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θ_ω–Connectedness and $_\omega$–R_1 properties

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Abstract:

We use the theta omega closure operator to define theta omega connectedness as a property which is weaker than connectedness and stronger than θ-connectedness. We give several sufficient conditions for the equivalence between θ_ω-connectedness and connectedness, and between θ_ω-connectedness and θ-connectedness. We give two results regarding the union of θ_ω-connected sets and also we show that the weakly θω-continuous image of a connected set is θ_ω-connected. We define and investigate V-θ_ω-connectedness as a strong form of V - θ-connectedness, and we show that the θω-connectedness and V-θ_ω-connectedness are independent. We continue the study of R_1 as a known topological property by giving several results regarding it. We introduce ω-R_1 (I), ω-R_1 (II), ω-R_1 (III) and weakly ω-R_1 as four weaker forms of R1 by utilizing ω-open sets, we give several relationships regarding them and we raise two open questions.

Keywords: Generalized open sets; θ-closure; θ_ω-closure; R_1.ittag-Leffler function; Laplace transforms; Hilfer derivatives.

MSC (2010): 54A10, 54D10.

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1. Introduction

Let (X, τ) be a topological space and let $A \subseteq X$. A is called an ω-open set in (X, τ) [1] if for each $x \in A$, there is $U \in \tau$ and a countable set $C \subseteq X$ such that $x \in U - C \subseteq A$. The family of all ω-open sets in (X, τ) is denoted by τ_ω. It is well known that τ_ω is a topology on X which contains τ. Denote the closure of A in (X, τ) (resp. (X, τ_ω)) by $\overline{A}$ (resp. $\overline{A}^{\omega}$). A point $x \in X$ is in θ-closure of A [2] ($x \in Cl_\theta (A)$) if for every $U \in \tau$ with $x \in U$, we have $\overline{U} \cap A \neq \emptyset$. A is called θ-closed [2] if $Cl_\theta (A) = A$. The complement of a θ-closed set is called a θ-open set. The family of all θ-open sets in (X, τ) is denoted by τ_θ. It is well known that $\tau_\theta \subseteq \tau$ and $\tau_\theta = \tau$ if and only if (X, τ) is regular. A topological space (X, τ) is called ω-regular [3] if for each closed set F in (X, τ) and $x \in X - F$, there exist $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$. In [4] the author defined θ_ω-closure operator as follows: A point $x \in X$ is in θ_ω-closure of A ($x \in Cl_{\theta_\omega} (A)$) if for any $U \in \tau$ with $x \in U$ we have $\overline{U}^{\omega} \cap A \neq \emptyset$. A is called θ_ω-closed if $Cl_{\theta_\omega} (A) = A$. The complement of a θ_ω-closed set is called a θ_ω-open set. The family of all θ_ω-open sets in (X, τ) is denoted by τ_{θ_ω}. It is proved in [4] that τ_{θ_ω} forms a topology on X which lies between τ_θ and τ, and that $\tau_{\theta_\omega} = \tau$ if and only if (X, τ) is ω-regular. Also, ω-T_2 topological spaces are characterized via θ_ω-open sets. Moreover, four new classes of functions, namely: θ_ω-continuous, ω-θ-continuous, weakly θ_ω-continuous and faintly θ_ω-continuous are defined and investigated. This paper is organized as follows:

In section two, we use the θ_ω-closure operator to define θ_ω-connectedness as a property which is weaker than connectedness and stronger than θ-connectedness. We give several sufficient conditions for the equivalence between θ_ω-connectedness and connectedness, and between θ_ω-connectedness and θ-connectedness. We give two results regarding the union of θ_ω-connected sets and also we show that the weakly θ_ω-continuous image of a connected set is theta omega connected.

In section three, we define and investigate V-θ_ω-connectedness as a strong form of V-θ-connectedness. We show that the θ_ω-connectedness and V-θ_ω-connectedness are independent.

In section four, we continue the study of R_1 as a known topological property by giving several results rgarding it. We introduce ω-$R_1 (I)$, ω-$R_1 (II)$, ω-$R_1 (III)$ and weakly ω-R_1 as four weaker forms of R_1 by utilizing ω-open sets. We give several relationships regarding them and we raise two open questions.

In this paper, $\mathbf{R}, \mathbf{Q}, \mathbf{Q}^c$ and $\mathbf{N}$ denote, respectively the set of real numbers, the set of rational numbers, the set of irrational numbers and the set of natural numbers.

The following definitions and theorems will be used in the sequel:

Definition 1.1. [5] Let (X, τ) be a topological space. A pair (P, Q) of non-empty subsets of X is said to be separation relative to (X, τ), if $(P \cap \overline{Q}) \cup (\overline{P} \cap Q) = \emptyset$.

Definition 1.2. [6] Let (X, τ) be a topological space. A pair (P, Q) of non-empty subsets of X is said to be θ-separation relative to (X, τ), if $(P \cap Cl_\theta(Q)) \cup (Cl_\theta(P) \cap Q) = \emptyset$.

Theorem 1.3. [4] Let (X, τ) be a topological space and let $A \subseteq X$. Then $\overline{A} \subseteq Cl_{\theta_\omega}(A) \subseteq Cl_\theta(A)$.

Definition 1.4. [7] Let (X, τ) be a topological space. Then (X, τ) is called locally countable if for each $x \in X$, there is $U \in \tau$ such that $x \in U$ and U is countable.

Theorem 1.5. [4] If (X, τ) is locally countable and $A \subseteq X$, then $\overline{A} = Cl_{\theta_\omega}(A)$.

Definition 1.6. [8] Let (X, τ) be a topological space. Then (X, τ) is called anti-locally countable if each $U \in \tau - \{\emptyset\}$ is uncountable.

Theorem 1.7. [4] If (X, τ) is anti-locally countable and $A \subseteq X$, then $Cl_\theta(A) = Cl_{\theta_\omega}(A)$.

Definition 1.8. [5] Let (X, τ) be a topological space. A subset A of X is said to be connected relative to (X, τ) if there is no separation relative to (X, τ), (P, Q), such that $A = P \cup Q$.

