Infinitely many solutions for anisotropic elliptic equations with variable exponent

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Abstract

In this article, we study the existence and multiplicity of solutions for a class of anisotropic elliptic equations

$$(P) \begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ - 2} u = \lambda(x) f(x, u) + \mu(x) g(x, u) \ in \ \Omega, \\ u = 0 \qquad on \qquad \partial\Omega. \end{cases}$$

First we establisch that anisotropic space is separable and by using the Fountain theorem, and dual Fountain theorem we prove, under suitable conditions, that the problem (P) admits two sequences of weak solutions.

Keywords: *Quasilinear elliptic equations, variable exponent Lebesgue space, anisotropic space, Fountain theorem, dual Fountain theorem.*

AMS Subject Classifications: *35J25, 35J62, 35D30, 46E35, 35J20.*

1. Introduction

Let $\Omega \subset \mathbf{R}^N (N \geq 3)$ be a bounded domain with smooth boundary. In this paper we will study the multiplicity of weak solutions of the anisotropic problem:

$$(P) \begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ - 2} u = \lambda(x) f(x, u) + \mu(x) g(x, u) \ in \ \Omega, \\ u = 0 \qquad on \qquad \partial\Omega, \end{cases}$$

where $\lambda \not\equiv 0$ and $\mu \not\equiv 0$, $b \in L^\infty(\Omega)$, $f, g : \Omega \times \mathbf{R} \to \mathbf{R}$, and $a_i : \Omega \times \mathbf{R} \to \mathbf{R}$ are Carathéodory functions fulfilling some natural hypotheses. The anisotropic differential operator $\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u)$ is a $\overrightarrow{p}(.)$-Laplace type operator, where $\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$ and $P_+^+ = \max_{i \in \{1,2,...,N\}} \sup_{x \in \Omega} p_i(x)$ for $i = 1, ..., N$, we assume that p_i is a continuous function on $\overline{\Omega}$ for all $i \in \{1, 2, ..., N\}$. We denote by $a_i(x, \eta)$ the continuous derivative with respect to η of the mapping $A_i : \Omega \times \mathbf{R} \to \mathbf{R}$, $A_i = A_i(x, \eta)$, that means $a_i(x, \eta) = \frac{\partial}{\partial \eta} A_i(x, \eta)$. We make the following assumptions on the mapping A_i:

(A_0) $A_i(x, 0) = 0$ for a.e. $x \in \Omega$.

(A_1) There exists a positive constant $\overline{c}_i$ such that a_i satisfies the growth condition

$$|a_i(x, \eta)| \leq \overline{c}_i(1 + |\eta|^{p_i(x) - 1}),$$

for all $x \in \Omega$ and $\eta \in \mathbf{R}$.

(A_2) The inequalities

$$|\eta|^{p_i(x)} \leq a_i(x, \eta)\eta \leq p_i(x) A_i(x, \eta),$$

are verified for all $x \in \Omega$ and $\eta \in \mathbf{R}$.

(A_3) assume that a_i is strictly monotone, that is,

$$(a_i(x, \eta) - a_i(x, \xi))(\eta - \xi) > 0,$$

for all $x \in \Omega$ and $\eta, \xi \in \mathbf{R}$, with $\eta \neq \xi$.

(A_4) $A_i(x, -\eta) = A_i(x, \eta)$ for all $x \in \Omega$, $\eta \in \mathbf{R}$.

Examples

1) If we take $a_i(x, \eta) = |\eta|^{p_i(x)-2}\eta$ for all $i \in \{1, ..., N\}$, we have $A_i(x, \eta) = \frac{1}{p_i(x)}|\eta|^{p_i(x)}$ for all $i \in \{1, ..., N\}$. Obviously, (A_0) - (A_4) are verified, and we obtain the $\overrightarrow{p}(x)$ -Laplace operator

$$\triangle_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u).$$

2) If we take $a_i(x, \eta) = (1 + \eta^2)^{\frac{p_i(x)-2}{2}}\eta$ for all $i \in \{1, ..., N\}$, we have $A_i(x, \eta) = \frac{1}{p_i(x)}[(1 + |\eta|^2)^{\frac{p_i(x)}{2}} - 1]$ for all $i \in \{1, ..., N\}$, then (A_0) - (A_4) are verified, and we find the anisotropic variable exponent mean curvature operator

$$\sum_{i=1}^{N} \partial_{x_i}(1 + |\partial_{x_i} u|^2)^{\frac{p_i(x)-2}{2}}\partial_{x_i} u).$$

Anisotropic elliptic equations has been received a rising interest, in the recent years. This is due to the some nonhomogeneous materials which could not be treated on the classical Sobolev space. For example we can point out to the electro-rheological fluids or to the thermo-rheological fluids (see for example [17], [19], [23]), image restoration [22], and contact mechanics [18]. We give in this work an extension of the article [1] where the authors considered the following problem

$$(S) \begin{cases} -\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{P_+^+ - 2}u = f(x, u) & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases}$$

and they considered $f(x, u) = \lambda|u|^{q(x)-2}u + \mu|u|^{\gamma(x)-2}u$, where λ and μ are constant, and they established the existence of two unbounded sequences of weak solutions, their proof is based on Fountain theorem [20].
This kind of equation are treated in several works by many authors , we refer here to the articles ([3], [6], [15], [16]). In [16], the authors considered the problem (S), without $b(x)|u|^{P_+^+ - 2}u$, where $f(x, u) = \lambda|u|^{q(x)-2}u$, and established the existence of a continuous spectrum in several distinct situations. But in [15], the authors took the same problem with λ depends on the variable x, using the mountain-pass theorem of Ambrosetti and Rabinowitz [3] and the Ekelandes variational principle, they proved that under suitable conditions, problem (S) has two nontrivial weak solutions. In [6],

M. M. Boureanu, P. Pucci, and V. Rădulescu proved that problem (S) has a sequence of weak solutions by means of the symmetric mountain-pass theorem .

Given $\Omega \subset \mathbf{R}^N$, we set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) | \min_{x \in \overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x \in \overline{\Omega}} h(x) \quad and \quad h^- = \inf_{x \in \overline{\Omega}} h(x).$$

Let $p \in C_+(\overline{\Omega})$, then $L^{p(x)}(\Omega)$ is called variable exponent Lebesgue space which is defined as follow

$$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function such that}$$

$$\int_\Omega |u(x)|^{p(x)} \, dx < \infty\},$$

endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\}$$

is a separable and reflexive Banach space (see [21]).

In order to study the problem (P) we have to introduce the vectors $\overrightarrow{P}_+$, $\overrightarrow{P}_- \in \mathbf{R}^N$, $p_+(x)$ and $p_-(x)$ which are defined in the following way

$$\overrightarrow{P}_+ = (p_1^+, p_2^+, ..., p_N^+), \ \overrightarrow{P}_- = (p_1^-, p_2^-, ..., p_N^-),$$

$$p_+(x) = \max\{p_1(x), ..., p_2(x)\}, \ p_-(x) = \min\{p_1(x), ..., p_2(x)\},$$

and the positive real numbers P_+^+, P_-^+, P_-^- as the following

$$P_+^+ = \max\{p_1^+, ..., p_N^+\}, \ P_-^+ = \max\{p_1^-, ..., p_N^-\}, \ P_-^- = \min\{p_1^-, ..., p_N^-\}.$$

We say that p is logarithmic Hölder continuous if

$$(1.1) \ |p(x) - p(y)| \le -\frac{M}{log(|x - y|)} \ \forall x, \ y \in \Omega \text{ such that } |x - y| \le 1/2.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \in [L^{p(x)}(\Omega)]^N\}.$$

For all $u \in W^{1,p(x)}(\Omega)$, we have $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. If p satisfies (1.1), the space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ under the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\|_{p(x)} = |\nabla u|_{p(x)}$.

