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On the maximal invariant set for the map $x^2 - 2$ restricted to intervals

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Abstract:

In this paper, we study the maximal invariant set of a quadratic family related to a class of unimodal maps. This family is very important and have direct application in many branches of science. In particular, we characterize when the maximal invariant of $f(x) = x^2 - 2$ (restricted to an interval) has a chaotic behavior.

Keywords: Chaos; Tent maps; Maximal invariant; Unimodal maps; Quadratic maps.

MSC (2020): 37E05.

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1. Introduction

Nonlinear dynamical systems are one of the most important tools to model a large number of physical systems in nature, ranging from biological populations, coupled networks, market crisis, brain dynamics, chemical systems, laser physics, granular dynamics, normal and anomalous transport, extreme events, weather forecast, among many others. One of the more important problem in nonlinear models is to identify what are the correct parameter values, which lead to the desired dynamics (or avoid it). We cite [7] for a nice overview about the subject.

In the main stream of the study of topological models in some applied sciences, there are the unimodal maps. Among them, one of the simplest continuous family of these maps are the family of tent maps which is the family $(T_\lambda)_{1<\lambda\leq 2}$ defined by

$$T_\lambda(x) = \begin{array}{ll} \lambda x, & \text{if } 0 \leq x \leq 1/2; \\ \lambda(1-x), & \text{if } 1/2 \leq x \leq 1. \end{array}$$

This family provides a topological model for the study of many differentiable unimodal maps (see the Figure 1 for the graph of T_2).

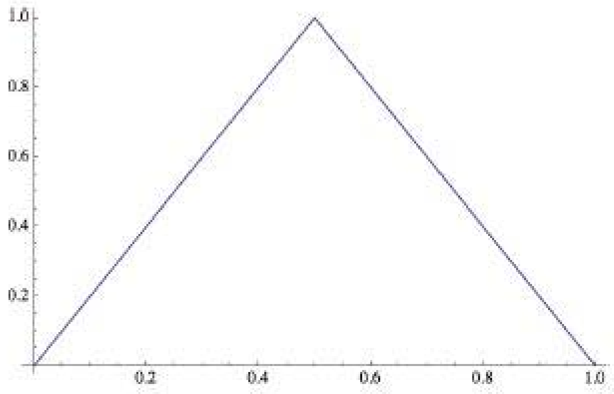

Figure 1.1: The tent map $T_2(x)$

In the last quarter of the twentieth century, the *real quadratic family* $f_c : \mathbf{R} \to \mathbf{R}$, $f_c : x \mapsto x^2 + c$ $(c \in \mathbf{R})$ was recognized as a very interesting and representative model of chaotic dynamics. It is a *full family* of unimodal maps: any C^1 unimodal map of the interval is semiconjugate to a quadratic map and the semi-conjugacy is strictly monotone in the backward orbit of the turning point (see Section 6 of Chapter II of [6]). Complexification of this family leads to a beautiful interplay between real and complex dynamics. If $c < -2$ then the set of orbits, which do not tend to infinity, is a (regular) Cantor set (see [2]); in this case it is not difficult to see that f_c is topologically conjugate via an affine map to a map of the kind $g_\mu(x) = \mu x(1-x)$, with $\mu > 4$. If $c > 1/4$ then all orbits tend to infinity (since in this case $f_c(x) > x$ for all x). For $c = 1/4$ the only orbit, which do not tend to infinity, is the constant orbit of $x = 1/2$. For $-2 \leq c < 1/4$, f_c leaves invariant the interval $[-p_c, p_c]$, where $p_c = \frac{1+\sqrt{1-4c}}{2}$ is a fixed point of f_c (and the orbits of points not belonging to this interval tend to infinity). For $c = -2$, we will see that the restriction of f_c to $[-p_c, p_c] = [-2, 2]$ is topologically conjugate to the complete tent map T_2. So, the dynamics of f_{-2} is an interesting limit case in the quadratic family. Notice also that f_{-2} is topologically conjugate via the affine map $h(x) = 2x$ to the Chebyshev polynomial $2x^2 - 1$.

Now, let x be a real number and define the sequence $(a_n(x))_{n\geq 0}$ by

$$a_0(x) = x \text{ and } a_{n+1}(x) = a_n(x)^2 - 2, \text{ for all } n \geq 0.$$

These sequences can be seen as iterations of the quadratic map $f(x) := f_{-2}(x) = x^2 - 2$. As we remark before, the choice of the initial parameter x can lead to some chaotic (or not) dynamics for $a_n(x)$. In a certain sense, the measure of this behavior can be seen in the problem of determining when $a_n(x)$ belongs to some prescribed interval for all $n \geq 1$. Surely, if the interval is very small, the possibilities for x can be also very small (or even empty).

