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Separation axioms of α^m-contra-open maps and b-ω-open sets in generalized topological spaces

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Abstract

In this paper, we introduce T_{α^m}-Super-Spaces, α^m-contra-closed maps, α^m-contra-open maps, α^m-contra-continuous maps, α^m-contra-irresolute maps, b-ω-open sets and Continuity via b-ω-open sets and studied some of their properties.

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Key words: *Generalized topological spaces, T_{α^m}-Super-Spaces, α^m-contra-closed maps, α^m-contra-open maps, α^m-contra-continuous maps, α^m-contra-irresolute maps and b-ω-open sets.*

1. Introduction

Separation axioms are among the most common, important and interesting concepts in Topology. They can be used to define more restricted classes of topological spaces. The separation axioms of topological spaces are ussually denoted with the letter "T" after the German "Trennung" which means separation. Most of the weak separation axioms are defined in terms of generalized closed sets and their definitions are deceptively simple. However, the structure and the properties of those spaces are not always that easy to comprehend. The separation axioms that were studied together in this way were the axioms for Hausdorff spaces, regular spaces and normal spaces.

Separation axioms and closed sets in topological spaces have been very useful in the study of certain objects in digital topology ([5], [7]). Khalimsky, Kopperman and Meyer [6] proved that the digital line is a typical asample of $T_{\frac{1}{2}}$-space. There were many definitions offered, some of which assumed to be separation axioms before the current general definition of a topological space. For example, the definition given by Felix Hausdorff in 1914 is equivalent to the modern definition plus the Hausdroff separation axiom. The first step of generalized closed sets was done by Levine in 1970 [5] and he initiated the notion of $T_{\frac{1}{2}}$-space in unital topology which is properly placed between T_0-space and T_1-space by defining $T_{\frac{1}{2}}$-space in which every generalized closed set is closed.

The concept of generalized closed sets (briefly g-closed) in topological spaces was introduced by Levine [9],[10] and a class of topological spaces called $T_{\frac{1}{2}}$-space. Arya and Nour[1],[2] introduced and investigated generalized semi-open sets, semi generalized open sets, generalized open sets, generalized open sets, semi-open sets, pre-open sets and α-open sets, semi pre-open sets and b-open sets which are some of the weak forms of open sets and the complements of these sets are called the same types of closed sets.

In 2014, [11] we introduced the concepts of α^m-closed sets and α^m-open set in topological spaces. Also we have introduced the concepts of α^m-contra-continuous functions and α^m-contra-irresolute functions. In this paper, based on the α^m-open and α^m-closed sets we introduce separation axioms of α^m-open set called T_{α^m}-space. Further various characterisation are studied are introduced.

Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a condensation point of A [34] if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. In 1982, Hdeib defined ω-closed sets and ω-open sets as follows: A is called ω-closed [35] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. The family of all ω-open subsets of X forms a topology on X, denoted by τ_ω. Many topological concepts and results related to ω-closed and ω-open sets appeared in [17], [18], [21], [22], [23], [24], [26], [27] and in the references therein. In 2002, Császár [28] defined generalized topological spaces as follows: the pair (X, μ) is a generalized topological space if X is a nonempty set and μ is a collection of subsets of X such that $\emptyset \in \mu$ and μ is closed under arbitrary unions. For a generalized topological space (X, μ), the elements of μ are called μ-open sets, the complements of μ-open sets are called μ-closed sets, the union of all elements of μ will be denoted by $\mathcal{M}_\mu$, and (X, μ) is said to be strong if $\mathcal{M}_\mu = X$. Recently many topological concepts have been modified to give new concepts in the structure of generalized topological spaces, see [19], [20], [25], [29], [30], [31], [32], [33], and others.

2. Preliminaires

In this paper, we introduce the separation axioms of α^m-contra-open maps and we study the notion of b-ω-open sets in generalized topological spaces, and we use them to introduce new classes of mappings in generalized topological spaces. We present several characterizations, properties, and examples related to the concepts. In Section 3, we introduce and study separation axioms of α^m-contra-open maps. In Section 4, we introduce and study b-ω-open sets in generalized topological spaces. In Section 5, we introduce and study the concept of b-ω-(μ_1, μ_2)-continuous function.

Definition 2.1. *A subset A of a toplogical space (X, τ) is called*

(a) *a preopen set* [10] *if $A \subseteq int(cl(A))$ and preclosed set if $cl(int(A)) \subseteq A$.*

(b) *a semiopen set* [8] *if $A \subseteq cl(int(A))$ and semiclosed set if $int(cl(A)) \subseteq A$.*

(c) *an α-open set* [13] *if $A \subseteq int(cl(int(A))) \subseteq A$ and an α-closed set if $cl(int(cl(A))) \subseteq A$.*

(d) *a semi-preopen set* [1] *(β-open set) if $A \subseteq cl(int(cl(A)))$ and semi-preclosed set if $int(cl(int(A))) \subseteq A$.*

(e) *an α^m-closed set* [11] *if $int(cl(A)) \subseteq U$, whenever $A \subseteq U$ and U is α-open.*

The complement of α^m-closed set is called an α^m-open set.