Definition 1.9. [6] Let (X, τ) be a topological space. A subset A of X is said to be θ-connected relative to (X, τ) if there is no θ-separation relative to (X, τ), (P, Q), such that $A = P \cup Q$.

Definition 1.10. [3]**.** A topological space (X, τ) is called ω-regular if for each closed set F in (X, τ) and $x \in X - F$, there exist $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.

Theorem 1.11. [4]**.** If (X, τ) is an ω-regular topological space and $A \subseteq X$, then $\overline{A} = Cl_{\theta_\omega}(A)$.

Theorem 1.12. [4] If (X, τ) is a regular topological space and $A \subseteq X$, then $\overline{A} = Cl_\theta(A) = Cl_{\theta_\omega}(A)$.

Theorem 1.13. [8] Let (X, τ) be a topological space and let A be a non-empty subset of X. Then $(\tau_A)_\omega = (\tau_\omega)_A$.

Definition 1.14. [4] A function $f : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be weakly θ_ω-continuous if for each $x \in X$ and $V \in \sigma$ containing $f(x)$, there is $U \in \tau$ such that $x \in U$ and $f(U) \subseteq \overline{V}^\omega$.

Definition 1.15. [2] Let (X, τ) be a topological space and let $A \subseteq X$. A is said to be V-θ-connected relative to (X, τ) if there are no disjoint non-empty sets P and Q and no open sets U and V such that $A = P \cup Q$, $P \subseteq U$, $Q \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Theorem 1.16. [3] Let (X, τ) be a topological space and let $A \subseteq X$. Then $\overline{A}^\omega \subseteq \overline{A}$.

Theorem 1.17. [3] If (X, τ) is an anti-locally countable topological space, then for all $U \in \tau_\omega$, $\overline{U} = \overline{U}^\omega$.

Definition 1.18. [10] A topological space (X, τ) is said to be R_1 if for any two points $x, y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$, there are $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 1.19. [4] A topological space (X, τ) is said to be ω-T_2 if for any pair (x, y) of distinct points in X there exist $U \in \tau$ and $V \in \tau_\omega$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 1.20. [4] (a) A topological space is ω-T_2 if and only if for each $x \in X$, $Cl_{\theta_\omega}(\{x\}) = \{x\}$.

(b) Every ω-T_2 topological space is T_1.

Theorem 1.21. [11] A topological space (X, τ) is T_2 if and only if it is R_1 and T_1.

Definition 1.22. [4] A topological space (X, τ) is said to be ω-locally indiscrete if every open set in (X, τ) is ω-closed.

Theorem 1.23. [4] a. If A is a subset of an ω-locally indiscrete topological space (X, τ), then $\overline{A} = Cl_{\theta_\omega}(A)$.

b. Every locally indiscrete topological space is ω-locally indiscrete.

c. Every locally countable topological space is ω-locally indiscrete.

Theorem 1.24. [12] A topological space (X, τ) is R_1 if and only if for each $x \in X$, $Cl_\theta(\{x\}) = \overline{\{x\}}$.

2. 2. θ_ω-Connectedness

In this section, we use the θ_ω-closure operator to define θ_ω-connectedness as a property which is weaker than connectedness and stronger than θ-connectedness. We give several sufficient conditions for the equivalence between θ_ω-connectedness and connectedness, and between θ_ω-connectedness and θ-connectedness. We give two results regarding the union of θ_ω-connected sets and also we show that the weakly θ_ω-continuous image of a connected set is θ_ω-connected.

Definition 2.1. Let (X, τ) be a topological space. A pair (P, Q) of non-empty subsets of X is a said to be a θ_ω-separation relative to (X, τ), if $(P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) = \emptyset$.

Theorem 2.2. Let (X, τ) be a topological space and let (P, Q) be a pair of non-empty subsets of X. If (P, Q) is a θ_ω-separation relative to (X, τ), then it is a separation relative to (X, τ).

Proof. Since the pair (P, Q) is a θ_ω-separation relative to (X, τ), then $(P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) = \emptyset$. By Theorem 1.3, we have

$(\mathrm{P} \cap \overline{Q}) \cup (\overline{P} \cap Q) \subseteq (P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) = \emptyset$.

It follows that $(P \cap \overline{Q}) \cup (\overline{P} \cap Q) = \emptyset$. Hence, the pair (P, Q) is a separation relative to (X, τ).

Theorem 2.3. Let (X, τ) be a locally countable topological space and let (P, Q) be a pair of non-empty subsets of X. Then (P, Q) is a θ_ω-separation relative to (X, τ) if and only if it is a separation relative to (X, τ).

Proof. *Necessity.* We can see it by Theorem 2.2.

Sufficiency. Suppose that the pair (P, Q) is a separation relative to (X, τ). Then $(P \cap \overline{Q}) \cup (\overline{P} \cap Q) = \emptyset$. Since (X, τ) is locally countable, then by Theorem 1.5 $\overline{P} = Cl_{\theta_\omega}(P)$ and $\overline{Q} = Cl_{\theta_\omega}(Q)$. Thus,

$$\begin{aligned}(P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) &= (P \cap \overline{Q}) \cup (\overline{P} \cap Q) \\ &= \emptyset.\end{aligned}$$

It follows that (P, Q) is a θ_ω-separation relative to (X, τ).

Example 2.11 will show that the condition 'locally countable' in Theorem 2.3 cannot be dropped.

Theorem 2.4. Let (X, τ) be a topological space and let (P, Q) be a pair of non-empty subsets of X. If (P, Q) is a θ-separation relative to (X, τ), then it is a θ_ω-separation relative to (X, τ).

Proof. Since the pair (P, Q) is a θ-separation relative to (X, τ), then $(P \cap Cl_\theta(Q)) \cup (Cl_\theta(P) \cap Q) = \emptyset$. By Theorem 1.3, we have

$(P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) \subseteq (P \cap Cl_\theta(Q)) \cup (Cl_\theta(P) \cap Q) = \emptyset$.

It follows that $(P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) = \emptyset$. Hence, the pair (P, Q) is a θ_ω-separation relative to (X, τ).

Theorem 2.5. Let (X, τ) be an anti-locally countable topological space and let (P, Q) be a pair of non-empty subsets of X. Then (P, Q) is a θ-separation relative to (X, τ) if and only if it is a θ_ω-separation relative to (X, τ).