Now, we introduce a natural generalization of the function space $W^{1,p(x)}(\Omega)$, which is called anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(x)}(\Omega)$ and we have

$$W^{1,\overrightarrow{p}(x)}(\Omega) = \{u \in L^{p_+(x)}(\Omega) : \partial_{x_i} u \in L^{p_i(x)}(\Omega), \forall i \in \{1, ..., N\}\}.$$

If $\overrightarrow{p} : \overline{\Omega} \to \mathbf{R}^N$; $\quad \overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$, and for each $i \in \{1, 2, ..., N\}$, we have $p_i \in C_+(\overline{\Omega})$, and satisfy (1.1), the anisotropic variable exponent Sobolev space $W_0^{1,\overrightarrow{p}(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = \|u\|_{\overrightarrow{p}(.)} = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(.)},$$

and ites a reflexive Banach space (see[9, 16]).

Throughout this paper, we assume that

$$(1.2) \qquad\qquad \sum_{i=1}^N \frac{1}{p_i^-} > 1.$$

Define $P_-^*, P_{-,\infty} \in \mathbf{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1}, \quad P_{-,\infty} = \max\{P_-^+, P_-^*\}.$$

In this paper, we have $P_+^+ < P_{-,\infty} = \max\{P_-^+, P_-^*\} = P_-^*$.

From now on, we put $X = W_0^{1,\overrightarrow{p}(x)}(\Omega)$.

Let us define $F(x,t) = \int_0^t f(x,s)\,ds$ and $G(x,t) = \int_0^t g(x,s)\,ds$, and we assume that the following hypotheses are fulfilled:

(f_0) $|f(x,t)| \leq c|t|^{\alpha(x)-1}$, for all $(x,t) \in \Omega \times \mathbf{R}$ where $c > 0$ is a constant, $\alpha \in C_+(\Omega)$ such that $\alpha^+ = \sup_{x \in \overline{\Omega}} \alpha(x) < P_-^- < P_+^+ \leq s(x) <$

$P_{-,\infty}$, $\forall x \in \Omega$ and $\lambda \in L^{\frac{s(x)}{s(x)-\alpha(x)}}(\Omega)$, with $s \in C_+(\overline{\Omega})$ and $\alpha(x)+1 \le s(x) < P_{-,\infty}$, $\forall x \in \Omega$.

(f_1) $F(x,t) \ge h_0(x)t^{\alpha_0(x)}$ when $t \to 0^+$, and $\alpha_0 \in C_+(\overline{\Omega})$ with $\alpha_0^+ < 1$, $h_0(x) \ge 0$ $\forall x \in \Omega$, $\not\equiv 0$, and also $h_0 \in C(\Omega, \mathbf{R})$.

(f_2) $f(x,-t) = -f(x,t)$ $\forall x \in \Omega$, $t \in \mathbf{R}$.

(g_0) $|g(x,t)| < d|t|^{\gamma(x)-1}$, for all $(x,t) \in \Omega \times \mathbf{R}$ where $d > 0$ is a constant, and $\gamma \in C_+(\overline{\Omega})$ satisfying $P_+^+ < \gamma^- < \gamma^+ < P_{-,\infty}$, and $\mu \in L^{\frac{r(x)}{r(x)-\gamma(x)}}(\Omega)$, with $r \in C_+(\overline{\Omega})$, and $\gamma(x)+1 \le r(x) < P_{-,\infty}$ $\forall x \in \Omega$.

(g_1) There exists a constant $\theta > P_+^+$ such that $0 < \theta G(x,t) \le tg(x,t)$, for a.e. $x \in \Omega$, $t \in \mathbf{R}$.

(g_1') There exist constants $\beta > \gamma(x) > \theta > P_+^+$ such that $0 < \theta G(x,t) \le tg(x,t) \le \beta G(x,t)$, for a.e. $x \in \Omega, t \in \mathbf{R}$.

(g_2) $g(x,-t) = -g(x,t)$ for all $x \in \Omega$, $t \in \mathbf{R}$.

(B) $b \in L^\infty(\Omega)$ and there exists $b_0 > 0$ such that $b(x) \ge b_0$ for all $x \in \Omega$.

The main result of this paper is as follows.

Theorem 1.1. *Assume* $(A_0) - (A_4)$, $(f_0) - (f_2)$ *and* $(g_0) - (g_2)$ *and* (B), *then*

1. *If* $\mu(x) > 0$ *a.e.* $x \in \Omega$ *and* $\lambda(x) \in \mathbf{R}$, *problem* (P) *has a sequence of weak solutions* $(\pm u_k)$ *such that* $\Phi(\pm u_k) \to +\infty$ *as* $k \to +\infty$.

2. *If* $\lambda(x) > 0$, $\mu(x) > 0$ *a.e.* $x \in \Omega$, *problem* (P) *has a sequence of weak solutions* $(\pm v_k)$ *such that* $\Phi(\pm v_k) \to 0$ *as* $k \to +\infty$.

3. *If* $\lambda(x) > 0$ *a.e.* $x \in \Omega$ *and* $\mu(x)$ *keeps a constant sign in* $\mathbf{R}$, *and* g *satisfies* (g_1') *instead of* (g_1), *problem* (P) *has a sequence of weak solutions* $(\pm v_k)$ *such that* $\Phi(\pm v_k) \to 0$ *as* $k \to +\infty$.

Remark 1.1.   *1. Among the difficulties we have raised in this pape is to prove that anisotropic space is separable. To my knowledge, I believe that the demonstration of this result is not found in the literature.*

2. *For $f(x,u) = |u|^{\alpha(x)-2}u$, $g(x,u) = |u|^{\gamma(x)-2}u$, $\lambda(x) = \lambda \in \mathbf{R}$ and $\mu(x) = \mu$ for all $x \in \Omega$ with $\alpha(x) < P_-^- < P_+^+ < P_{-,\infty}$, $\forall x \in \Omega$, and $P_+^+ < \gamma^- < \gamma^+ < P_{-,\infty}$, we obtain the result in the article* [1].

This paper is divided into two sections. In the first section we will give some known results, in the second we will give the proof of our main results.

2. Preliminaries

We recall some important definitions and properties of the Lebesgue and Sobolev spaces with variable exponent $L^{p(x)}(\Omega)$ and $W_0^{1,\overrightarrow{p}(x)}(\Omega)$, where Ω is a bounded domain in $\mathbf{R}^N$.

Proposition 2.1.   *(see* [8, 13, 12]*)*

1. *The space $(L^{p(x)}(\Omega), |u|_{p(x)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv \, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

2. *If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$, $\forall x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Remark: If $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{r(x)} = 1$, then for any $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$, $w \in L^{r(x)}(\Omega)$, we have

$$\left| \int_\Omega uvw \, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} + \frac{1}{r^-} \right) |u|_{p(x)}|v|_{q(x)}|w|_{r(x)} \le 3|u|_{p(x)}|v|_{q(x)}|w|_{r(x)}.$$
(2.1)

Lemma 2.1. $(see[2])$ *Let* $q, s \in C_+(\overline{\Omega})$ *with* $q(x) \le s(x)$ *for all* $x \in \overline{\Omega}$, *and* $u \in L^{s(x)}(\Omega)$, *then* $|u|^{q(x)} \in L^{\frac{s(x)}{q(x)}}(\Omega)$ *and*

$$\left| |u|^{q(x)} \right|_{\frac{s(x)}{q(x)}} \le |u|^{q^+}_{s(x)} + |u|^{q^-}_{s(x)},$$

or there exists a number $\tilde{q} \in [q^-, q^+]$ *such that*

$$\left| |u|^{q(x)} \right|_{\frac{s(x)}{q(x)}} = |u|^{\tilde{q}}_{s(x)}.$$

Proposition 2.2. $(see[11])$ *Denote* $\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx$. *Then for* $u \in L^{p(x)}(\Omega)$, $(u_n) \subset L^{p(x)}(\Omega)$ *we have*

1. $|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1(= 1; > 1)$,

2. $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}$,

3. $|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}$,

4. $|u|_{p(x)} \to 0(\to \infty) \Leftrightarrow \rho_{p(x)}(u) \to 0(\to \infty)$,

5. $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

We recall now some results which concerning the embedding theorem.