Definition 2.2. *A space (X, τ_X) is called a $T_{\frac{1}{2}}$-space* [9] *if every g-closed set is closed.*

Definition 2.3. *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called*

(a) *an α^m-continuous* [12] *if $f^{-1}(V)$ is α^m-closed in (X, τ) for every closed set V of (Y, σ).*

(b) *an α^m-irresolute* [12] *if $f^{-1}(V)$ is α^m-closed in (X, τ) for every α^m-closed set V of (Y, σ).*

Definition 2.4. *A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be contra-continuous [15] if $f^{-1}(V)$ is closed for every open set V of Y.*

Definition 2.5. *A space (X, τ) is called T_{α^m}-space [12] if every α^m-closed set is closed.*

Definition 2.6. *([19]). Suppose (X, μ) is a generalized topological space and A a nonempty subset of X. The subspace generalized topology of A on X is generalized topological $\mu_A = \{A \cap U : U \in \mu\}$ on A. The pair (A, μ_A) is called a subspace generalized topological space of (X, μ). A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called a function on generalized topological spaces. From now on, each function is a function on generalized topological spaces unless otherwise stated.*

Definition 2.7. ([*28*]). *A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called (μ_1, μ_2)-continuous at a point $x \in X$, if for every μ_2-open set V containing $f(x)$ there is a μ_1-open set U containing x such that $f(U) \subseteq V$. If f is (μ_1, μ_2)-continuous at each point of X, then f is said to be (μ_1, μ_2)-continuous.*

Definition 2.8. ([*32*]). *A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called (μ_1, μ_2)-closed if $f(C)$ is μ_2-closed in (Y, μ_2) for each μ_1-closed set C.*

3. Separation axioms of α^m-contra-open maps

In this section (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$, $int(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively.

Definition 3.1. *A function $f : (X,\tau) \longrightarrow (Y,\sigma)$ is called an α^m-contra-continuous if $f^{-1}(V)$ is α^m-closed in (X,τ) for every open set V of (Y,σ).*

Proposition 3.2. *If $f : (X,\tau) \longrightarrow (Y,\sigma)$ be an α^m-contra-continuous function and (X,τ) be a T_{α^m}-space. Then f is contra-continuous.*

Proof. Let V be open set in (Y,σ). Since f is an α^m-contra-continuous function, $f^{-1}(V)$ is an α^m-closed set in (X,τ). Since (X,τ) is a T_{α^m}-space, $f^{-1}(V)$ is closed set in (X,τ). Hence f is contra-continuous. □

Definition 3.3. *A function $f : (X,\tau) \longrightarrow (Y,\sigma)$ is called an α^m-contra-irresolute if $f^{-1}(V)$ is α^m-closed in (X,τ) for every α^m-open set V of (Y,σ).*

Theorem 3.4. *Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be a mapping and (X,τ) be a T_{α^m}-space, then f is contra-continuous if one of the following conditions is satisfied.*

(a) *f is α^m-contra-continuous.*

(b) *f is α^m-contra-irresolute.*

Proof. Obvious. □

Theorem 3.5. *A map $f : (X,\tau) \longrightarrow (Y,\sigma)$ is an α^m-contra-continuous function if and only if the inverse image of every open set in (Y,σ) is an α^m-closed set in (X,τ)*

Proof. Necessity: Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be an α^m-contra-continuous function and U be an open set in (Y,σ). Then $Y\backslash U$ is closed in (Y,σ). Since f is an α^m-contra-continuous function, $f^{-1}(Y\backslash U) = X\backslash f^{-1}(U)$ is an α^m-open in (X,τ) and hence $f^{-1}(U)$ is an α^m-closed in (X,τ).

Sufficiency: Assume that $f^{-1}(U)$ is an α^m-closed set in (X,τ) for each open set V in (Y,σ). Let V be a closed set in (Y,σ). Then $Y\backslash V$ is an open set in (Y,σ). By assumption, $f^{-1}(Y\backslash V) = X\backslash f^{-1}(V)$ is an α^m-closed in (X,τ), which implies that $f^{-1}(V)$ is an α^m-open set in (X,τ). Hence f is an α^m-contra-continuous function. □

Proposition 3.6. *Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ be any topological space and (Y,σ) be a T_{α^m}-space. If $f : (X,\tau) \longrightarrow (Y,\sigma)$ is α^m-continuous function and $g : (Y,\sigma) \longrightarrow (Z,\eta)$ is α^m-contra-continuous function, then their composition $g \circ f : (X,\tau) \longrightarrow (Z,\eta)$ is an α^m-contra-continuous function.*