Proof. *Necessity.* We can see it by Theorem 2.4.

Sufficiency. Suppose that the pair (P, Q) is a θ_ω-separation relative to (X, τ). Then $(P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) = \emptyset$. Since (X, τ) is anti-locally countable, then by Theorem 1.7 $Cl_\theta(P) = Cl_{\theta_\omega}(P)$ and $Cl_\theta(Q) = Cl_{\theta_\omega}(Q)$.

Thus,

$$\begin{aligned}(P \cap Cl_\theta(Q)) \cup (Cl_\theta(P) \cap Q) &= (P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) \\ &= \emptyset.\end{aligned}$$

It follows that (P, Q) is a θ-separation relative to (X, τ).

Example 2.12 will show that the condition 'anti-locally countable' in Theorem 2.5 cannot be dropped.

Definition 2.6. Let (X,τ) be a topological space. A subset A of X is said to be θ_ω-connected relative to (X,τ) if there is no θ_ω-separation relative to (X,τ), (P,Q), such that $A = P \cup Q$.

Theorem 2.7. Let (X,τ) be a topological space and $A \subseteq X$. If A is connected relative to (X,τ), then A is θ_ω-connected relative to (X,τ).

Proof. Suppose A is not θ_ω-connected relative to (X,τ). Then there is a θ_ω-separation relative to (X,τ), (P,Q), such that $P \cup Q = A$. By Theorem 2.2, (P,Q) is a separation relative to (X,τ) with $P \cup Q = A$. Therefore, A is not connected relative to (X,τ), a contradiction.

Theorem 2.8. Let (X,τ) be a topological space and $A \subseteq X$. If A is θ_ω-connected relative to (X,τ), then A is θ-connected relative to (X,τ).

Proof. Suppose A is not θ-connected relative to (X,τ). Then there is a θ-separation relative to (X,τ), (P,Q), such that $P \cup Q = A$. By Theorem 2.4, (P,Q) is a θ_ω-separation relative to (X,τ) with $A = P \cup Q$. Therefore, A is not θ_ω-connected relative to (X,τ), a contradiction.

Theorem 2.9. Let (X,τ) be a locally countable topological space and $A \subseteq X$. Then A is connected relative to (X,τ) if and only if A is θ_ω-connected relative to (X,τ).

Proof. *Necessity.* We can see it by Theorem 2.7.

Sufficiency. Suppose that A is θ_ω-connected relative to (X,τ) and suppose to the contrary that A is not connected relative to (X,τ). Since A is not connected relative to (X,τ), then there is a separation relative to (X,τ), (P,Q), such that $P \cup Q = A$. By Theorem 2.3, the pair (P,Q) is a θ_ω-separation relative to (X,τ) with $P \cup Q = A$. Therefore, A is not θ_ω-connected relative to (X,τ), a contradiction.

Theorem 2.10. Let (X,τ) be an anti-locally countable topological space and $A \subseteq X$. Then A is θ_ω-connected relative to (X,τ) if and only if A is θ-connected relative to (X,τ).

Proof. *Necessity.* We can see it by Theorem 2.8.

Sufficiency. Suppose that A is θ-connected relative to (X,τ) and suppose to the contrary that A is not θ_ω-connected relative to (X,τ). Since A is not θ_ω-connected relative to (X,τ), then there is a θ_ω-separation relative to (X,τ), (P,Q), such that $P \cup Q = A$. By Theorem 2.5, the pair (P,Q)

is a θ-separation relative to (X, τ) with $P \cup Q = A$. Therefore, A is not θ-connected relative to (X, τ), a contradiction.

The following example shows in Theorem 2.9 that the condition 'locally countable' cannot be dropped:

Example 2.11. (Example 2.4 of [6]) Let $I = [0, 1], X = I \times I$, and $A = I \times \{0\}$. Let τ be the topology on X generated by the following base on X: (1) the relative open sets from the plane in $X - A$ and (2) for $x \in A$, sets of the form $(V \cap (X - A)) \cup \{x\}$ where V is open in the plane with $x \in V$. It is proved in [15] that A is θ-connected relative to (X, τ) but A is not connected relative to (X, τ). Since (X, τ) is anti-locally countable, then by Theorem 2.10, A is θ_ω-connected relative to (X, τ).

The following example shows in Theorem 2.10 that the condition 'anti-locally countable' cannot be dropped:

Example 2.12. (Example 4 of [9]) Let $X = \mathbf{Z}$ and τ be the topology generated by $\mathcal{B} = \{\{2m-1\}, m \in \mathbf{Z}\} \cup \{\{2m-1, 2m, 2m+1\}, m \in \mathbf{Z}\}$ as a base. Let $A = \{2m, 2m+2\}$. It is proved in [9] that A is θ-connected relative to (X, τ). On the other hand, it is not difficult to check that the pair $(\{2m\}, \{2m+2\})$ is a separation of A relative to (X, τ) and so A is not connected relative to (X, τ). Since (X, τ) is locally countable, then by Theorem 2.9, A is not θ_ω-connected relative to (X, τ).

For ω-regular topological spaces, the concepts connectedness relative to (X, τ) and θ_ω-connectedness relative to (X, τ) are equivalent:

Theorem 2.13. Let (X, τ) be an ω-regular topological space and $A \subseteq X$. Then the following are equivalent:

(a) A is θ_ω-connected relative to (X, τ).

(b) A is connected relative to (X, τ).

Proof. The definitions and Theorem 1.11.

For regular topological spaces, the concepts connectedness relative to (X, τ), θ_ω-connectedness relative to (X, τ) and θ-connectedness relative to (X, τ) are equivalent:

Theorem 2.14. Let (X, τ) be a regular topological space and $A \subseteq X$. Then the following are equivalent:

(a) A is connected relative to (X,τ)
(b) A is θ_ω-connected relative to (X,τ)
(c) A is θ-connected relative to (X,τ).

Proof. The definitions and Theorem 1.12.

Notation 2.15. Let (X,τ) be a topological space and let $B \subseteq A \subseteq X$ with A is non-empty.

