$$

*where $a$ is real constant and the function $g=f^{-1}$ is given by (1.2).*

**Theorem 2.1.** *For $0\le\eta\le\lambda\le1$ and $r\in\mathbf{R}$, let $f\in\mathcal{A}$ with $a_n\in\mathbf{R}$ be in the class $\mathcal{G}_\Sigma(\lambda,\eta,r)$. Then*

$$|a_2|\le\frac{|br|\sqrt{|br|}}{\sqrt{2\left|\left[(6\lambda\eta+2(\lambda-\eta)+1)\,b-2p\,(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa\,(2\lambda\eta+\lambda-\eta+1)^2\right|}}$$

*and*

$$|a_3|\le\frac{|br|}{2\,(6\lambda\eta+2(\lambda-\eta)+1)}+\frac{b^2r^2}{4\,(2\lambda\eta+\lambda-\eta+1)^2}.$$

**Proof.** Let $f \in \mathcal{G}_\Sigma(\lambda, \eta, r)$. Then there are two holomorphic functions $u, v : \mathbf{D} \longrightarrow \mathbf{D}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in \mathbf{D}) \tag{2.1}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \quad (w \in \mathbf{D}), \tag{2.2}$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in \mathbf{D}$ such that

$$\frac{2\left[\lambda\eta z^3 f'''(z) + (\lambda + \eta(2\lambda - 1))\, z^2 f''(z) + z f'(z)\right]}{\lambda\eta z^2 \left(f(z) - \overline{f(-\overline{z})}\right)'' + (\lambda - \eta) z \left(f(z) - \overline{f(-\overline{z})}\right)' + (1 - \lambda + \eta)\left(f(z) - \overline{f(-\overline{z})}\right)} = \Pi(r, u(z)) + 1 - a$$

and

$$\frac{2\left[\lambda\eta w^3 g'''(w) + (\lambda + \eta(2\lambda - 1))\, w^2 g''(w) + w g'(w)\right]}{\lambda\eta w^2 \left(g(w) - \overline{g(-\overline{w})}\right)'' + (\lambda - \eta) w \left(g(w) - \overline{g(-\overline{w})}\right)' + (1 - \lambda + \eta)\left(g(w) - \overline{g(-\overline{w})}\right)} = \Pi(r, v(w)) + 1 - a.$$

Or, equivalently

$$\frac{2\left[\lambda\eta z^3 f'''(z) + (\lambda + \eta(2\lambda - 1))\, z^2 f''(z) + z f'(z)\right]}{\lambda\eta z^2 \left(f(z) - \overline{f(-\overline{z})}\right)'' + (\lambda - \eta) z \left(f(z) - \overline{f(-\overline{z})}\right)' + (1 - \lambda + \eta)\left(f(z) - \overline{f(-\overline{z})}\right)}$$
$$= 1 + h_1(r) + h_2(r) u(z) + h_3(r) u^2(z) + \cdots \tag{2.3}$$

and

$$\frac{2\left[\lambda\eta w^3 g'''(w) + (\lambda + \eta(2\lambda - 1))\, w^2 g''(w) + w g'(w)\right]}{\lambda\eta w^2 \left(g(w) - \overline{g(-\overline{w})}\right)'' + (\lambda - \eta) w \left(g(w) - \overline{g(-\overline{w})}\right)' + (1 - \lambda + \eta)\left(g(w) - \overline{g(-\overline{w})}\right)}$$
$$= 1 + h_1(r) + h_2(r) v(w) + h_3(r) v^2(w) + \cdots. \tag{2.4}$$