Proof. Let V be a open set in (Z, η). Since $g : (Y, \sigma) \longrightarrow (Z, \eta)$ is an α^m-contra-continuous function, $g^{-1}(V)$ is an α^m-closed in (Y, σ). Since (Y, σ) is a T_{α^m}-space, $g^{-1}(V)$ is a closed set in (Y, σ). Since $f : (X, \tau) \longrightarrow (Y, \sigma)$ is an α^m-continuous function, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is an α^m-closed set in (X, τ). Hence $g \circ f : (X, \tau) \longrightarrow (Z, \eta)$ is an α^m-contra-continuous function. □

Definition 3.7. *A map $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be*

(a) *α^m-closed map* [15] *if $f(V)$ is α^m-closed in (Y, σ) for every closed set V of (X, τ).*

(b) *α^m-open map* [15] *if $f(V)$ is α^m-open in (Y, σ) for every open set V of (X, τ).*

(c) *α^m-contra-closed map if $f(V)$ is α^m-closed in (Y, σ) for every open set V of (X, τ).*

(d) *α^m-contra-open map if $f(V)$ is α^m-open in (Y, σ) for every closed set V of (X, τ).*

Theorem 3.8. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ be two mappings and (Y, σ) be a T_{α^m}-space, then*

(a) *$g \circ f$ is α^m-contra-continuous, if f is an α^m-continuous and g is α^m-contra-continuous.*

(b) *$g \circ f$ is α^m-contra-closed, if f is an α^m-contra-closed and g is an α^m-closed.*

Proof. (a) Let V be a open set of (Z, η), then $g^{-1}(V)$ is α^m-closed set in (Y, σ). Since (Y, σ) is a T_{α^m}-space, then $g^{-1}(V)$ is a closed set in (Y, σ). But f is α^m-continuous, then $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is α^m-closed in (X, τ) this implies that $(g \circ f)$ is α^m-contra-continuous mappings.

(b) Let V be a open set of (Z, η), then $f(V)$ is α^m-closed set in (Y, σ). Since (Y, σ) is a T_{α^m}-space, then $f(V)$ is a closed set in (Y, σ). But g is α^m-closed, then $(g \circ f)(V) = g(f(V))$ is α^m-closed in (Z, η) this implies that $(g \circ f)$ is α^m-contra-closed mappings. □

Theorem 3.9. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a mapping from a T_{α^m}-space (X, τ) into a space (Y, σ), then*

(a) *f is contra-continuous mapping if, f is α^m-contra-continuous.*

(b) *f is contra-closed mapping if, f is α^m-contra-closed.*

Proof. Obvious. □

Definition 3.10. *A space (X,τ) is called T_{α^m}-super-space if every α^m-open set is closed.*

Theorem 3.11. *Let $f : (X,\tau) \longrightarrow (Y,\sigma)$ is surjective closed and α^m-contra-irresolute, then (Y,σ) is T_{α^m}-super-space if (X,τ) is T_{α^m}-space.*

Proof. Let V be an α^m-open subset of (Y,σ). Then $f^{-1}(V)$ is α^m-closed set in (X,τ). Since, (X,τ) is a T_{α^m}-space, then $f^{-1}(V)$ is closed set in (X,τ). Hence V is closed set in (Y,σ) and so, (Y,σ) is T_{α^m}-super-space. □

Proposition 3.12. *For any bijection $f : (X,\tau) \longrightarrow (Y,\sigma)$ the following statments are equivalent:*

(a) *$f^{-1} : (X,\tau) \longrightarrow (Y,\sigma)$ is α^m-contra-continuous.*

(b) *f is α^m-contra-open-map.*

(c) *f is α^m-contra-closed-map.*

Proof. $(a) \Rightarrow (b)$ Let F be an closed set of (X,τ). By assumption, $(f^{-1})^{-1}(U) = f(U)$ is α^m-open in (Y,σ) and so f is α^m-contra-open.

$(b) \Rightarrow (c)$ Let U be a open set of (X,τ). Then U^c is closed set in (X,τ). By assumption, $f(F^c)$ is α^m-open in (Y,σ). That is $f(F^c) = (f(F))^c$ is α^m-open in (Y,σ) and there for $f(F)$ is α^m-closed in (Y,σ). Hence f is α^m-contra-closed.

$(c) \Rightarrow (a)$ Let F be a open set of (X,τ). By assumption, $f(F)$ is α^m-closed in (Y,σ) and therefore f^{-1} is α^m-contra-continuous. □

Proposition 3.13. *If $f : (X,\tau) \longrightarrow (Y,\sigma)$ be an α^m-continuous function and (X,τ) be a T_{α^m}-super-space. Then f is contra-continuous.*

Proof. Let U be open set in (Y,σ). Since f is an α^m-continuous function, $f^{-1}(U)$ is an α^m-open set in (X,τ). Since (X,τ) is a T_{α^m}-super-space, $f^{-1}(U)$ is closed set in (X,τ). Hence f is contra-continuous. □

Theorem 3.14. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a mapping and (X, τ) be a T_{α^m}-super-space, then f is continuous if one of the following conditions is satisfied.*