(a) The closure of B in (A,τ_A) will be denoted by $\left(\overline{B}\right)_A$.
(b) The θ-closure of B in (A,τ_A) will be denoted by $(Cl_\theta B)_A$.
(c) The θ_ω-closure of B in (A,τ_A) will be denoted by $(Cl_{\theta_\omega} B)_A$.
(d) The ω-closure of B in (A,τ_A) will be denoted by $\left(\overline{B}^\omega\right)_A$.

Theorem 2.16. Let (X,τ) be a topological space and let $B \subseteq A \subseteq X$ with A is non-empty. Then $(Cl_{\theta_\omega} B)_A \subseteq Cl_{\theta_\omega}(B) \cap A$.

Proof. $(Cl_{\theta_\omega} B)_A \subseteq A$ is obvious. To see that $(Cl_{\theta_\omega} B)_A \subseteq Cl_{\theta_\omega}(B)$, let $x \in (Cl_{\theta_\omega} B)_A$ and $U \in \tau$ with $x \in U$. Let $V = U \cap A$, then $V \in \tau_A$ and $x \in V$. Since $x \in (Cl_{\theta_\omega} B)_A$, then $\left(\overline{V}^\omega\right)_A \cap B \neq \emptyset$. By Theorem 1.13, $\left(\overline{V}^\omega\right)_A = \overline{V}^\omega \cap A$. Thus, $\left(\overline{V}^\omega\right)_A \cap B = \overline{V}^\omega \cap A \cap B \subseteq \overline{U}^\omega \cap A \cap B \subseteq \overline{U}^\omega \cap B$ and hence $\overline{U}^\omega \cap B \neq \emptyset$. It follows that $x \in Cl_{\theta_\omega}(B)$.

Theorem 2.17. Let (X,τ) be a topological space and let $A \subseteq X$. If A is θ_ω-connected relative to (A,τ_A), then A is θ_ω-connected relative to (X,τ).

Proof. Suppose to the contrary that A is not θ_ω-connected relative to(X,τ). Then there is a θ_ω-separation relative to (X,τ), (P,Q), such that $A = P \cup Q$. By Theorem 2.16,

$$\begin{aligned}(P \cap (Cl_{\theta_\omega} Q)_A) \cup ((Cl_{\theta_\omega} P)_A \cap Q) &\subseteq (P \cap (Cl_{\theta_\omega}(Q) \cap A)) \cup (Cl_{\theta_\omega}(P) \cap A \cap Q) \\ &\subseteq (P \cap (Cl_{\theta_\omega} Q)) \cup ((Cl_{\theta_\omega} P) \cap Q) \\ &= \emptyset.\end{aligned}$$

It follows that A is not θ_ω-connected relative to (A,τ_A), a contradiction.

Theorem 2.18. Let (X,τ) be a topological space and $A \subseteq X$, let D be the closure of A in (X,τ_{θ_ω}). Then $Cl_{\theta_\omega}(A) \subseteq D$.

Proof. Let $x \in Cl_{\theta_\omega}(A)$ and let $U \in \tau_{\theta_\omega}$ with $x \in U$. Choose $V \in \tau$ such that $x \in V \subseteq \overline{V}^{\omega} \subseteq U$. Since $x \in Cl_{\theta_\omega}(A)$, then $\overline{V}^{\omega} \cap A \neq \emptyset$ and so $U \cap A \neq \emptyset$. Thus, $x \in D$.

Theorem 2.19. Let (X, τ) be a topological space and let $A \subseteq X$. If A is θ_ω-connected relative to (X, τ), then A is connected relative to $(X, \tau_{\theta_\omega})$.

Proof. Suppose to the contrary that A is not connected relative to $(X, \tau_{\theta_\omega})$. Then there is a separation, (P, Q), relative to $(X, \tau_{\theta_\omega})$ such that $A = P \cup Q$. Let C and D be the closures of P and Q in $(X, \tau_{\theta_\omega})$, respectively. By Theorem 2.18,

$$\begin{aligned}(P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q) &\subseteq (P \cap D) \cup (C \cap Q) \\ &= \emptyset.\end{aligned}$$

It follows that A is not θ_ω-connected relative to (X, τ), a contradiction.

The next example shows that the converse of Theorem 2.19 is not true in general:

Example 2.20. (Example 2.26 of [4]) Let $\tau = \{\emptyset, \mathbf{R}, \mathbf{N}, \mathbf{Q}^c, \mathbf{N} \cup \mathbf{Q}^c\}$. It is proved in [4] that $\tau_{\theta_\omega} = \{\mathbf{R}, \emptyset, \mathbf{N}\}$. Let $A = \mathbf{N} \cup \{\sqrt{2}\}$, then

(a) A is connected relative to $(X, \tau_{\theta_\omega})$.

(b) A is not θ_ω-connected relative to (X, τ).

Proof. a) Suppose to the contrary that A is not connected relative to $(X, \tau_{\theta_\omega})$. Then there is a separation, (P, Q), relative to $(X, \tau_{\theta_\omega})$ such that $A = P \cup Q$. Let C and D be the closures of P and Q in $(X, \tau_{\theta_\omega})$, respectively. Without loss of generality let us assume that $\sqrt{2} \in P$. Then $\sqrt{2} \notin D$. So, there is $U \in \tau_{\theta_\omega}$ such that $\sqrt{2} \in U$ and $U \cap Q = \emptyset$. Since $U = \mathbf{R}$, then $U \cap Q = Q \neq \emptyset$, a contradiction.

b) Let $P = \mathbf{N}$ and $Q = \{\sqrt{2}\}$.To see that $P \cap Cl_{\theta_\omega}(Q) = \emptyset$, let $x \in P$. Since $P \in \tau$ and $\overline{P}^{\omega} \cap Q = \overline{\mathbf{N}}^{\omega} \cap \{\sqrt{2}\} = \mathbf{N} \cap \{\sqrt{2}\} = \emptyset$, then $x \notin Cl_{\theta_\omega}(Q)$. To see that $Cl_{\theta_\omega}(P) \cap Q = \emptyset$ we show that $\sqrt{2} \notin Cl_{\theta_\omega}(\mathbf{N})$. Since $\sqrt{2} \in \mathbf{Q}^c \in \tau$ and $\overline{\mathbf{Q}^c}^{\omega} \cap \mathbf{N} = (\mathbf{R} - \mathbf{N}) \cap \mathbf{N} = \emptyset$, then $\sqrt{2} \notin Cl_{\theta_\omega}(\mathbf{N})$.