Proposition 2.3. $(see[16])$ *Suppose that* $\Omega \subset \mathbf{R}^N(N > 3)$ *is a bounded domain with smooth boundary and relation (1.2) is fulfilled.*

1. *For any* $q \in C(\overline{\Omega})$ *verifying*

$$1 < q(x) < P_{-,\infty} \ \forall x \in \overline{\Omega},$$

   the embedding
$$W_0^{1, \overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

   is continuous and compact.

2. *Assume that* $P_-^- > N$, *then the embedding*

$$W_0^{1, \overrightarrow{p}(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$$

   is continuous and compact.

Under the conditions (A_i), $i = 0, 1, 2, 3$, we have the proposition below which is useful.

Proposition 2.4. $(cf.[14, \ 5])$ *Let*

$$\mathcal{A}_i(u) = \int_\Omega A_i(x, \partial_{x_i} u) \, dx$$

For $i \in \{1, 2, ..., N\}$, we have:

- $\mathcal{A}_i$ *is well defined on X,*

- *the functional $\mathcal{A}_i \in C^1(X, \mathbf{R})$ and*

$$\langle \mathcal{A}_i'(u), \varphi \rangle = \int_\Omega a_i(x, \partial_{x_i} u) \partial_{x_i} \varphi \, dx,$$

  for all $u, \varphi \in X$.

- $\mathcal{A}_i$ *is weakly lower semi-continuous.*

- *Let*

$$\mathcal{A}(u) = \int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u) \, dx,$$

  then $\mathcal{A}'$ is an operator of type (S_+) $(cf.[5])$.

The following lemma plays an important role to define the weak solution.

Lemma 2.2. *Let $B(u) = \int_\Omega [\lambda(x) F(x, u) + \mu(x) G(x, u)] \, dx$. Suppose that (f_0) and (g_0) are verified, then we have*

1. *B is well defined and $B \in C^1(X, \mathbf{R})$.*

2. *B, B' are weakly-strongly continuous.*

Proof. 1.) As f and g satisfy (f_0) and (g_0), using lemma 2.1 and proposition 2.3, we have

$$
\begin{aligned}
\int_\Omega |\lambda(x)F(x,u) + \mu(x)G(x,u)|\, dx \quad &\leq \quad \int_\Omega |\lambda(x)||F(x,u)|\, dx + \int_\Omega |\mu(x)||G(x,u)|\, dx \\
&\leq \quad 2\frac{c}{\alpha^-}|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}}|u^{\alpha(x)}|_{\frac{s(x)}{\alpha(x)}} \\
&+2 \quad \frac{d}{\gamma^-}|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}}|u^{\gamma(x)}|_{\frac{s(x)}{\gamma(x)}} \\
&\leq \quad 2c_1\frac{c}{\alpha^-}|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}}\|u\|^{\tilde{\alpha}} + 2d_1\frac{d}{\gamma^-}|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}}\|u\|^{\tilde{\gamma}}.
\end{aligned}
$$

The last expression is obtained by the compact embeddings $X \hookrightarrow L^{s(x)}(\Omega)$ and $X \hookrightarrow L^{r(x)}(\Omega)$, which means that there exist constants $c_1 > 0$ and $d_1 > 0$ such that $|u|_{s(x)} \leq c_1\|u\|$, and $|u|_{r(x)} \leq d_1\|u\|$, so B is well defined. Let us prove that $B \in C^1(X, \mathbf{R})$. We show first that B is *Gâteaux* differentiable. Using the Mean-value theorem, we have

$$
\begin{aligned}
DB(u,\varphi) \quad &= \quad \lim_{t\to 0}\frac{B(u+t\varphi) - B(u)}{t} \\
&= \quad \lim_{t\to 0}\int_\Omega \frac{\lambda(x)F(x,u+t\varphi) - \lambda(x)F(x,u) + \mu(x)G(x,u+t\varphi) - \mu(x)G(x,u)}{t}\, dx \\
&= \quad \lim_{t\to 0}\int_\Omega [\lambda(x)f(x,u+t\theta\varphi) + \mu(x)g(x,u+t\theta\varphi)]\,\varphi(x)\, dx,
\end{aligned}
$$

where $0 \leq \theta \leq 1$. From (f_0), (g_0), Young's inequality, and the convexity of the function $h(a) = |a|^p$ with $p \geq 1$, for $|t| \leq 1$, we have

$[\lambda(x)f(x,u+t\theta\varphi) + \mu(x)g(x,u+t\theta\varphi)]\,\varphi(x)$
$\leq c|\lambda(x)||u+t\theta\varphi|^{\alpha(x)-1}|\varphi(x)| + d|\mu(x)||u+t\theta\varphi|^{\gamma(x)-1}|\varphi(x)|,$
$\leq \frac{c(s(x)-\alpha(x))}{s(x)}|\lambda(x)|^{\frac{s(x)}{s(x)-\alpha(x)}} + \frac{c(\alpha(x)-1)}{s(x)}[|u+t\theta\varphi|^{\alpha(x)-1}]^{\frac{s(x)}{\alpha(x)-1}}$
$+ c|\varphi(x)|^{s(x)} + \frac{d(r(x)-\gamma(x))}{r(x)}|\mu(x)|^{\frac{r(x)}{r(x)-\gamma(x)}}$
$+ \frac{d(\gamma(x)-1)}{r(x)}[|u+t\theta\varphi|^{\gamma(x)-1}]^{\frac{r(x)}{\gamma(x)-1}} + d|\varphi(x)|^{r(x)},$
$\leq \frac{c(s(x)-\alpha(x))}{s(x)}|\lambda(x)|^{\frac{s(x)}{s(x)-\alpha(x)}} + \frac{c(\alpha(x)-1)}{s(x)}2^{s(x)-1}[|u|^{s(x)} + |\varphi|^{s(x)}]$
$+ c|\varphi(x)|^{s(x)} + \frac{d(r(x)-\gamma(x))}{r(x)}|\mu(x)|^{\frac{r(x)}{r(x)-\gamma(x)}}$
$+ \frac{d(\gamma(x)-1)}{r(x)}2^{r(x)-1}[|u|^{r(x)} + |\varphi|^{r(x)}] + d|\varphi(x)|^{r(x)}.$

The last right expression is independent on t and it is in $L^1(\Omega)$, then by the Lebesgue dominated convergence theorem, we have

$$(2.2) \qquad DB(u, \varphi) = \int_\Omega \left[\lambda(x) f(x, u) + \mu(x) g(x, u) \right] \varphi(x) \, dx.$$

So using the fact that $N_f : L^{s(x)}(\Omega) \to L^{\frac{s(x)}{\alpha(x)-1}}(\Omega)$; $u \mapsto f(x, u)$ and $N_g : L^{r(x)}(\Omega) \to L^{\frac{r(x)}{\gamma(x)-1}}(\Omega)$; $u \mapsto g(x, u)$ are continuous bounded operators. Then, by (f_0), (g_0), and Proposition 2.1, we obtain

$$
\begin{aligned}
DB(u, \varphi) &= \int_\Omega \left[\lambda(x) f(x, u) + \mu(x) g(x, u) \right] \varphi(x) \, dx \\
&\leq \int_\Omega c |\lambda(x)| |f(x, u)| |\varphi(x)| \, dx + \int_\Omega d |\mu(x)| |g(x, u)| |\varphi(x)| \, dx \\
&\leq 3c |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |f(x, u)|_{\frac{s(x)}{\alpha(x)-1}} |\varphi(x)|_{s(x)} \\
&+ 3d |\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} |g(x, u)|_{\frac{r(x)}{\gamma(x)-1}} |\varphi(x)|_{r(x)}.
\end{aligned}
$$