(a) *f is α^m-contra-continuous.*

(b) *f is α^m-contra-irresolute.*

Proof. Obvious. □

Proposition 3.15. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be any topological space and (Y, σ) be a T_{α^m}-super-space. If $f : (X, \tau) \longrightarrow (Y, \sigma)$ and $g : (Y, \sigma) \longrightarrow (Z, \eta)$ are α^m-continuous function, then their composition $g \circ f : (X, \tau) \longrightarrow (Z, \eta)$ is an α^m-contra-continuous function.*

Proof. Let U be a open set in (Z, η). Since $g : (Y, \sigma) \longrightarrow (Z, \eta)$ is an α^m-continuous function, $g^{-1}(U)$ is an α^m-open set in (Y, σ). Since (Y, σ) is a T_{α^m}-super-space, $g^{-1}(U)$ is a closed set in (Y, σ). Since $f : (X, \tau) \longrightarrow (Y, \sigma)$ is an α^m-continuous function, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is an α^m-closed set in (X, τ). Hence $g \circ f : (X, \tau) \longrightarrow (Z, \eta)$ is an α^m-contra-continuous function. □

Theorem 3.16. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a mapping from a T_{α^m}-super-space (X, τ) into a space (Y, σ), then*

(a) *f is contra-continuous mapping if, f is α^m-continuous.*

(b) *f is contra-closed mapping if, f is α^m-closed.*

Proof. Obvious. □

Theorem 3.17. *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ is surjective closed and α^m-contra-irresolute, then (Y, σ) T_{α^m}-super-space if (X, τ) is T_{α^m}-space.*

Proof. Let U be an α^m-open subset of (Y, σ). Then $f^{-1}(U)$ is α^m-closed in (X, τ). Since, (X, τ) is a T_{α^m}-space, then $f^{-1}(U)$ is closed set in (X, τ). Hence, U is closed set in (Y, σ) and so (Y, σ) is T_{α^m}-super-space. □

4. b-ω-Open sets in generalized topological spaces

In this section, we introduce and study b-ω-open sets in generalized topological spaces. We obtain several characterizations of b-omega-open sets in generalized topological spaces and prove that they form a generalized topology.

Definition 4.1. *Let (X,μ) be a generalized topological space and B a subset of X.*

(a) *A points $\{x_1,\ldots,x_n\} \subset X$ is called base of condensation points of B if for all family $\{A_i \in \mu,\ 1 \leq i \leq n$ such that $x_i \in A_i\}$, $\cap_{i=1}^n A_i \cap B$ is uncountable.*

(b) *The set of all base of condensation points of B is denoted by $cond_b(B)$.*

(c) *B is b-ω-μ-closed if $cond_b(B) \subseteq B$.*

(d) *B is b-ω-μ-open if $X\backslash B$ is b-ω-μ-closed.*

(e) *The family of all b-ω-μ-open sets of (X,μ) will be denoted by $\mu_{b-\omega}$.*

Theorem 4.2. *A subset G of a generalized topological space (X,μ) is b-ω-μ-open if and only if for every collection of points $\{x_1,\ldots,x_n\} \subseteq G$, there exists a $\{U_1,\ldots,U_n\} \in \mu$ such that $x_i \in U_i$, $1 \leq i \leq n$ and $\cup_{i=1}^n U_i \backslash G$ is countable.*

Proof. G is b-ω-μ-open if and only if $X\backslash G$ is b-ω-μ-closed if and only if $\text{cond}_b(X\backslash G) \subseteq X\backslash G$ if and only if for each collection of points $\{x_1,\ldots,x_n\} \subseteq G$; $\{x_1,\ldots,x_n\} cond_b(X\backslash G)$, if and only if for each collection of points $\{x_1,\ldots,x_n\} \subseteq G$, there exists a $\{U_1,\ldots,U_n\} \in \mu$ such that $x_i \in U_i$, for each $1 \leq i \leq n$ and $\cup_{i=1}^n U_i \cap (X\backslash G) = \cup_{i=1}^n U_i \backslash G$ is countable.

Corollary 4.3. *A subset G of a generalized topological space (X,μ) is b-ω-μ-open if and only if for every collection of points $\{x_1,\ldots,x_n\} \subseteq G$, there exists a $\{U_1,\ldots,U_n\} \in \mu$ and a countable set $N \subseteq \mathcal{M}_\mu$ such that $\cup_{i=1}^n \{x_i\} \subseteq \cup_{i=1}^n U_i \backslash N \subseteq G$.*

Proof. $\Rightarrow$) Suppose G is b-ω-μ-open and let $\{x_1,\ldots,x_n\} \subseteq G$. By Theorem 4.2, there exists a $\{U_1,\ldots,U_n\} \in \mu$ such that $x_i \in U_i$, for each $1 \leq i \leq n$ and $\cup_{i=1}^n U_i \backslash G$ is countable.

Let $N = \cup_{i=1}^{n} U_i \backslash G$. Then N is countable, $N \subseteq \mathcal{M}_\mu$ and $\cup_{i=1}^{n}\{x_i\} \subseteq \cup_{i=1}^{n} U_i \backslash N = \cup_{i=1}^{n} U_i \backslash (\cup_{i=1}^{n} U_i \backslash G) \subseteq G$.