Combining (2.1), (2.2), (2.3) and (2.4) yields

$$\frac{2\left[\lambda\eta z^3 f'''(z) + (\lambda + \eta(2\lambda - 1))\, z^2 f''(z) + z f'(z)\right]}{\lambda\eta z^2 \left(f(z) - \overline{f(-\overline{z})}\right)'' + (\lambda - \eta) z \left(f(z) - \overline{f(-\overline{z})}\right)' + (1 - \lambda + \eta)\left(f(z) - \overline{f(-\overline{z})}\right)}$$

$$= 1 + h_2(r)u_1 z + \left[h_2(r)u_2 + h_3(r)u_1^2\right] z^2 + \cdots \tag{2.5}$$

and

$$\frac{2\left[\lambda\eta w^3 g'''(w) + (\lambda + \eta(2\lambda - 1))\, w^2 g''(w) + w g'(w)\right]}{\lambda\eta w^2 \left(g(w) - \overline{g(-\overline{w})}\right)'' + (\lambda - \eta) w \left(g(w) - \overline{g(-\overline{w})}\right)' + (1 - \lambda + \eta)\left(g(w) - \overline{g(-\overline{w})}\right)}$$

$$= 1 + h_2(r)v_1 w + \left[h_2(r)v_2 + h_3(r)v_1^2\right] w^2 + \cdots. \tag{2.6}$$

It is quite well-known that if $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in \mathbf{D}$, then

$$|u_i| \le 1 \quad and \quad |v_i| \le 1 \; for\ all\ i \in \mathbf{N}. \tag{2.7}$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$2\left(2\lambda\eta + \lambda - \eta + 1\right) a_2 = h_2(r)u_1, \tag{2.8}$$

$$2\left(6\lambda\eta + 2(\lambda - \eta) + 1\right) a_3 = h_2(r)u_2 + h_3(r)u_1^2, \tag{2.9}$$

$$-2\left(2\lambda\eta + \lambda - \eta + 1\right) a_2 = h_2(r)v_1 \tag{2.10}$$

and

$$2\left(6\lambda\eta + 2(\lambda - \eta) + 1\right)\left(2a_2^2 - a_3\right) = h_2(r)v_2 + h_3(r)v_1^2. \tag{2.11}$$

In view of (2.8) and (2.10), we conclude that

$$u_1 = -v_1 \tag{2.12}$$

and

$$8\left(2\lambda\eta + \lambda - \eta + 1\right)^2 a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \tag{2.13}$$

If we add (2.9) to (2.11), we find that

$$4\left(6\lambda\eta + 2(\lambda - \eta) + 1\right) a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \tag{2.14}$$

Substituting the value of $u_1^2 + v_1^2$ from (2.13) into (2.14), it follows that

$$(2.15) \quad a_2^2 = \frac{h_2^3(r)(u_2+v_2)}{4\left[h_2^2(r)\left(6\lambda\eta+2(\lambda-\eta)+1\right)-2h_3(r)\left(2\lambda\eta+\lambda-\eta+1\right)^2\right]}.$$

Further computations using (1.3), (2.7) and (2.15), we deduce that

$$|a_2| \leq \frac{|br|\sqrt{|br|}}{\sqrt{2\left|\left[\left(6\lambda\eta+2(\lambda-\eta)+1\right)b-2p\left(2\lambda\eta+\lambda-\eta+1\right)^2\right]br^2-2qa\left(2\lambda\eta+\lambda-\eta+1\right)^2\right|}}.$$

To determinate the bound on $|a_3|$, by subtracting (2.11) from (2.9), we can easily see that

$$(2.16) \quad 4\left(6\lambda\eta+2(\lambda-\eta)+1\right)(a_3-a_2^2) = h_2(r)(u_2-v_2)+h_3(r)(u_1^2-v_1^2).$$

Also, by using (2.12) and (2.13) together with (2.16), we conclude that

$$a_3 = \frac{h_2(r)(u_2-v_2)}{4\left(6\lambda\eta+2(\lambda-\eta)+1\right)} + \frac{h_2^2(r)(u_1^2+v_1^2)}{8\left(2\lambda\eta+\lambda-\eta+1\right)^2}.$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{|br|}{2\left(6\lambda\eta+2(\lambda-\eta)+1\right)} + \frac{b^2r^2}{4\left(2\lambda\eta+\lambda-\eta+1\right)^2}.$$

This completes the proof of Theorem 2.1 □

In the next theorem, we discuss the "Fekete-Szegö problem" for the family $\mathcal{G}_\Sigma(\lambda,\eta,r)$.