Theorem 2.21. Let (X, τ) be a topological space and let $A \subseteq X$. Then A is θ_ω-connected relative to (X, τ) if and only if when (P, Q) is a θ_ω-separation relative to (X, τ) and $A \subseteq P \cup Q$, then either $A \subseteq P$ or $A \subseteq Q$.

Proof. *Necessity.* Suppose that A is θ_ω-connected relative to (X,τ). Let (P,Q) be θ_ω-separation relative to (X,τ) such that $A \subseteq P \cup Q$. Let $G = A \cap P$ and $H = A \cap Q$. Then

$$\begin{aligned}(G \cap Cl_{\theta_\omega}(H)) \cup (Cl_{\theta_\omega}(G) \cap H) &\subseteq (P \cap Cl_{\theta_\omega}(Q)) \cup (Cl_{\theta_\omega}(P) \cap Q)\\ &= \emptyset.\end{aligned}$$

Thus, (G,H) is a θ_ω-separation relative to (X,τ). Since $A = G \cup H$ and A is θ_ω-connected relative to (X,τ), either $G = \emptyset$ or $H = \emptyset$. Thus either$A = A \cap P$ or $A = A \cap Q$. Hence either $A \subseteq P$ or $A \subseteq Q$.

Sufficiency. Suppose to the contrary that A is not θ_ω-connected relative to (X,τ). Then there is a θ_ω-separation, (P,Q), relative to (X,τ) such that $A = P \cup Q$. By assumption either $A \subseteq P$ or $A \subseteq Q$. Hence $Q = \emptyset$ or $P = \emptyset$, a contradiction.

Theorem 2.22. Let (X,τ) be a topological space and let A be θ_ω-connected relative to (X,τ) and let $B \subseteq X$ such that $A \subseteq B \subseteq Cl_{\theta_\omega}(A)$. Then B is θ_ω-connected relative to (X,τ).

Proof. Suppose to the contrary that B is not θ_ω-connected relative to (X,τ). Then there exists, (P,Q), θ_ω-separation relative to (X,τ) such that $B = P \cup Q$. Since $A \subseteq P \cup Q$, then by Theorem 2.21 either $A \subseteq P$ or $A \subseteq Q$. Thus, either we have $Q \subseteq Cl_{\theta_\omega}(A) \subseteq Cl_{\theta_\omega}(P)$ or $P \subseteq Cl_{\theta_\omega}(A) \subseteq Cl_{\theta_\omega}(Q)$. So either we have $Q \subseteq Cl_{\theta_\omega}(P) \cap Q = \emptyset$ or $P \subseteq P \cap Cl_{\theta_\omega}(Q) = \emptyset$. Therefore, either $P = \emptyset$ or $Q = \emptyset$, a contradiction.

Corollary 2.23. Let (X,τ) be a topological space. If A is θ_ω-connected relative to (X,τ), then $Cl_{\theta_\omega}(A)$ is θ_ω-connected relative to (X,τ).

Theorem 2.24. Let (X,τ) be a topological space. If for each $\alpha \in \triangle$, the set A_α is θ_ω-connected relative to (X,τ) such that $A_\alpha \cap A_\beta \neq \emptyset$ for each α, $\beta \in \triangle$, then $\bigcup_{\alpha \in \triangle} A_\alpha$ is θ_ω-connected relative to (X,τ).

Proof. Suppose to the contrary that $\bigcup_{\alpha \in \triangle} A_\alpha$ is not θ_ω-connected relative to (X,τ). Then there exists (P,Q), θ_ω-separation relative to (X,τ) such that $\bigcup_{\alpha \in \triangle} A_\alpha = P \cup Q$. For each $\beta \in \triangle$, A_β is θ_ω-connected relative to (X,τ) and $A_\beta \subseteq \bigcup_{\alpha \in \triangle} A_\alpha = P \cup Q$, so by Theorem 2.21 either $A_\beta \subseteq P$ or $A_\beta \subseteq Q$. Without loss of generality we may assume that $A_\beta \subseteq P$.

Claim. Either we have $A_\alpha \subseteq P$ for all $\alpha \in \triangle$ or we have $A_\alpha \subseteq Q$ for all $\alpha \in \triangle$.

Proof of Claim. Suppose to the contrary that there are $\alpha, \beta \in \triangle$ such that $A_\alpha \subseteq P$ and $A_\beta \subseteq Q$. Then $A_\alpha \cap A_\beta \subseteq P \cap Q = \emptyset$, a contradiction.

By the above claim either ($\bigcup\limits_{\alpha \in \triangle} A_\alpha = P$ and $Q = \emptyset$) or ($\bigcup\limits_{\alpha \in \triangle} A_\alpha = Q$ and $P = \emptyset$) which is a contradiction.

Theorem 2.25. Let f_n be a sequance of θ_ω-connected relative to (X, τ) subsets such that for all $n \in \mathbf{N}$, $f_n \cap f_{n+1} \neq \emptyset$. Then $\bigcup\limits_{n \in \mathbf{N}} f_n$ is θ_ω-connected relative to (X, τ).

Proof. For each $n \in \mathbf{N}$, let $B_n = \bigcup\limits_{i=1}^{n} f_n$.

Claim. For all $n \in \mathbf{N}$, B_n is θ_ω-connected relative to (X, τ).

Proof of Claim. By induction.

$B_1 = f_1$ is θ_ω-connected relative to (X, τ). Suppose B_n is θ_ω-connected relative to (X, τ). Since $f_n \cap f_{n+1} \neq \emptyset$ and $f_n \cap f_{n+1} \subseteq B_n \cap f_{n+1}$ then $B_n \cap f_{n+1} \neq \emptyset$ and by Theorem 2.24 $B_{n+1} = B_n \cup f_{n+1}$ is θ_ω-connected relative to (X, τ).