$\Leftarrow$) Let $\{x_1, \ldots, x_n\} \subseteq G$. Then by assumption there exists a $\{U_1, \ldots, U_n\} \in \mu$ such that $x_i \in U_i$, for each $1 \leq i \leq n$ and a countable set $N \subseteq \mathcal{M}_\mu$ such that $\cup_{i=1}^{n}\{x_i\} \subseteq \cup_{i=1}^{n} U_i \backslash N \subseteq G$. Since $\cup_{i=1}^{n} U_i \backslash G \subseteq N$, then $\cup_{i=1}^{n} U_i \backslash G$ is countable, which ends the proof. □

Corollary 4.4. *A subset G of a generalized topological space (X, μ) is ω-μ-open [16] if and only if for every $x \in G$ there exists a $U \in \mu$ and a countable set $C \subseteq \mathcal{M}_\mu$ such that $x \in U \backslash C \subseteq G$.*

Corollary 4.5. *Let (X, μ) be a generalized topological space. Then $\mu_\omega \subseteq \mu_{b-\omega}$*

Proof. Let $G \in \mu_\omega$ and $\{x_1, \ldots, x_n\} \subseteq G$. By corollary 4.4, for each $x_i \in G$, $1 \leq i \leq n$, there exists a $U_i \in \mu$, $1 \leq i \leq n$ and a countable set $C_i \subseteq \mathcal{M}_\mu$ such that $x_i \in U_i \subseteq G$ for each $1 \leq i \leq n$. Then for every $\{x_1, \ldots, x_n\} \subseteq G$, there exists a $\{U_1, \ldots, U_n\} \in \mu$ and a countable set $N = \cup_{i=1}^{n} C_i \subseteq \mathcal{M}_\mu$ such that $\cup_{i=1}^{n}\{x_i\} \subseteq \cup_{i=1}^{n} U_i \backslash N \subseteq G$. Therefore, by corollary 4.3, it follows that $G \in \mu_{b-\omega}$. □

Corollary 4.6. *Let (X, μ) be a generalized topological space [16]. Then $\mu \subseteq \mu_\omega$*

Remark 4.7. *For any generalized topological space (X, μ), $\mu \subseteq \mu_\omega \subseteq \mu_{b-\omega}$*

Theorem 4.8. *For any generalized topological space $(X,_m u)$, $\mu_{b-\omega}$ is a generalized topology on X.*

Proof. By corollary 4.5,$\emptyset \in \mu_{b-\omega}$. Let $\{G_\alpha : \alpha \in J\}$ be a collection of b-ω-μ-open subsets of (X, μ) and $\{x_1, \ldots, x_n\} \subseteq \bigcup_{\alpha \in J} G_\alpha$. There exists an $\alpha_0 \in J$ such that $\{x_1, \ldots, x_n\} \subseteq G_{\alpha_0}$. Since G_{α_0} is b-ω-μ-open set, then by corollary 4.3, there exists a $\{U_1, \ldots, U_n\} \in \mu$ and a countable set $N \subseteq \mathcal{M}_\mu$ such that $\cup_{i=1}^{n}\{x_i\} \subseteq \cup_{i=1}^{n} U_i \backslash N \subseteq G_{\alpha_0} \subseteq \bigcup_{\alpha \in J} G_\alpha$. By corollary 4.5, it follows that $\bigcup_{\alpha \in J} G_\alpha$ is b-ω-μ-open. □ The following example shows that $\mu \neq \mu_{b-\omega}$ in general.

Example 4.9. *Consider $X = \mathbf{R}$ and $\mu = \{\emptyset, [-3, -1], [-2, 0] \cup \mathbf{N}, [-3, 0] \cup \mathbf{N}\}$.*

Then (X, μ) is a generalized topological space. Let $A_1 = [-2, 0]$, $A_2 = [-3, -1]$ and $A_3 = A_1 \cap A_2$. It is easy to check $cond_b(\mathbf{R} \backslash A_3) = ((\mathbf{R} \backslash A_3) \backslash \mathbf{N}) \subseteq \mathbf{R} \backslash A_3$. Then $A_3 \in \mu_{b-\omega} \backslash \mu$.

Theorem 4.10. *Let (X, μ) be a generalized topological space. Then $\mathcal{M}_{\mu_\omega} = \mathcal{M}_{\mu_{b-\omega}}$.*

Proof. Since $\mu_\omega \subseteq \mu_{b-\omega}$, then $\mathcal{M}_{\mu_\omega} \subseteq \mathcal{M}_{\mu_{b-\omega}}$. On the other hand, let $\{x_1, \dots, x_n\} \in \mathcal{M}_{\mu_{b-\omega}}$. Since $\mathcal{M}_{\mu_{b-\omega}} \in \mu_{b-\omega}$ by corollary 4.3, there exists a $\{U_1, \dots, U_n\} \in \mu$ and a countable set $N \subseteq \mathcal{M}_\mu$ such that $\cup_{i=1}^n \{x_i\} \subseteq \cup_{i=1}^n U_i \backslash N \subseteq \mathcal{M}_{\mu_{b-\omega}}$. Since $\cup_{i=1}^n U_i \subseteq \mathcal{M}_\mu$, it follows that $\cup_{i=1}^n \{x_i\} \subseteq \mathcal{M}_\mu \subseteq \mathcal{M}_{\mu_\omega}$, it follows that $\cup_{i=1}^n \{x_i\} \in \mathcal{M}_{\mu_\omega}$. □

For a nonempty set X, we denote the cocountable topology on X by (b-$\tau_{coc})_X$.