In this work, we are interested in a critical interval I where uncountably many of these sequences belong completely to an interval if and only if this interval contains I. This leads to an uncountable profusion of such values of x. This can be of wide interest in many fields and we leave to other specialist the study of this model. Now, let us pose the mathematical problem that we are interested.

We call the *maximal invariant* of the sequence $a_n(x)$, restricted to an interval $[c, d]$, the set

$$\begin{aligned}\mathcal{K}_{c,d} &= \{x \in [-2,2] : c \leq a_n(x) \leq d, \forall n \geq 1\} \\ &= \{x \in [-2,2] : f^n(x) \in [c,d], \forall n \geq 1\}.\end{aligned}$$

Also, for tents maps, we have: Let J be a closed subinterval of $[0,1]$, the *maximal invariant* $M_{\lambda,J}$ of $T_\lambda|_J$ is defined by

$$\mathcal{M}_{\lambda,J} := \bigcap_{n=0}^{\infty} T_\lambda^{-n}(J).$$

In other words, $\mathcal{M}_{\lambda,J}$ is the set of all real numbers $y \in [0,1]$ such that the $T_\lambda^n(y) \in J$, for all $n \geq 0$. We point out to the reader the strong relation of the maximal invariant and the *basin of attraction* (i.e., the set of all initial conditions in the phase space whose trajectories go to the set of all attractors). We point the reader to some basic literature on the invariant sets of one-dimensional maps (see [4, 5, 1, 3] and references therein).

We must point out that the set invariance theory has been subject to an extensive study over the last 50 years due to its close relationship with basic concepts of control theory, some of which are control synthesis under uncertainty, reachability analysis and stability theory (see more in [9]).

The distribution of the iterations of $f_c(x)$ (the values of $a_n(x)$) is scattered and appears to be random. The next figure shows the orbit of $x_0 = 0.1$ under $f_{-2}(x)$ (with $10,000$ iterations). From this graphic, we can appreciate that the distribution of the orbit does not run evenly over the interval of $(-2,2)$. It is clear that the orbit favors points along the edges of the interval. If the orbit was truly distributed randomly, we would see no such patterns. This occurrence suggests the presence of chaos within the orbit.

We remark the importance of the study of these sequences. For example, the search for big prime numbers has been a very popular area of research and many mathematicians and computer scientists have devoted time to find these numbers (since they are very important in cryptography). The bigger prime numbers known to this date are Mersenne primes, i.e., primes of the form $2^p - 1$, where p is prime. A very useful test for primality for Mersenne numbers (the Lucas-Lehmer test) works as follows: Let $M_p = 2^p - 1$ be the Mersenne number and p be an odd prime. Then M_p is prime if and only if $a_{p-2}(4) \equiv 0 \pmod{M_p}$. The Lucas-Lehmer test is the primality test used by the *Great Internet Mersenne Prime Search (GIMPS)* to locate large primes.

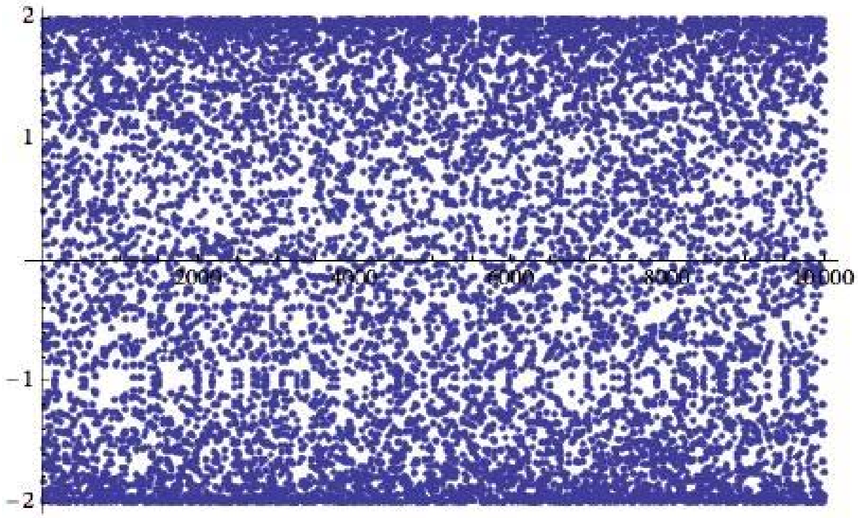

Figure 1.2: Orbit of $x_0 = 0.1$ under $f_{-2}(x)$ or $a_n(0.1)$, for $1 \leq n \leq 10,000$