So $DB(u, \varphi)$, as a function of φ, is a continuous linear functional on X, then it is the *Gâteaux* differential of B. Let's prove that it's continuous, then for u, v, $\varphi \in X$, from (2.2), we get

$$
\begin{aligned}
|\langle DB(u) - DB(v), \varphi \rangle| &\leq 3|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |f(x, u) - f(x, v)|_{\frac{s(x)}{\alpha(x)-1}} |\varphi|_{s(x)} \\
&+ 3|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} |g(x, u) - g(x, v)|_{\frac{r(x)}{\gamma(x)-1}} |\varphi|_{r(x)} \\
&\leq K_1 |f(x, u) - f(x, v)|_{\frac{s(x)}{\alpha(x)-1}} \|\varphi\| + K_2 |g(x, u) \\
&- g(x, v)|_{\frac{r(x)}{\gamma(x)-1}} \|\varphi\|,
\end{aligned}
$$

where $K_1 > 0$ and $K_2 > 0$ are constants. Then,

$$\|DB(u) - DB(v)\|_{X^*} \leq K_1 |f(x, u) - f(x, v)|_{\frac{s(x)}{\alpha(x)-1}} + K_2 |g(x, u) - g(x, v)|_{\frac{r(x)}{\gamma(x)-1}}.$$

Thus, $DB(u)$ is continuous, so B is Frèchet differentiable and $B \in C^1(X, \mathbf{R})$ with

$$\langle B'(u), \varphi \rangle = \int_\Omega \left[\lambda(x) f(x, u) + \mu(x) g(x, u) \right] \varphi(x) \, dx.$$

2.) Suppose by contradiction that there exists a sequence $(u_n) \subset X$ such that $u_n \rightharpoonup u$ and $B(u_n)B(u)$, then there exists ε_0 and subsequence still denoted (u_n) such that:

$$0 < \varepsilon_0 \leq |B(u_n) - B(u)|.$$

For $0 < \theta_n < 1$, and by finite increment theorem we have

$$0 < \varepsilon_0 \leq |\langle B'(u_n + \theta_n(u_n - u)), u_n - u \rangle|.$$

Put $w_n = u_n + \theta_n(u_n - u)$. As $B'(u)(w) = \int_\Omega [\lambda(x) f(x, u) w + \mu(x) g(x, u) w]\, dx$, using (2.1), proposition 2.3, (f_0) and (g_0) we obtain

$$
\begin{aligned}
|\langle B'(w_n), (u_n - u) \rangle| &= \int_\Omega |\lambda(x) f(x, w_n)(u_n - u) + \mu(x) g(x, w_n)(u_n - u)|\, dx \\
&\leq \int_\Omega \{|\lambda(x)||u_n - u||f(x, w_n)| + |\mu(x)||u_n - u||g(x, w_n)|\}\, dx \\
&\leq \int_\Omega c|\lambda(x)||w_n|^{\alpha(x)-1}|u_n - u|\, dx|| \\
&\quad + \int_\Omega d|\mu(x)||w_n|^{\gamma(x)-1}|u_n - u|\, dx \\
&\leq 3c|\lambda(x)|_{\frac{s(x)}{s(x)-\alpha(x)}} ||w_n|^{\alpha(x)-1}|_{\frac{s(x)}{\alpha(x)-1}} |u_n - u|_{s(x)} \\
&\quad + 3d|\mu(x)|_{\frac{r(x)}{r(x)-\gamma(x)}} ||w_n|^{\gamma(x)-1}|_{\frac{r(x)}{\gamma(x)-1}} |u_n - u|_{r(x)}.
\end{aligned}
$$

Since $\lim\limits_{n \to +\infty} |w_n|_{s(x)} \neq \infty$ and $\lim\limits_{n \to +\infty} |w_n|_{r(x)} \neq \infty$, then by the proposition 2.2, we deduce that

$$\lim_{n \to +\infty} ||w_n|^{\alpha(x)-1}|_{\frac{s(x)}{\alpha(x)-1}} \neq \infty \text{ and } \lim_{n \to +\infty} ||w_n|^{\gamma(x)-1}|_{\frac{r(x)}{\gamma(x)-1}} \neq \infty.$$

So, as the embeddings $X \hookrightarrow L^{s(x)}(\Omega)$ and $X \hookrightarrow L^{r(x}(\Omega)$ are compact, then the last expression on the right goes to 0 as $n \to +\infty$. Finally, B is weakly-strongly continuous.

Let's prove that B' is also weakly-strongly continuous. We know that

$$\langle B'(u), v \rangle = \int_\Omega \lambda(x) f(x, u) v\, dx + \int_\Omega \mu(x) g(x, u) v\, dx,$$

where $v \in X$. For $u_n \rightharpoonup u$, then (u_n) is bounded, using relation (2.1), we have

$$
\begin{aligned}
|\langle B^{'}(u_n) - B^{'}(u), v\rangle| \;\leq\; & \int_\Omega |\lambda(x)||(f(x,u_n) - f(x,u))v|\,dx, \\
+ \;& \int_\Omega |\mu(x)||(g(x,u_n) - g(x,u))v|\,dx, \\
\leq \;& 3|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |f(x,u_n) - f(x,u)|_{\frac{s(x)}{\alpha(x)-1}} |v|_{s(x)} \\
+ \;& 3|\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} |g(x,u_n) - g(x,u)|_{\frac{r(x)}{\gamma(x)-1}} |v|_{r(x)}.
\end{aligned}
$$

The compact embedding $X \hookrightarrow L^{s(x)}(\Omega)$ (respectively $X \hookrightarrow L^{r(x)}(\Omega)$) guarantees the existence of subsequence (u_n) which converges to u in $L^{s(x)}(\Omega)$ (respectively $L^{r(x)}(\Omega)$). So, using the continuity of N_f and N_g, we deduce easily that $B^{'}$ is weakly-strongly continuous.

Let us define now the functional Φ associated with the problem (P): $\Phi : X \longrightarrow \mathbf{R}$

$$
\Phi(u) = \int_\Omega \left[\sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{P_+^+}|u|^{P_+^+} - \lambda(x)F(x,u) - \mu(x)G(x,u) \right] dx.
$$

Under assumptions (A_0), (A_1), (f_0), (B) and (g_0), we have Φ is well defined on X and $\Phi \in C^1(X, \mathbf{R})$, moreover the conditions (A_4), (f_2) and (g_2) imply that Φ is even. So we can define a weak solution as below.

Definition 2.1. *A function u is a weak solution of the problem (P) if and only if*

$$
\int_\Omega \left[\sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i}\varphi + b(x)|u|^{P_+^+ - 2}u\varphi - \lambda(x)f(x,u)\varphi - \mu(x)g(x,u)\varphi \right] dx = 0,
$$

for all $\varphi \in X$.

3. Existence of sequences of weak solutions

We start with proving some results and lemmas before to deduce the proof of our main results.