**Theorem 2.2.** *For $0 \leq \eta \leq \lambda \leq 1$ and $r,\mu \in \mathbf{R}$, let $f \in \mathcal{A}$ with $a_n \in \mathbf{R}$ be in the family $\mathcal{G}_\Sigma(\lambda,\eta,r)$. Then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{|br|}{2(6\lambda\eta+2(\lambda-\eta)+1)} \\ \quad for \ |\mu-1| \leq \frac{\left|\left[(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{b^2r^2(6\lambda\eta+2(\lambda-\eta)+1)}, \\ \frac{|br|^3|\mu-1|}{2\left|\left[(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|} \\ \quad for \ |\mu-1| \geq \frac{\left|\left[(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{b^2r^2(6\lambda\eta+2(\lambda-\eta)+1)}. \end{cases}$$

**Proof.** In the light of (2.15) and (2.16), we find that

$$\begin{aligned} a_3-\mu a_2^2 &= \tfrac{h_2(r)(u_2-v_2)}{4(6\lambda\eta+2(\lambda-\eta)+1)}+(1-\mu)\,a_2^2 \\ &= \tfrac{h_2(r)(u_2-v_2)}{4(6\lambda\eta+2(\lambda-\eta)+1)}+\tfrac{h_2^3(r)(u_2+v_2)(1-\mu)}{4\left[h_2^2(r)(6\lambda\eta+2(\lambda-\eta)+1)-2h_3(r)(2\lambda\eta+\lambda-\eta+1)^2\right]} \\ &= \tfrac{h_2(r)}{4}\left[\left(\psi(\mu,r)+\tfrac{1}{(6\lambda\eta+2(\lambda-\eta)+1)}\right)u_2+\left(\psi(\mu,r)-\tfrac{1}{(6\lambda\eta+2(\lambda-\eta)+1)}\right)v_2\right], \end{aligned}$$

where

$$\psi(\mu,r)=\frac{h_2^2(r)\,(1-\mu)}{h_2^2(r)\,(6\lambda\eta+2(\lambda-\eta)+1)-2h_3(r)\,(2\lambda\eta+\lambda-\eta+1)^2}.$$

According to (1.3), we deduce that

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{|br|}{2(6\lambda\eta+2(\lambda-\eta)+1)}, & 0\le|\psi(\mu,r)|\le\frac{1}{6\lambda\eta+2(\lambda-\eta)+1},\\ \frac{1}{2}\,|br|\,|\psi(\mu,r)|, & |\psi(\mu,r)|\ge\frac{1}{6\lambda\eta+2(\lambda-\eta)+1}.\end{cases}$$

After some computations, we obtain

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{|br|}{2(6\lambda\eta+2(\lambda-\eta)+1)} \\ \quad for\;\;|\mu-1|\le\frac{\left|\left[(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{b^2r^2(6\lambda\eta+2(\lambda-\eta)+1)} \\ \frac{|br|^3|\mu-1|}{2\left|\left[(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|} \\ \quad for\;\;|\mu-1|\ge\frac{\left|\left[(6\lambda\eta+2(\lambda-\eta)+1)b-2p(2\lambda\eta+\lambda-\eta+1)^2\right]br^2-2qa(2\lambda\eta+\lambda-\eta+1)^2\right|}{b^2r^2(6\lambda\eta+2(\lambda-\eta)+1)}.\end{cases}$$

□

Putting $\mu=1$ in Theorem 2.2, we obtain the following result:

**Corollary 2.1.** *For $0\le\eta\le\lambda\le1$ and $r\in\mathbf{R}$, let $f\in\mathcal{A}$ be in the family $\mathcal{G}_\Sigma(\lambda,\eta,r)$. Then*

$$\left|a_3-a_2^2\right|\le\frac{|br|}{2\,(6\lambda\eta+2(\lambda-\eta)+1)}.$$

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