The aim of this work is to study the dynamics of the unimodal map $f_{-2}(x)$ and to find the interval of critical invariance for this map. We will study maximal invariant sets $\mathcal{K}_{c,d}$ of the sequence $a_n(x)$, restricted to intervals $[c, d] \subset \mathbf{R}$. There is no loss of generality in assuming $[c, d] \subset [-2, 2]$, since $|x| > 2 \lim a_n(x) = +\infty$. More precisely, we proved that

Theorem 1. *(i) If $[c, d] \subseteq [-2, 2]$, then $\mathcal{K}_{c,d} \neq \emptyset$ if and only if $[c, d] \cap \{-1, 2\} \neq \emptyset$.*
(ii) If $[c, d] \subseteq [-2, 2]$, then $\mathcal{K}_{c,d}$ is infinite if and only if $[c, d] \supseteq [-1, 2]$ or $[c, d] \supseteq [-(\frac{1+\sqrt{5}}{2}), \frac{\sqrt{5}-1}{2}] = [-1.61803\ldots, 0.61803\ldots]$.

Theorem 2. *If $[c, d] \subseteq [-2, 2]$, then there exists an explicit interval $[c_0, d_0]$ such that $\mathcal{K}_{c,d}$ is uncountable if and only if $[c, d] \supseteq [c_0, d_0]$. In fact, $c_0 = 2\cos(2m\pi) = -1.70497\ldots$ and $d_0 = 2\cos((2 - 4m)\pi) = 0.90695\ldots$, where m is the Thue-Morse constant.*

2. Proof of the theorems

Set $f(x) = x^2 - 2$ and let $h : [0, 1] \to [-2, 2]$ be the homomorphism defined by $h(y) = 2\cos(\pi y)$. We have the following commutative diagram:

$$\begin{tikzcd}
{[0,1]} \arrow[r, "T_2"] \arrow[d, "h"'] & {[0,1]} \arrow[d, "h"] \\
{[-2,2]} \arrow[r, "f"] & {[-2,2]}
\end{tikzcd}$$

In fact, this diagram commutes since

$$\begin{aligned} f(h(y)) &= 4\cos^2(\pi y) - 2 = 2(2\cos^2(\pi y) - 1) = 2\cos(2\pi y) \\ &= 2\cos(\pi(2-2y)) = 2\cos(\pi T_2(y)) = h(T_2(y)). \end{aligned}$$

More generally, $f^n \circ h = h \circ T_2^n$, for all $n \geq 1$. Indeed, to prove that we shall use induction on n. The case $n = 1$ is already proved. So, suppose that this equality is true for n. Then $f^{n+1} \circ h = f \circ (f^n \circ h) = f \circ (h \circ T_2^n) = (f \circ h) \circ T_2^n = (h \circ T_2) \circ T_2^n = h \circ T_2^{n+1}$. In particular, the study of dynamics of $f(x)$ is related to the one of $T_2(x)$. Therefore, we have that

$$\begin{aligned} \mathcal{K}_{c,d} &= \{x \in [-2,2] : c \leq a_n(x) \leq d, \forall n \geq 1\} \\ &= \{x \in [-2,2] : f^n(x) \in [c,d], \forall n \geq 1\} \\ &= h(\{y \in [0,1] : T_2^n(y) \in h^{-1}([c,d]) = [h^{-1}(d), h^{-1}(c)], \forall n \geq 1\}). \end{aligned}$$

Since $\mathcal{M}_{2,[a,b]} = \bigcap_{n=0}^{\infty} T_2^{-n}([a,b]) = \{y \in [0,1] : T_2^n(y) \in [a,b], \forall n \geq 1\}$, it follows that $\mathcal{K}_{c,d} = h(\mathcal{M}_{2,[a,b]})$, where $a = h^{-1}(d)$ and $b = h^{-1}(c)$. So, from now on, in order to study $\mathcal{K}_{c,d}$, we set $a = h^{-1}(d)$ and $b = h^{-1}(c)$.

In order to prove Theorem 1, notice that, if $-1 < c \leq d < 2$, then $0 < a \leq b < 2/3$. If $1/2 \leq y < 2/3$ then $T_2(y) > 2/3$, so $T_2(y) \notin [a,b]$. This implies that $\mathcal{M}_{2,[a,b]} \cap [1/2,1] = \emptyset$. On the other hand, if $0 < y < 1/2$, there is a positive integer k such that $2^{-k-1} \leq y < 2^{-k}$, and thus $T_2^k(y) = 2^k \cdot y \in [1/2,1)$, and so, since $\mathcal{M}_{2,[a,b]} \cap [1/2,1] = \emptyset$, $y \notin \mathcal{M}_{2,[a,b]}$. Therefore $\mathcal{M}_{2,[a,b]} = \emptyset$. On the other hand, since $f(-1) = -1$ and $f(2) = 2$, if $-1 \in [c,d]$ then $-1 \in \mathcal{K}_{c,d}$ and, if $2 \in [c,d]$ then $2 \in \mathcal{K}_{c,d}$. This proves (i).