Proposition 3.1. *X is a separable Banach space.*

Proof Let us define $Y = L^{p^+(x)}(\Omega) \times L^{p_1(x)}(\Omega) \times L^{p_2(x)}(\Omega) \times ... \times L^{p_N(x)}(\Omega)$, endowed with the product norm is a separable Banach space, and consider the operator $T : W^{1,\vec{p}(x)}(\Omega) \to Y$, defined by $T(u) = (u, \nabla u)$. We have $\|T(u)\|_Y = |u|_{p^+(x)} + \sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)} = \|u\|_{W^{1,\vec{p}(x)}(\Omega)}$. Thus T is an isometric isomorphism of $W^{1,\vec{p}(x)}(\Omega)$ onto a subspace $W = T(W^{1,\vec{p}(x)}(\Omega))$ of Y. As $W^{1,\vec{p}(x)}(\Omega)$ is complete, then W is a closed subspace of Y. As Y is a separable banach space, we will show that W is a separable space. Let us consider a sequence $(y_n)_{n>0}$ which is dense in Y. For all $n \in \mathbf{N}^*$, let us fix a point $w_n \in W$ which verifies $\|y_n - w_n\| < \inf_{w \in W} \|y_n - w\| + \frac{1}{n}$. For $w \in W$, for all $\varepsilon > 0$ and there exists $n > \frac{3}{\varepsilon}$ such that $\|y_n - w\| < \frac{\varepsilon}{3}$, so by the definition of the sequence (w_n), we have $\|y_n - w_n\| < \frac{\varepsilon}{3} + \frac{1}{n} < \frac{2}{3}\varepsilon$, therefore

$$\|w_n - w\| < \|y_n - w\| + \|y_n - w_n\| < \varepsilon.$$

Thus, (w_n) is a countable dense sequence in W. Consequently, W is a separable Banach space. Thus W and $W^{1,\vec{p}(x)}(\Omega) = T^{-1}(W)$ are separable. By the definition of X, and the equivalence of the norms $\|.\|$ and $\|.\|_{W^{1,\vec{p}(x)}(\Omega)}$, we can prove easily that X is a closed subspace of $W^{1,\vec{p}(x)}(\Omega)$ and then X is a separable Banach space.

Lemma 3.1. *(see[6]) Let $u \in X$.*

1. *When $\|u\| < 1$, we have $\sum_{i=1}^N \int_\Omega |\partial_{x_i}(u)|^{p_i(x)} dx \geq \frac{\|u\|^{P_+^+}}{N^{P_+^+ - 1}}$.*

2. *When $\|u\| > 1$, we have $\sum_{i=1}^N \int_\Omega |\partial_{x_i}(u)|^{p_i(x)} dx \geq \frac{\|u\|^{P_-^-}}{N^{P_-^- - 1}} - N$.*

Lemma 3.2. *The functional Φ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence, namely, $|\Phi(u_n)| \leq R$, where $R > 0$, and $\Phi'(u_n) \to 0$, as $n \to \infty$, then when $\|u_n\| \geq 1$, and $\lambda(x) \in \mathbf{R}$, for all $x \in \Omega$, using (f_0), (g_1), (B) and $\theta > P_+^+$, we have

$$
\begin{aligned}
1 + R + \|u_n\| &\geq \Phi(u_n) - \frac{1}{\theta}\langle \Phi'(u_n), u_n \rangle \\
&\geq \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u_n) + \frac{b(x)}{P_+^+} |u_n|^{P_+^+} - \lambda(x) F(x, u_n) \right.
\end{aligned}
$$

$$- \quad \mu(x)G(x,u_n)\} \, dx$$

$$- \quad \frac{1}{\theta} \int_{\Omega} \left\{ \sum_{i=1}^{N} a_i(x,\partial_{x_i} u_n)\partial_{x_i} u_n + b(x)|u_n|^{P_+^+} - \lambda(x)f(x,u_n)u_n \right.$$

$$- \quad \mu(x)g(x,u_n)u_n\} \, dx$$

$$\geq \quad \int_{\Omega} \left\{ \sum_{i=1}^{N} A_i(x,\partial_{x_i} u_n) - \frac{1}{\theta} a_i(x,\partial_{x_i} u_n)\partial_{x_i} u_n \right\} dx$$

$$+ \quad \left(\frac{1}{P_+^+} - \frac{1}{\theta} \right) \int_{\Omega} b(x)|u_n|^{P_+^+} \, dx + \int_{\Omega} \mu(x) \left(\frac{1}{\theta} g(x,u_n)u_n - G(x,u_n) \right) dx$$

$$+ \quad \int_{\Omega} \lambda(x) \left(\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n) \right) dx$$

$$\geq \quad \int_{\Omega} \left\{ \sum_{i=1}^{N} A_i(x,\partial_{x_i} u_n) - \frac{1}{\theta} a_i(x,\partial_{x_i} u_n)\partial_{x_i} u_n \right\} dx$$

$$- \quad c \left(\frac{1}{\theta} + \frac{1}{\alpha^-} \right) \int_{\Omega} |\lambda(x)||u_n|^{\alpha(x)} \, dx.$$

From (A_2), for all $x \in \Omega$ and $i \in \{1,...,N\}$ we have

$$(3.1) \qquad - \frac{1}{\theta} a_i(x,\partial_{x_i} u_n)\partial_{x_i} u_n \geq - \frac{P_+^+}{\theta} A_i(x,\partial_{x_i} u_n).$$

On the other hand, we have by the Lemma 2.1 and Proposition 2.1

$$\int_{\Omega} \lambda(x)|u_n|^{\alpha(x)} \quad \leq \quad 2|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \left| |u_n|^{\alpha(x)} \right|_{\frac{s(x)}{\alpha(x)}}$$

$$\leq \quad 2|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} |u_n|^{\tilde{\alpha}}_{s(x)},$$

where $\tilde{\alpha} \in [\alpha^-, \alpha^+]$.
Since the embedding $X \hookrightarrow L^{s(x)}(\Omega)$ is compact, then there exists a constant $c_1 > 0$ such that

$$|u_n|_{s(x)} \leq c_1 \|u_n\|.$$

Then,

$$(3.2) \qquad \int_{\Omega} \lambda(x)|u_n|^{\alpha(x)} \leq C|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}},$$

where $C > 0$. From (3.1) and (3.2), we get

$$1 + R + \|u_n\| \geq \left(1 - \frac{p_+^+}{\theta} \right) \sum_{i=1}^{N} \int_{\Omega} A_i(x,\partial_{x_i} u_n) \, dx - C_1|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}},$$

where $C_1 > 0$. Again from (A_2) we have

$$A_i(x, \partial_{x_i} u_n) \geq \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)} \geq \frac{1}{P_+^+} |\partial_{x_i} u_n|^{p_i(x)},$$

for all $x \in \Omega$ and $i \in \{1, ..., N\}$, so

$$1 + R + \|u_n\| \geq \left(\frac{1}{P_+^+} - \frac{1}{\theta}\right) \sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)} \, dx - C_1 |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}}.$$

Using 2. of Lemma 3.1, we get

$$1 + R + \|u_n\| \geq \left(\frac{1}{P_+^+} - \frac{1}{\theta}\right) \left(\frac{\|u_n\|^{P_-^-}}{N^{P_-^- - 1}} - N\right) - C_1 |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_n\|^{\tilde{\alpha}},$$

and consequently $\{u_n\}$ is bounded because $\theta > P_+^+$ and $P_-^- > \tilde{\alpha}$. As X is reflexive, then there exists a subsequence still denoted by $\{u_n\}$ which converges weakly to u_0 in X.