Theorem 4.11. *Let (X, μ) be a generalized topological space. Then $(b - \tau_{coc})_{\cup_{i=1}^n U_i} \subseteq \mu_{b-\omega}$ for all collection of sets $\{U_1, \dots, U_n\} \in \mu \backslash \emptyset$*

Proof. Let $\{U_1, \dots, U_n\} \in \mu \backslash \emptyset$, $V \in (b - \tau_{coc})_{\cup_{i=1}^n U_i}$ and $\{x_1, \dots, x_n\} \subseteq V$. Since $V \subseteq \cup_{i=1}^n U_i$. Also, as $\cup_{i=1}^n U_i \backslash V$ is countable, then by theorem 4.2, it follows that $V \in \mu_{b-\omega}$. □

Theorem 4.12. *Let (X, μ) be a generalized topological space. Then $\mu = \mu_\omega$ if and only if $(\tau_{coc})_{\cup_{i=1}^n U_i} \subseteq \mu$ for all collection of sets $\{U_1, \dots, U_n\} \in \mu \backslash \emptyset$.*

Proof. $\Rightarrow$) Suppose $\mu = \mu_{b-\omega}$ and $\{U_1, \dots, U_n\} \in \mu \backslash \emptyset$. Then by theorem 4.11, $(b - \tau_{coc})_{\cup_{i=1}^n U_i} \subseteq \mu_{b-\omega} = \mu$

$\Leftarrow$) Suppose $(b - \tau_{coc})_{\cup_{i=1}^n U_i} \subseteq \mu$ for all collection of sets $\{U_1, \dots, U_n\} \in \mu \backslash \emptyset$. It is enough to show that $\mu_{b-\omega} \subseteq \mu$. Let $A \in \mu_{b-\omega} \backslash \emptyset$. By corollary 4.3, for each collection of points $\{x_1, \dots, x_n\} \subseteq A$ there exists a $\{U_{x_1}, \dots, U_{x_n}\} \in \mu$ and a countable set $N \subseteq \mathcal{M}_\mu$ such that $\cup_{i=1}^n \{x_i\} \subseteq \cup_{i=1}^n U_{x_i} \backslash N \subseteq A$. Thus, $\cup_{i=1}^n U_{x_i} \backslash N \in$ (b-$\tau_{coc})_{\cup_{i=1}^n U_{x_i}}$ for all $\{x_1, \dots, x_n\} \subseteq A$, and so $\cup_{i=1}^n U_{x_i} \backslash N \in \mu$. It follows that $A = \bigcup \{\cup_{i=1}^n U_{x_i} \backslash N : \{x_1, \dots, x_n\} \subseteq A \in \mu$. □

Definition 4.13. *A generalized topological space (X, μ) is called b-locally countable if $\mathcal{M}_\mu$ is nonempty and for every collection of points $\{x_1, \dots, x_n\} \in \mathcal{M}_\mu$, there exists a $\{U_1, \dots, U_n\} \in \mu$ such that $x_i \in U_i$, $1 \leq i \leq n$ and $\cup_{i=1}^n U_i$ is countable.*

Theorem 4.14. *If $(X,_m u)$ is a b-locally countable generalized topological space, then $\mu_{b-\omega}$ is the discrete topology on $\mathcal{M}_\mu$.*

Proof. We show that every collection of singleton subset of $\mathcal{M}_\mu$ is b-ω-μ-open. For $\{x_1,\ldots,x_n\} \in \mathcal{M}_\mu$, since (X,μ) is b-locally countable, there exists a $\{U_1,\ldots,U_n\} \in \mu$ such that $x_i \in U_i$, $1 \leq i \leq n$ and $\cup_{i=1}^n U_i$ is countable. By theorem 4.12, $(b-\tau_{coc})_{\cup_{i=1}^n U_i} \subseteq \mu_{b-\omega}$. Hence $\cup_{i=1}^n U_i \backslash (\cup_{i=1}^n U_i \backslash \cup_{i=1}^n \{x_i\}) = \cup_{i=1}^n \{x_i\} \in \mu_{b-\omega}$. □

Corollary 4.15. *If (X,μ) is generalized topological space such that $\mathcal{M}_\mu$ is a countable nonempty set, then $\mu_{b-\omega}$ is the discrete topology on $\mathcal{M}_\mu$.*