Since for each $n, m \in \mathbf{N}$, $B_n \cap B_m = B_t \neq \emptyset$ where $t = \min\{n, m\}$, then again by Theorem 2.24, $\bigcup\limits_{n \in \mathbf{N}} B_n = \bigcup\limits_{n \in \mathbf{N}} f_n$ is θ_ω-connected relative to (X, τ).

Theorem 2.26. Let (X, τ) be a topological space and let $A\$beanon-emptysubsetof$X$.Then$A$is\theta_\omega$-connected relative to (X, τ) if and only if for each two points $x, y \in A$, there is a subset $B \subseteq A$ with $x, y \in B$, such that B is θ_ω-connected relative to (X, τ).

Proof. *Necessity.* Obvious.

Sufficiency. Choose $x \in A$. By assumption, for each $y \in A$ there is θ_ω-connected relative to (X, τ) B_{xy} which contains x and y and $B_{xy} \subseteq A$. Since $A = y \in A \cup B_{xy}$ and $x \in \bigcap\limits_{y \in A} B_{xy}$, then by Theorem 2.24, A is θ_ω-connected relative to (X, τ).

Theorem 2.27. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a weakly θ_ω-continuous function. If K is connected relative to (X, τ), then $f(K)$ is θ_ω-connected relative to (Y, σ).

Proof. Suppose on the contrary that there is, (P, Q), θ_ω-separation relative to (Y, σ) such that $f(K) = P \cup Q$. Let $A = K \cap f^{-1}(P)$ and $B = K \cap f^{-1}(Q)$. Then $A \neq \emptyset, B \neq \emptyset$ and $K = A \cup B$.

Claim. (A, B) is separation relative to (X, τ).

Proof of claim. Assume not. Without loss of generality we may assume that there is $x \in A \cap \overline{B}$. Then $x \in K$ and $f(x) \in P \subseteq Y - Cl_{\theta_\omega}(Q) \in \tau_{\theta_\omega}$. Choose $V \in \sigma$ such that $f(x) \in V \subseteq \overline{V}^{\omega} \subseteq Y - Cl_{\theta_\omega}(Q)$. Since f is weakly θ_ω-continuous, there is $U \in \tau$ such that $x \in U$ and $f(U) \subseteq \overline{V}^{\omega}$. So $f(U) \cap Q = \emptyset$. Since $x \in \overline{B}$, then $U \cap B \neq \emptyset$. Choose $t \in U \cap B = U \cap K \cap f^{-1}(Q)$, then $f(t) \in f(U) \cap Q$, a contradiction.

By the above claim we conclude that K is connected relative to (X, τ) which is a contradiction.

3. 3. V-θ_ω-Connectedness

In this section, we define and investigate V-θ_ω-connectedness as a strong form of V-θ-connectedness. We show that the θ_ω-connectedness and V-θ_ω-connectedness are independent.

We start by the following relationship between connectedness and V-θ-connectedness.

Theorem 3.1. Let (X, τ) be a topological space and let $A \subseteq X$. If A is connected relative to (X, τ), then A is V-θ-connected relative to (X, τ).

Proof. Suppose to the contrary that A is not V-θ-connected relative to (X, τ), then there are disjoint non-empty sets P and Q and open sets U, V such that $A = P \cup Q, P \subseteq U, Q \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$. Thus,

$$\begin{aligned}
\left(P \cap \overline{Q}\right) \cup \left(\overline{P} \cap Q\right) &\subseteq \left(U \cap \overline{V}\right) \cup \left(\overline{U} \cap V\right) \\
&\subseteq \left(\overline{U} \cap \overline{V}\right) \cup \left(\overline{U} \cap \overline{V}\right) \\
&= \emptyset \cup \emptyset \\
&= \emptyset.
\end{aligned}$$

Thus the pair (P, Q) is a separation relative to (X, τ), and so A is not connected relative to (X, τ). This is a contradiction.

Example 3.6 will show that the converse of Theorem 3.1 is not true in general.

Definition 3.2. Let (X, τ) be a topological space and let $A \subseteq X$. A is V-θ_ω-connected if there are no disjoint non-empty sets P and Q and no open sets U and V such that $A = P \cup Q$, $P \subseteq U$, $Q \subseteq V$ and $\overline{U}^\omega \cap \overline{V}^\omega = \emptyset$.

Theorem 3.3. Let (X, τ) be a topological space and let $A \subseteq X$. If A is connected relative to (X, τ_ω), then A is V-θ_ω-connected relative to (X, τ).

Proof. Suppose to the contrary that A is not V-θ_ω-connected relative to (X, τ). Then there are disjoint non-empty sets P and Q and open sets U,V such that $A = P \cup Q$, $P \subseteq U$, $Q \subseteq V$, and $\overline{U}^\omega \cap \overline{V}^\omega = \emptyset$. Thus,

$$\begin{aligned}
\left(P \cap \overline{Q}^\omega\right) \cup \left(\overline{P}^\omega \cap Q\right) &\subseteq \left(U \cap \overline{V}^\omega\right) \cup \left(\overline{U}^\omega \cap V\right) \\
&\subseteq \left(\overline{U}^\omega \cap \overline{V}^\omega\right) \cup \left(\overline{U}^\omega \cap \overline{V}^\omega\right) \\
&= \emptyset \cup \emptyset \\
&= \emptyset.
\end{aligned}$$

Thus the pair (P, Q) is a separation relative to (X, τ_ω), and so A is not connected relative to (X, τ_ω). This is a contradiction.

Question 3.4. Let (X, τ) be a topological space. Is it true that every V-θ_ω-connected relative to (X, τ) is connected relative to (X, τ_ω)?

Theorem 3.5. Let (X, τ) be a topological space and let $A \subseteq X$. If A is V-θ_ω-connected relative to (X, τ), then A is V-θ-connected relative to (X, τ).

Proof. Suppose to the contrary that A is not V-θ-connected relative to (X, τ). Then there are disjoint non-empty sets P and Q and open sets U,V such that $A = P \cup Q$, $P \subseteq U$, $Q \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$. By Theorem 1.16, $\overline{U}^\omega \cap \overline{V}^\omega \subseteq \overline{U} \cap \overline{V} = \emptyset$, and so A is not V-θ_ω-connected relative to (X, τ), a contradiction.