Using the fact that $\Phi'(u_n) \to 0$, as $n \to +\infty$, we can deduce that:

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u_0 \rangle = 0,$$

more precisely,

$$\lim_{n\to\infty} \int_\Omega [\sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0)$$

$$+ b(x)|u_n|^{P_+^+ - 2} u_n(u_n - u_0) - \lambda(x) f(x, u_n)(u_n - u_0) - \mu(x) g(x, u_n)(u_n - u_0)] \, dx = 0.$$

Using Hölder inequality we have

$$\int_\Omega \lambda(x) f(x, u_n)(u_n - u_0) \leq 3c |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \left| |u|^{\alpha(x)-1} \right|_{\frac{s(x)}{\alpha(x)-1}} |u_n - u_0|_{s(x)}.$$

$$\int_\Omega \mu(x) g(x, u_n)(u_n - u_0) \leq 3d |\mu|_{\frac{r(x)}{r(x)-\gamma(x)}} \left| |u|^{\gamma(x)-1} \right|_{\frac{r(x)}{\gamma(x)-1}} |u_n - u_0|_{r(x)}.$$

$$\int_\Omega b(x)|u_n|^{P_+^+ - 2} u_n(u_n - u_0) \leq 2|b|_\infty \left| |u_n|^{P_+^+ - 1} \right|_{\frac{P_+^+}{P_+^+ - 1}} |u_n - u_0|_{P_+^+}.$$

As $s(x)$, $r(x)$ and P_+^+ fulfill Proposition 2.3, thus (u_n) converges strongly to u_0 in $L^{s(x)}(\Omega)$, $L^{r(x)}(\Omega)$ and $L^{P_+^+}(\Omega)$. By these facts the relation above reduces to

$$\lim_{n \to \infty} \int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u_n)(\partial_{x_i} u_n - \partial_{x_i} u_0) \, dx = 0.$$

Using Proposition 2.4, we deduce that (u_n) converges strongly to u_0 in X, that is to say that Φ satisfies Palais-Smale condition.

Since X is a reflexive and separable Banach space, then X^* is too. Then there exist (see[11]) $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2.....\}}, \ X^* = \overline{span\{e_j^* : j = 1, 2.....\}},$$

and

$$\langle e_i, e_j^* \rangle = \delta_{ij}.$$

Now, we define

$$X_j = span\{e_j\}, \ Y_k = \bigoplus_{j=1}^k X_j, \ Z_k = \overline{\bigoplus_{j=k}^\infty X_j}.$$

Lemma 3.3. *(see[10]) Suppose that $r(x)$, $s(x) \in C_+(\overline{\Omega})$ and $r(x)$, $s(x) < P_{-,\infty}$, for all $x \in \overline{\Omega}$. Denote*

$$\beta_k = \sup\{|u|_{r(x)} : \|u\| = 1, \ u \in Z_k\}, \ \theta_k = \sup\{|u|_{s(x)} : \|u\| = 1, \ u \in Z_k\}.$$

Then,

$$\lim_{k \to \infty} \beta_k = 0, \ \lim_{k \to \infty} \theta_k = 0.$$

Lemma 3.4. *For every $k \in \mathbf{N}$, there exists $r_k > 0$ such that*

$$\inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \to \infty, \ as \ k \to +\infty.$$

Proof. By (A_2), (B), (f_0), (g_0), Proposition 2.1, Lemma 2.1 and 2. Lemma 3.1, for any $u \in Z_k$, $\|u\| = r_k > 1$, when $\mu(x) > 0$, $a.e.x \in \Omega$, and $\lambda(x) \in \mathbf{R}$, we have

$$\Phi(u) = \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(u)) + \frac{b(x)}{P_+^+} |u|^{P_+^+} - \lambda(x) F(x, u) - \mu(x) G(x, u) \right\} dx$$

$$\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i}(u)|^{p_i(x)} dx + \frac{b_0}{P_+^+} \int_\Omega |u|^{P_+^+} dx - c \int_\Omega \frac{|\lambda(x)|}{\alpha(x)} |u|^{\alpha(x)} dx$$

$$- d \int_\Omega \frac{\mu(x)}{\gamma(x)} |u|^{\gamma(x)} dx,$$

$$\geq \frac{\|u\|^{P_-^-}}{P_+^+ N^{P_-^- - 1}} - C_1' |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} \|u\|^{\tilde{\alpha}} - \frac{2d}{\gamma^-} |\mu|_{\frac{r(x)}{r(x) - \gamma(x)}} |u|_{r(x)}^{\tilde{\gamma}} - \frac{N}{P_+^+},$$

where $C_1' = \frac{2cc_1}{\alpha^-}$. Since $P_-^- > \alpha^+ > \tilde{\alpha}$, there exists r_0 large enough such that

$$C_1' |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} \|u\|^{\tilde{\alpha}} \leq \frac{\|u\|^{P_-^-}}{2 P_+^+ N^{P_-^- - 1}} \ as \ r = \|u\| \geq r_0.$$

If $|u|_{r(x)} \leq 1$ then $|u|_{r(x)}^{r^+} \leq |u|_{r(x)}^{\tilde{\gamma}} \leq 1$. But, if $|u|_{r(x)} > 1$ using Lemma 3.3, we obtain $|u|_{r(x)}^{\tilde{\gamma}} \leq |u|_{r(x)}^{r^+} \leq (\beta_k \|u\|)^{r^+}$.

so

$$\Phi(u) \geq \left\{ \begin{array}{l} \|u\|^{P_-^-} 2 P_+^+ N^{P_-^- - 1} - \frac{2d}{\gamma^-} |\mu|_{\frac{r(x)}{r(x) - \gamma(x)}} - \frac{N}{P_+^+} \text{if } |u|_{r(x)} \leq 1, \\ \frac{\|u\|^{P_-^-}}{2 P_+^+ N^{P_-^- - 1}} - \frac{2d}{\gamma^-} |\mu|_{\frac{r(x)}{r(x) - \gamma(x)}} (\beta_k \|u\|)^{r^+} - \frac{N}{P_+^+} \text{if } |u|_{r(x)} > 1, \end{array} \right.$$

then

$$\Phi(u) \geq \frac{\|u\|^{P_-^-}}{2 P_+^+ N^{P_-^- - 1}} - \frac{2d}{\gamma^-} |\mu|_{\frac{r(x)}{r(x) - \gamma(x)}} (\beta_k \|u\|)^{r^+} - C_{12}.$$

Let's take $r_k = \left(\frac{4d}{\gamma^-} |\mu|_{\frac{r(x)}{r(x) - \gamma(x)}} N^{P_-^- - 1} r^+ \beta_k^{r^+} \right)^{\frac{1}{P_-^- - r^+}}$. Then,

$$\Phi(u) \geq \frac{1}{2 N^{P_-^- - 1}} \left(\frac{1}{P_+^+} - \frac{1}{r^+} \right) r_k^{P_-^-} - C_{12},$$

consequently, $\lim_{k \to +\infty} \Phi(u) = +\infty$, because $P_+^+ < r^+$ and $\beta_k \to 0$ when $k \to +\infty$.

Lemma 3.5. *For every $k \in \mathbf{N}$, there exists $\rho_k > r_k$ (r_k given as above) such that*

$$\max_{u \in Y_k, \|u\|=\rho_k} \Phi(u) \leq 0.$$

Proof. From (A_0) and (A_1), (f_0), (g_1) and for any $u \in Y_k - \{0\}$ with $\|u\| = 1$ and $1 < \rho_k$, $\mu(x) > 0$ a.e. $x \in \Omega$, and $\lambda(x) \in \mathbf{R}$, we have

$$\begin{aligned}
\Phi(\rho_k u) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(\rho_k u)) + \frac{b(x)}{P_+^+} |\rho_k u|^{P_+^+} - \lambda(x) F(x, \rho_k u) - \mu(x) G(x, \rho_k u) \right\} dx, \\
&\leq C \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i}(\rho_k u)| + \frac{|\partial_{x_i}(\rho_k u)|^{p_i(x)}}{p_i(x)} \right) dx + \frac{\rho_k^{P_+^+}}{P_+^+} \int_\Omega b(x) |u|^{P_+^+} dx \\
&\quad - \int_\Omega \lambda(x) F(x, \rho_k u) \, dx - \int_\Omega \mu(x) G(x, \rho_k u) \, dx, \\
&\leq C \rho_k^{P_+^+} \sum_{i=1}^N \int_\Omega \left(|\partial_{x_i} u| + \frac{|\partial_{x_i} u|^{p_i(x)}}{P_-^-} \right) dx + \frac{\rho_k^{P_+^+}}{P_+^+} \int_\Omega b(x) |u|^{P_+^+} dx \\
&\quad + \frac{c}{\alpha^-} \rho_k^{\alpha^+} \int_\Omega |\lambda(x)| |u|^{\alpha(x)} \, dx, - M \rho_k^\theta \int_\Omega \mu(x) |u|^\theta \, dx.
\end{aligned}$$

We have Y_k is a finite dimensional space then all norms are equivalent. Since $\theta > P_+^+ > \alpha^+$ and $\rho_k \to +\infty$ as $k \to +\infty$. Hence, for ρ_k large enough $(\rho_k > r_k)$, $\max_{u \in Y_k, \|u\|=\rho_k} \Phi(u) \leq 0$.