To prove (ii), notice that, since $\cos(0) = 1$, $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$, $\cos(\frac{2\pi}{5}) = \frac{\sqrt{5}-1}{4}$ and $\cos(\frac{4\pi}{5}) = -(\frac{1+\sqrt{5}}{4})$, we have $[c,d] \supseteq [-1,2] \iff [a,b] \supset [0,2/3]$ and $[c,d] \supseteq [-(\frac{1+\sqrt{5}}{2}), \frac{\sqrt{5}-1}{2}] \iff [a,b] \supset [2/5,4/5]$. If $a > 2/5$ and $y \in [a,1/2]$ then $T_2(y) > 4/5$ and $T_2^2(y) < 2/5 < a$, so $y \notin \mathcal{M}_{2,[a,b]}$. This implies that $\mathcal{M}_{2,[a,b]} \cap [0,1/2] = \emptyset$, and so $\mathcal{M}_{2,[a,b]} \subset \mathcal{M}_{2,[1/2,b]} \subset \mathcal{M}_{2,[1/2,1]} = \{2/3\}$

(indeed, if there was an element $z \in \mathcal{M}_{2,[1/2,1]} \setminus \{2/3\}$, we would have $|T_2^n(z) - 2/3| = 2^n|z - 2/3|$ for all positive integer n, a contradiction). If $b < 4/5$ and $y \in [1/2, b]$ then $T_2(y) > 2/5$. Since $\mathcal{M}_{2,[1/2,1]} = \{2/3\}$, given any $z \in [1/2, b] \cap \mathcal{M}_{2,[a,b]}$ with $z \neq 2/3$, there is a natural number n with $T_2^n(z) \in [1/2, b]$, $T_2^{n+1}(z) \in [0, 1/2)$, but then $T_2^{n+1}(z) = T_2(T_2^n(z)) > 2/5$ and so $T_2^{n+2}(z) > 4/5 > b$, and thus $z \notin \mathcal{M}_{2,[a,b]}$. This implies that, if $w \in \mathcal{M}_{2,[a,b]} \setminus \{2/3\}$, then $w = 0$ or there is a natural number n such that $T_2^j(w) \in (0, 1/2]$ for $0 \leq j \leq n$ and $T_2^{n+1}(w) = 2/3$, thus $w = 2^{-n}/3$. If $a > 0$ then there are only finitely many values of n for which $2^{-n}/3 \geq a$, and so $\mathcal{M}_{2,[a,b]}$ is finite. On the other hand, if $b \geq 2/3$, then $\mathcal{M}_{2,[0,b]} \supset \{2^{-n}/3, n \in \mathbf{N}\}$ is infinite. And if $[a, b] \supset [2/5, 4/5]$ then $\mathcal{M}_{2,[a,b]} \supset \mathcal{M}_{2,[2/5,4/5]} = \{2/5\} \cup \{2/3\} \cup \{\frac{2}{3} + (-\frac{1}{2})^n \cdot \frac{2}{15}; n \geq 0\}$ is infinite. This concludes the proof of Theorem 1.

In order to prove our Theorem 2, we first notice that, by [8, Theorem 4], there exists an interval $[a_0, b_0]$ such that

$$\bigcap_{n=0}^{\infty} T_2^{-n}([a, b]) = \{y \in [0, 1]; T_2^n(y) \in [a, b], \forall n \geq 1\}$$

is uncountable if and only if $[a, b] \supseteq [a_0, b_0]$. Thus, $\mathcal{K}_{c,d}$ is uncountable if and only if $[h^{-1}(d), h^{-1}(c)] \supseteq [a_0, b_0]$, that is, if and only if $[c, d] \supseteq [h(b_0), h(a_0)]$. So, $[c_0, d_0] := [h(b_0), h(a_0)]$ is our desired interval. In fact, it was proved also in [8] that $[a_0, b_0] = [2 - 4m, 2m]$, where $m = (0.0110100110010110\ldots)_2$ is the binary representation of the Thue-Morse constant which is approximately $0.41245\ldots$. Thus, $c_0 = h(2m)$ and $d_0 = h(2 - 4m)$. The proof is then complete. ∎

3. Conclusion

In this paper, we study the maximal invariant set of a quadratic map related to a class of unimodal maps. This map is very important and has direct application in many branches of science (e.g., computer science search for big primes). In particular, our result asserts the existence of a critical interval I with the property that the maximal invariant of $f|_J$ has a chaotic behavior for any interval J containing I. This implies the existence of uncountably many sequences $a_n(x)$ lying inside of $[c, d]$, for all $n \geq 1$ (this shows the highly sensitive behavior of the dynamics in relation to the initial conditions).

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