The following example shows that the converse of Theorem 3.5 is not true in general:

Example 3.6. (Example 2.26 of [4]) Consider $(\mathbf{R}, \tau)$ where $\tau = \{\emptyset, \mathbf{R}, \mathbf{N}, \mathbf{Q}^c, \mathbf{N} \cup \mathbf{Q}^c\}$. It is proved in [4] that $\tau_{\theta_\omega} = \{\mathbf{R}, \emptyset, \mathbf{N}\}$. Let $A = \mathbf{N} \cup \mathbf{Q}^c$. Then

(a) A is V-θ-connected relative to $(\mathbf{R}, \tau)$.

(b) A is not V-θ_ω-connected relative to $(\mathbf{R}, \tau)$.

(c) A is θ_ω-connected relative to $(\mathbf{R}, \tau)$.

Proof . a) Suppose to the contrary that A is not V-θ-connected relative to $(\mathbf{R}, \tau)$. Then there are disjoint non-empty sets P and Q and open sets U,V such that $A = P \cup Q$, $P \subseteq U$, $Q \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$. If $U = \mathbf{N} \cup \mathbf{Q}^c$ or $V = \mathbf{N} \cup \mathbf{Q}^c$, then $\overline{U} \cap \overline{V} = \overline{\mathbf{N} \cup \mathbf{Q}^c} \cap \overline{V} = \mathbf{R} \cap \overline{V} = \overline{V} \neq \emptyset$ or $\overline{U} \cap \overline{V} = \overline{U} \cap \overline{\mathbf{N} \cup \mathbf{Q}^c} = \overline{U} \cap \mathbf{R} = \overline{U} \neq \emptyset$. So either ($U = \mathbf{N}$ and $V = \mathbf{Q}^c$) or ($V = \mathbf{N}$ and $U = \mathbf{Q}^c$). Thus, $\overline{U} \cap \overline{V} = \mathbf{Q} \cap (\mathbf{R} - \mathbf{N}) \neq \emptyset$, a contradiction.

b) Note that $\tau_\omega = \tau_{coc} \cup \{C : C \subseteq \mathbf{N}\}$ where τ_{coc} denotes the cocountable topology on $\mathbf{R}$, and $\{\mathbf{R}\} \cup \{M : M \text{ is countable}\} \cup \{\mathbf{R} - C : C \subseteq \mathbf{N}\}$ is the collection of ω-closed sets of $(\mathbf{R}, \tau)$. Let $P = \mathbf{N} = U$ and $Q = \mathbf{Q}^c = V$. Then $\overline{U}^\omega = \mathbf{N}$ and $\overline{V}^\omega = \overline{\mathbf{Q}^c} = \mathbf{R} - \mathbf{N}$ and so $\overline{U}^\omega \cap \overline{V}^\omega = (\mathbf{R} - \mathbf{N}) \cap \mathbf{N} = \emptyset$. This shows that A is not V-θ_ω-connected relative to (X, τ).

c) Suppose to the contrary that there is a pair, (P, Q), that is θ_ω-separation relative to (X, τ) such that $\mathbf{N} \cup \mathbf{Q}^c = P \cup Q$. Note that $\{\emptyset, \mathbf{R}, \mathbf{R} - \mathbf{N}\}$ is the collection θ_ω-closed sets of $(\mathbf{R}, \tau)$. Thus, $Cl_{\theta_\omega}(P) = Cl_{\theta_\omega}(Q) = \mathbf{R} - \mathbf{N}$. Since $P \cap Cl_{\theta_\omega}(Q) = Q \cap Cl_{\theta_\omega}(P) = \emptyset$, then $P \subseteq \mathbf{N}$ and $Q \subseteq \mathbf{N}$ and so $P \cup Q \subseteq \mathbf{N}$, a contradiction.

Theorem 3.7. Let (X, τ) be an anti-locally countable topological space and $A \subseteq X$. Then A is V-θ_ω-connected relative to (X, τ) if and only if A is V-θ-connected relative to (X, τ).

Proof. *Necessity.* Theorem 3.5.

Sufficiency. We can see it by Theorem 1.17.

The following example, together with Example 3.6, shows that θ_ω-connectedness relative to (X, τ) and V-θ_ω-connectedness relative to (X, τ) are independent:

Example 3.8. (Example 1 of [9]) Consider that topological space $(\mathbf{R}, \tau_u)$ where τ_u is the usual topology on $\mathbf{R}$. Let $A = (0, 1) \cup (1, 2)$. Then

(a) A is not θ_ω-connected relative to $(\mathbf{R}, \tau_u)$.

(b) A is V-θ_ω-connected relative to $(\mathbf{R}, \tau_u)$.

Proof. a) Since A is not interval, then A is not connected relative to $(\mathbf{R}, \tau_u)$. Since $(\mathbf{R}, \tau_u)$ is regular, then by Theorem 2.14 A is not θ_ω-connected relative to $(\mathbf{R}, \tau_u)$.

b) It is proved in [9] that A is V-θ-connected relative to $(\mathbf{R}, \tau_u)$. Since $(\mathbf{R}, \tau_u)$ is anti-locally countable, then by Theorem 3.7, A is V-θ_ω-connected relative to $(\mathbf{R}, \tau_u)$.

4. 4. R_1 and ω-R_1 Separation Axioms

In this section, we continue the study of R_1 as a known topological property by giving several results regarding it. We introduce ω-$R_1\,(I)$, ω-$R_1\,(II)$, ω-$R_1\,(III)$ and weakly ω-R_1 as four weaker forms of R_1 by utilizing ω-open sets.

Definition 4.1. A topological space (X, τ) is said to be ω-$R_1\,(I)$ if for any two points $x, y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$, there are $U \in \tau$, $V \in \tau_\omega$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Definition 4.2. A topological space (X, τ) is said to be ω-$R_1\,(II)$ if $Cl_{\theta_\omega}(\{x\}) = \overline{\{x\}}$ for all $x \in X$.