Lemma 3.6. *There is k_0 such that for each $k > k_0$, there exists $\rho_k > 0$ such that* $\inf_{u \in Z_k, \|u\|=\rho_k} \Phi(u) \geq 0$.

Proof. From (A_2), (B), (f_0), (g_0) and Proposition 2.1, Lemma 2.1, and 1. of Lemma 3.1, for any $u \in Z_k$, with $\lambda(x) > 0$ a.e. $x \in \Omega$, and $\mu(x) \in \mathbf{R}$, one has

$$\begin{aligned}
\Phi(u) &= \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(u)) + \frac{b(x)}{P_+^+} |u|^{P_+^+} - \lambda(x) F(x, u) - \mu(x) G(x, u) \right\} dx \\
&\geq \frac{1}{P_+^+} \sum_{i=1}^N \int_\Omega |\partial_{x_i}(u)|^{p_i(x)} \, dx + \frac{b_0}{P_+^+} \int_\Omega |u|^{P_+^+} \, dx - c \int_\Omega \frac{\lambda(x)}{\alpha(x)} |u|^{\alpha(x)} \, dx
\end{aligned}$$

$$- \quad d \int_\Omega \frac{|\mu(x)|}{\gamma(x)} |u|^{\gamma(x)} \, dx,$$

$$\geq \quad \frac{\|u\|^{P_+^+}}{P_+^+ N^{P_+^+ - 1}} - \frac{2c}{\alpha^-} |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} |u|_{s(x)}^{\tilde{\alpha}} - C_2' |\mu|_{\frac{r(x)}{r(x) - \gamma(x)}} \|u\|^{\tilde{\gamma}},$$

where $C_2' = \frac{2dd_1}{\gamma^-}$. As $P_+^+ < \gamma^- \leq \tilde{\gamma}$, then there exists ρ_0 small enough such that $C_2' |\mu|_{\frac{r(x)}{r(x) - \gamma(x)}} \|u\|^{\tilde{\gamma}} \leq \frac{\|u\|^{P_+^+}}{2P_+^+ N^{P_+^+ - 1}}$ as $0 < \rho = \|u\| < \rho_0$. So using lemma 3.3, we have

$$\Phi(u) \geq$$

$$\begin{cases} \|u\|^{P_+^+} 2P_+^+ N^{P_+^+ - 1} - \frac{2c}{\alpha^-} |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} \text{ if } |u|_{s(x)} \leq 1, \\ \frac{\|u\|^{P_+^+}}{2P_+^+ N^{P_+^+ - 1}} - \frac{2c}{\alpha^-} |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} (\theta_k \|u\|)^{\alpha^+} \text{ if } |u|_{s(x)} > 1. \end{cases}$$
$$(3.3)$$

Let's take $\rho_k = \left(\frac{4c}{\alpha^-} P_+^+ N^{P_+^+ - 1} |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} \theta_k^{\alpha^+} \right)^{\frac{1}{P_+^+ - \alpha^+}}$. As $P_+^+ > P_-^- > \alpha^+$, then $\theta_k \to 0$ means $\rho_k \to 0$ as k goes to $+\infty$. It follows from the inequality above that

$$\Phi(u) \geq \frac{\rho_k^{P_+^+}}{2P_+^+ N^{P_+^+ - 1}} - \frac{\rho_k^{P_+^+}}{2P_+^+ N^{P_+^+ - 1}} = 0.$$

Finally, we deduce that for $u \in Z_k$ with $\|u\| = \rho_k$, $\Phi(u) \geq 0$.

Lemma 3.7. *for each $k > k_0$, there exists $r_k < \rho_k$ (ρ_k given by Lemma 3.6) such that* $\max\limits_{u \in Y_k, \|u\| = r_k} \Phi(u) < 0$, *as* $k \to +\infty$.

Proof. From (A_0), (A_1), (f_0), (f_1), (g_0), and Proposition 2.1, for any $v \in Y_k$ with $\|v\| = 1$ and $0 < t < \rho_k < 1$, with $\lambda(x) > 0$ a.e. $x \in \Omega$, and $\mu(x) \in \mathbf{R}$, we have

$$\Phi(tv) \quad = \quad \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i}(tv)) + \frac{b(x)}{P_+^+} |tv|^{P_+^+} - \lambda(x) F(x, tv) - \mu(x) G(x, tv) \right\} dx$$

$$\leq \ C \sum_{i=1}^{N} \int_{\Omega} \left(|\partial_{x_i}(tv)| + \frac{|\partial_{x_i}(tv)|^{p_i(x)}}{p_i(x)} \right) dx + \frac{t^{P_+^+}}{P_+^+} \int_{\Omega} b(x)|v|^{P_+^+} dx$$

$$- \ \int_{\Omega} \lambda(x) F(x, tv) \, dx - \int_{\Omega} \mu(x) G(x, tv) \, dx,$$

$$\leq \ Ct \sum_{i=1}^{N} \int_{\Omega} \left(|\partial_{x_i} v| + \frac{|\partial_{x_i} v|^{p_i(x)}}{P_-^-} \right) dx + \frac{t^{P_+^+}}{P_+^+} \int_{\Omega} b(x)|v|^{P_+^+} dx$$

$$- \ t^{\alpha_0^+} \int_{\Omega} \lambda(x) h_0(x)|v|^{\alpha_0(x)} \, dx + \frac{d}{\gamma^-} t^{\gamma^-} \int_{\Omega} |\mu(x)||v|^{\gamma(x)} \, dx.$$

Since $\gamma^- > 1 > \alpha_0^+$ and $dim Y_k = k$, then there exists $0 < r_k < \rho_k$ such that $\Phi(u) < 0$ if $\|u\| = r_k$. Hence $b_k = \max\limits_{u \in Y_k, \|u\| = r_k} \Phi(u) < 0$.

Lemma 3.8. *For each $k > k_0$, and $\rho_k > 0$ given by Lemma 3.6, one has*

$$\inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi(u) \to 0, \ as \ k \to +\infty.$$

Proof. From a definition of Y_k and Z_k, we have $Y_k \cap Z_k \neq$ and $r_k < \rho_k$, then

$$d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi(u) \leq b_k = \max_{u \in Y_k, \|u\| = r_k} \Phi(u) < 0.$$

From (3.3), for $u \in Z_k, \|u\| \leq \rho_k$ small enough we have

$$\Phi(u) \ \geq \ \frac{\|u\|^{P_+^+}}{2 P_+^+ N^{P_+^+ - 1}} - \frac{2c}{\alpha^-} |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} (\theta_k \|u\|)^{\alpha^+},$$

$$\geq \ -\frac{2c}{\alpha^-} |\lambda|_{\frac{s(x)}{s(x) - \alpha(x)}} \theta_k^{\alpha^+} \|u\|^{\alpha^+}.$$

We deduce that $d_k \to 0$, because $\theta_k \to 0$ and $\rho_k \to 0$ as $k \to +\infty$.