Proof. Since $\mathcal{M}_\mu$ is countable, it follows directly that (X,μ) is b-locally countable. By theorem 4.14, it follows that $\mu_{b-\omega}$ is the discrete topology on $\mathcal{M}_\mu$. □

Corollary 4.16. *If (X,μ) is a generalized topological space such that X is a countable nonempty set and $\mathcal{M}_\mu$ is nonempty, then $\mu_{b-\omega}$ is the discrete topology on $\mathcal{M}_\mu$.*

Theorem 4.17. *Let A be a subset of a generalized topological space (X,μ). Then $(\mu_A)_{b-\omega} = (\mu_{b-\omega})_A$.*

Proof. $(\mu_A)_{b-\omega} \subseteq (\mu_{b-\omega})_A$. Let $B \in (\mu_A)_{b-\omega}$ and $\{x_1,\ldots,x_n\} \in B$. By corollary 4.3, there exists a $\{V_1,\ldots,V_n\} \in \mu_A$ and a subset $N \subseteq \mathcal{M}_{mu_A}$ such that $\cup_{i=1}^n \{x_i\} \subseteq \cup_{i=1}^n V_i \backslash N \subseteq B$. Choose $\{U_1,\ldots,U_n\} \in \mu$ such that $\cup_{i=1}^n V_i = \cup_{i=1}^n U_i \cap A$. Then $\cup_{i=1}^n U_i \backslash N \in \mu_{b-\omega}$, $\cup_{i=1}^n \{x_i\} \subseteq \cup_{i=1}^n U_i \backslash N$, and $(\cup_{i=1}^n U_i \backslash N) \cap A = \cup_{i=1}^n V_i \backslash N \subseteq B$. Therefore, $B \in (\mu_{b-\omega})_A$. $(\mu_{b-\omega})_A \subseteq (\mu_A)_{b-\omega}$. Let $G \in (\mu_{b-\omega})_A$. Then there exists an $H \in \mu_{b-\omega}$ such that $G = H \cap A$. If $\{x_1,\ldots,x_n\} \subseteq G$, then $\{x_1,\ldots,x_n\} \subseteq H$ and there exists a collection of subset $\{U_1 \ldots, U_n\} \in \mu$ and a countable subset $D \subseteq \mathcal{M}_\mu$ such that $\cup_{i=1}^n \{x_i\} \subseteq \cup_{i=1}^n U_i \backslash D \subseteq H$. We put $V = \cup_{i=1}^n U_i \cap A$. Then $V \in \mu_A$ and $\cup_{i=1}^n \{x_i\} \subseteq V \backslash D \subseteq G$. It follows that $G \in (\mu_A)_{b-\omega}$. □

5. Continuity via b-ω-open sets in generalized topological spaces

In this section, we introduce b-ω-(μ_1,μ_2)-continuous functions between generalized spaces. We obtain several characterizations of them and we introduce composition and restriction theorems.

Definition 5.1. *Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces. A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is called b-ω-(μ_1, μ_2)-continuous at a point $x \in X$, if for every μ_2-open sets V containing $f(x)$ there is an b-ω-μ_1 open set U containing x such that $f(U) \subseteq V$. If f is b-ω-(μ_1, μ_2)-continuous at each point of X, then f is said to be b-ω-(μ_1, μ_2)-continuous.*

Theorem 5.2. *Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces. If $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is ω-(μ_1, μ_2)-continuous at $x \in X$, then f is b-ω-(μ_1, μ_2)-continuous at x.*

Proof. Let V be μ_2-open set with $f(x) \in V$. Since f is ω-(μ_1, μ_2)-continuous at x, there is a ω-μ_1-open set U containing x such that $f(U) \subseteq V$. By corollary 4.5, U is b-ω-μ_1-open. It follows that f is b-ω-(μ_1, μ_2) continuous at x. □

It is clear that every ω-(μ_1, μ_2)-continuous function is b-ω-(μ_1, μ_2) continuous. The proof of the following theorem is obvious and left to the reader.

Theorem 5.3. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be a function. Then the following conditions are equivalent:*

a) *The function f is b-ω-(μ_1, μ_2)-continuous.*

b) *For each μ_2-open set $V \subseteq Y$, $f^{-1}(V)$ is b-ω-μ_1-open in X.*

c) *For each μ_2-closed set $M \subseteq Y$, $f^{-1}(M)$ is b-ω-μ_1-closed in X.*

The following theorem is an immediate consequence of Theorem 5.3.