Definition 3.1. *The functional Φ satisfies the $(PS)_c^*$ condition (with respect to (Y_n)), if any sequence $(u_{n_j}) \subset X$ such that $n_j \to +\infty, u_{n_j} \in Y_{n_j}, \Phi(u_{n_j}) \to c$ and $(\Phi|_{Y_{n_j}})'(u_{n_j}) \to 0$, has a subsequence converging to a critical point of Φ.*

Lemma 3.9. *The functional Φ satisfies the $(PC)_c^*$ for every $c \in [d_{k_0}, 0[$.*

Proof. Assume that $(u_{n_j}) \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $\Phi(u_{n_j}) \to c$ and $(\Phi|_{Y_{n_j}})'(u_{n_j}) \to 0$. Suppose that $\|u_{n_j}\| > 1$. If $\mu(x) > 0$ a.e. $x \in \Omega$, for n_j large enough, we have

$$
\begin{aligned}
1 + R_1 + \|u_{n_j}\| \;\geq\;& \Phi(u_{n_j}) - \frac{1}{\theta}\langle \Phi'(u_{n_j}), u_{n_j}\rangle, \\
\geq\;& \left(\frac{1}{P_+^+} - \frac{1}{\theta}\right)\left(\frac{\|u_{n_j}\|^{P_-^-}}{N^{P_-^- - 1}} - N\right) - M_1|\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}}\|u_{n_j}\|^{\tilde{\alpha}}.
\end{aligned}
$$

Since $\theta > P_+^+$ and $P_-^- > \tilde{\alpha}$, $\{u_{n_j}\}$ is bounded in X.

For a subsequence $\{u_{n_j}\}$ we have $u_{n_j} \rightharpoonup u \in X$. As $X = \overline{\bigcup_{n_j} Y_{n_j}}$, we can take $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Then

$$
\begin{aligned}
\lim_{n_j \to +\infty}\langle \Phi'(u_{n_j}), u_{n_j} - u\rangle \;=\;& \lim_{n_j \to +\infty}\langle \Phi'(u_{n_j}), u_{n_j} - v_{n_j}\rangle \\
& + \lim_{n_j \to +\infty}\langle \Phi'(u_{n_j}), v_{n_j} - u\rangle, \\
=\;& \lim_{n_j \to +\infty}\langle (\Phi|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\rangle, \\
=\;& 0.
\end{aligned}
$$

We follow the similar procedure in the proof of (PS) condition in the Lemma 3.2. We conclude that $u_{n_j} \to u$, therefore we have $\Phi'(u_{n_j}) \to \Phi'(u)$. Let's show that $\Phi'(u) = 0$. Let us take $\varphi_k \in Y_k$, notice that if $n_j \geq k$ we have

$$
\begin{aligned}
\langle \Phi'(u), \varphi_k\rangle \;=\;& \langle \Phi'(u) - \Phi'(u_{n_j}), \varphi_k\rangle + \langle \Phi'(u_{n_j}), \varphi_k\rangle, \\
=\;& \langle \Phi'(u) - \Phi'(u_{n_j}), \varphi_k\rangle + \langle (\Phi|_{Y_{n_j}})'(u_{n_j}), \varphi_k\rangle.
\end{aligned}
$$

Then, when $n_j \to +\infty$, we obtain

$$
\langle \Phi'(u), \varphi_k\rangle = 0 \;\forall\; \varphi_k \in Y_k,
$$

so $\Phi'(u) = 0$. Finally, Φ satisfies $(PS)_c^*$ condition for every $c \in \mathbf{R}$.

Proof theorem 1.1.

By using Fountain theorem (see[20], [4]), Lemma 3.2, Lemma 3.4 and Lemma 3.5, and as Φ is even the proof of 1. Theorem 1.1 is finished. Now, we apply the dual fountain theorem (see[20], [4]) in order to prove 2. and 3. of Theorem 1.1, first by using the Lemma 3.6, Lemma 3.7, Lemma 3.8 and Lemma 3.9, the conditions of dual fountain theorem are fulfilled. As Φ is even, this achieved the proof of 2. of Theorem 1.1.

Proof of 3. of theorem 1.1.
In order to prove 3. of theorem 1.1, we apply the dual fountain theorem, and we use the condition (g_1'), because the only problem here is to demonstrate that every $(PS)^*$ sequence is bounded. In the fact, assume that $(u_{n_j}) \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $\Phi(u_{n_j}) \to c$ and $(\Phi|_{Y_{n_j}})'(u_{n_j}) \to 0$. Suppose that $\|u_{n_j}\| > 1$. For the case $\lambda(x) > 0 \, and \, \mu(x) > 0$ a.e. $x \in \Omega$, the proof is the same as above. Then we will treat only the case when $\lambda(x) > 0 \, and \, \mu(x) < 0$ a.e. $x \in \Omega$.
For n_j large enough, we have

$$
\begin{aligned}
1 + R_2 + \|u_{n_j}\| \; \geq \; & \Phi(u_{n_j}) - \frac{1}{\beta}\langle \Phi'(u_{n_j}), u_{n_j} \rangle \\[2mm]
\geq \; & \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u_{n_j}) + \frac{b(x)}{P_+^+}|u_{n_j}|^{P_+^+} - \lambda(x)F(x, u_{n_j}) \right. \\[2mm]
- \; & \left. \mu(x)G(x, u_{n_j}) \right\} dx \\[2mm]
- \; & \frac{1}{\beta}\int_\Omega \left\{ \sum_{i=1}^N a_i(x, \partial_{x_i} u_{n_j})\partial_{x_i} u_{n_j} + b(x)|u_{n_j}|^{P_+^+} \right. \\[2mm]
- \; & \left. \lambda(x)f(x, u_{n_j})u_{n_j} - \mu(x)g(x, u_{n_j})u_{n_j} \right\} dx \\[2mm]
\geq \; & \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u_{n_j}) - \frac{1}{\beta}a_i(x, \partial_{x_i} u_{n_j})\partial_{x_i} u_{n_j} \right\} dx \\[2mm]
+ \; & \left(\frac{1}{P_+^+} - \frac{1}{\beta}\right) \int_\Omega b(x)|u_{n_j}|^{P_+^+} \, dx \\[2mm]
+ \; & \int_\Omega \mu(x) \left(\frac{1}{\beta}g(x, u_{n_j})u_{n_j} - G(x, u_{n_j})\right) dx \\[2mm]
+ \; & \int_\Omega \lambda(x) \left(\frac{1}{\beta}f(x, u_{n_j})u_{n_j} - F(x, u_{n_j})\right) dx
\end{aligned}
$$

$$\geq \int_\Omega \left\{ \sum_{i=1}^N A_i(x, \partial_{x_i} u_{n_j}) - \frac{1}{\beta} a_i(x, \partial_{x_i} u_{n_j}) \partial_{x_i} u_{n_j} \right\} dx$$
$$- c\left(\frac{1}{\beta} + \frac{1}{\alpha^-}\right) \int_\Omega |\lambda(x)| |u_{n_j}|^{\alpha(x)}.$$

Then,

$$1 + R_2 + \|u_{n_j}\| \geq \Phi(u_{n_j}) - \frac{1}{\beta}\langle \Phi'(u_{n_j}), u_{n_j}\rangle,$$
$$\geq \left(\frac{1}{P_+^+} - \frac{1}{\beta}\right)\left(\frac{\|u_{n_j}\|^{P_-^-}}{N^{P_-^- -1}} - N\right) - M_2 |\lambda|_{\frac{s(x)}{s(x)-\alpha(x)}} \|u_{n_j}\|^{\tilde{\alpha}}.$$

Since $\beta > P_+^+$ and $P_-^- > \tilde{\alpha}$, $\{u_{n_j}\}$ is bounded in X. This achieve the proof.

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