Theorem 5.4. *A function $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is b-ω-(μ_1, μ_2)-continuous if and only if $f : (X, (\mu_1)_{b-\omega}) \longrightarrow (Y, \mu_2)$ is $((\mu_1)_{b-\omega}, \mu_2)$-continuous.*

Theorem 5.5. *If $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is b-ω-(μ_1, μ_2)-continuous and $g : (Y, \mu_2) \longrightarrow (Z, \mu_3)$ is (μ_1, μ_2)-continuous, then $g \circ f : (X, \mu_1) \longrightarrow (Z, \mu_3)$ is b-ω-(μ_1, μ_2)-continuous.*

Proof. Let $V \in \mu_3$. Since g is a (μ_1, μ_2)-continuous function, then $g^{-1}(V) \in \mu_2$. Since f is b-ω-(μ_1, μ_2)-continuous, then $f^{-1}(g^{-1}(V))$ is b-ω-μ_1-open in X. Thus $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is b-ω-μ_1-open and hence $(g \circ f)$ is b-ω-(μ_1, μ_2)-continuous. □

Theorem 5.6. *If A is a subset of a generalized topological space (X, μ_1) and $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ is b-ω-(μ_1, μ_2)-continuous, then the restriction of f to A, $f\backslash_A : (A, (\mu_1)_A) \longrightarrow (Y, \mu_2)$ is an b-ω-$((\mu_1)_A, \mu_2)$-continuous function.*

Proof. Let V be any μ_2-open set in Y. Since f is b-ω-(μ_1, μ_2)-continuous, then $f^{-1}(V) \in \mu_{b-\omega}$ and so $(f\backslash_A)^{-1}(V) = f^{-1}(V) \cap A \in (\mu_{b-\omega})\backslash_A$. Therefore, by Theorem 4.17, $(f\backslash_A)^{-1}(V) \in (\mu_A)_{b-\omega}$. It follows that $f\backslash_A$ is b-ω-$((\mu_1)_A, \mu_2)$-continuous. □

Lemma 5.7. *Let (X, μ) be a strong generalized topological space and A a nonempty subset of X. Then [16] a subset $C \subseteq A$ is μ_A-closed, if and only if there exists a μ-closed set H such that $C = H \cap A$.*

Theorem 5.8. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ be a function and $X = A \cup B$, where A and B are b-ω-μ_1-closed subsets of (X, μ_1) and $f\backslash_A : (A, (\mu_1)_A) \longrightarrow (Y, \mu_2)$, $f\backslash_B : (B, (\mu_1)_B) \longrightarrow (Y, \mu_2)$ are b-ω-(μ_1, μ_2)-continuous functions. Then f is b-ω-(μ_1, μ_2)-continuous.*

Proof. We will use Theorem 5.3. Let C be a μ_2-closed subset of (Y, μ_2). Then

$$f^{-1}(C) = f^{-1}(C) \cap X = f^{-1}(C) \cap (A \cup B) = (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B)$$

. Since $f\backslash_A : (X, (\mu_1)_A) \longrightarrow (Y, \mu_2)$, $f\backslash_B : (X, (\mu_1)_B) \longrightarrow (Y, \mu_2)$ are b-ω-(μ_1, μ_2)-continuous functions, then $(f\backslash_A)^{-1}(C) = f^{-1}(C) \cap A$ is b-ω-$(\mu_1)_A$-closed in : $(A, (\mu_1)_A)$ and $(f\backslash_B)^{-1}(C) = f^{-1}(C) \cap B$ is b-ω-$(\mu_1)_B$-closed. By Lemma 5.7, it follows that $(f\backslash_A)^{-1}(C)$ and $(f\backslash_B)^{-1}(C)$ are b-ω-μ_1-closed in (X, μ_1). It follows that f is b-ω-(μ_1, μ_2)-continuous. □

For any two generalized topological spaces (X, μ_1) and (Y, μ_2), we call the generalized topology on $X \times Y$ having the family $\{A \times B : A \in \mu_1$ and $B \in \mu_2\}$ as a base, the product of (X, μ_1) and (Y, μ_2) and denote it by μ_{prod} [17].

Lemma 5.9. *Let (X, μ_1) and (Y, μ_2) be two generalized topological spaces. Then [16] the projection functions $\pi_x : (X \times Y, \mu_{prod}) \longrightarrow (X, \mu_1)$ on X and $\pi_y : (X \times Y, \mu_{prod}) \longrightarrow (Y, \mu_2)$ on Y are (μ_{prod}, μ_1)-continuous and (μ_{prod}, μ_2)-continuous, respectively.*

Theorem 5.10. *Let $f : (X, \mu_1) \longrightarrow (Y, \mu_2)$ and $g : (X, \mu_1) \longrightarrow (Z, \mu_3)$ be two functions. If the function $h : (X, \mu_1) \longrightarrow (Y \times Z, \mu_{prod})$ defined by $h(x) = (f(x), g(x))$ is b-ω-(μ_1, μ_{prod})-continuous, then f is b-ω-(μ_1, μ_2)-continuous and g is b-ω-(μ_1, μ_3)-continuous.*

Proof. Assume that h is b-ω-(μ_1, μ_{prod})-continuous. Since $f = \pi_y \circ h$, where $\pi_y : (Y \times Z, \mu_{prod}) \longrightarrow (Y, \mu_2)$ is the projection function on Y, by Lemma 5.9 and Theorem 5.5, it follows that f is b-ω-(μ_1, μ_2)-continuous. Similarly we can show that g is b-ω-(μ_1, μ_3)-continuous. □

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