Definition 4.3. A topological space (X, τ) is said to be ω-$R_1\,(III)$ if for any two points $x, y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$, there are disjoint sets $U \in \tau$ and $V \in \tau_\omega$ such that $(x \in U\ \text{ and } y \in V)$ or $(y \in U\ \text{ and } x \in V)$.

Theorem 4.4. Every R_1 topological space is ω-$R_1\,(I)$.

Proof. Follows directly from the definitions.

Theorem 4.5. Every ω-$R_1\,(I)$ topological space is ω-$R_1\,(II)$.

Proof. Let (X, τ) be ω-$R_1\,(I)$. Suppose to the contrary that there is $x \in X$ such that $Cl_{\theta_\omega}(\{x\}) \neq \overline{\{x\}}$. Since $\overline{\{x\}} \subseteq Cl_{\theta_\omega}(\{x\})$ is always true, then there is $y \in Cl_{\theta_\omega}(\{x\}) - \overline{\{x\}}$. Since $y \notin \overline{\{x\}}$, then $\overline{\{x\}} \neq \overline{\{y\}}$ and by assumption, there exist $U \in \tau_\omega$ and $V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since $y \in Cl_{\theta_\omega}(\{x\})$ and $y \in V \in \tau$, then $\overline{V}^{\omega} \cap \{x\} \neq \emptyset$. So $x \in \overline{V}^{\omega}$. Since $x \in U \in \tau_\omega$ and $x \in \overline{V}^{\omega}$, then $U \cap V \neq \emptyset$, a contradiction.

Theorem 4.6. Every ω-$R_1\,(II)$ topological space is ω-$R_1\,(III)$.

Proof. Let (X,τ) be ω-$R_1\,(II)$. Let $x,y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$. Since (X,τ) is ω-$R_1\,(II)$, then $Cl_{\theta_\omega}(\{x\}) = \overline{\{x\}}$ and $Cl_{\theta_\omega}(\{y\}) = \overline{\{y\}}$ and so $Cl_{\theta_\omega}(\{x\}) \neq Cl_{\theta_\omega}(\{y\})$. We have two cases to be considered:

Case 1. There is $z \in Cl_{\theta_\omega}(\{x\}) - Cl_{\theta_\omega}(\{y\})$. Since $z \notin Cl_{\theta_\omega}(\{y\})$, then there exists $U \in \tau$ such that $z \in U$ and $\overline{U}^{\omega} \cap \{y\} = \emptyset$. Since $z \in U \in \tau$ and $z \in Cl_{\theta_\omega}(\{x\}) = \overline{\{x\}}$, then $U \cap \{x\} \neq \emptyset$ and so $x \in U$. Set $V = X - \overline{U}^{\omega}$. Then we have $x \in U \in \tau$, $y \in V \in \tau_\omega$ and $U \cap V = \emptyset$.

Case 2. There is $z \in Cl_{\theta_\omega}(\{y\}) - Cl_{\theta_\omega}(\{x\})$. As Case 1, we can find $U \in \tau$, $V \in \tau_\omega$ such that $y \in U \in \tau$, $x \in V \in \tau_\omega$ and $U \cap V = \emptyset$.

Theorem 4.7. Every ω-T_2 topological space is ω-$R_1\,(I)$.

Proof. Let (X,τ) be ω-T_2 and let $x,y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$. Then $x \neq y$. Since (X,τ) is ω-T_2, there are $U \in \tau$, $V \in \tau_\omega$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore, (X,τ) is ω-$R_1\,(I)$.

Theorem 4.8. A topological space (X,τ) is ω-T_2 if and only if it is ω-$R_1\,(II)$ and T_1.

Proof. Suppose that (X,τ) is ω-T_2. Then by Theorems 4.7 and 4.5 it is ω-$R_1\,(II)$. Also, by Theorem 1.20 (b) it is T_1. Conversely, suppose that (X,τ) is ω-$R_1\,(II)$ and T_1. To show that (X,τ) is ω-T_2, we apply Theorem 1.20 (a). Let $x \in X$. By the definition of ω-$R_1\,(II)$, $Cl_{\theta_\omega}(\{x\}) = \overline{\{x\}}$. Since (X,τ) is T_1, then $\overline{\{x\}} = \{x\}$. It follows that $Cl_{\theta_\omega}(\{x\}) = \{x\}$. Hence by Theorem 1.20 (a), it follows that (X,τ) is ω-T_2.

Corollary 4.9. A topological space (X,τ) is ω-T_2 if and only if it is ω-$R_1\,(I)$ and T_1.

Proof. Suppose that (X,τ) is ω-T_2. Then by Theorems 4.7 and 4.5 it is ω-$R_1\,(II)$. Also, by Theorem 4.8 it is T_1. The converse follows from Theorems 4.5 and 4.8.

Recall that a proper non-empty open subset U of a topological space (X,τ) is said to be a minimal open set if any open set which is contained in U is $\emptyset$ or U.

Theorem 4.10. Let U be a minimal open set of a topological space (X,τ). If for some $x \in U$ we have $\overline{\{x\}} \subseteq U$, then $\overline{\{x\}} = U$. In particular, U is clopen.

Proof. Suppose to the contrary that $\overline{\{x\}} \neq U$. Then there is $y \in U - \overline{\{x\}}$. Choose $V \in \tau$ such that $y \in V$ and $\{x\} \cap V = \emptyset$. Since $y \in U \cap V \in \tau - \{\emptyset\}$, $U \cap V \subseteq U$ and U is minimal open, then $U \cap V = U$ and hence $U \subseteq V$. Since $x \in \overline{\{x\}} \subseteq U$, then $x \in V$, a contradiction.

Recall that a space (X, τ) is locally indiscrete if every open subset of X is closed. It is known that a topological space (X, τ) is locally indiscrete if and only if (X, τ) is generated by a partition of X as a base. If $\mathcal{B}$ is a partition of a non-empty set X and (X, τ) is the topological space generated by $\mathcal{B}$ as a base, then U is a minimal open set of (X, τ) if and only if $U \in \mathcal{B}$.

Theorem 4.11. Every locally indiscrete topological space is R_1.

Proof. Let (X, τ) be locally indiscrete. Let $\mathcal{B}$ be a a partition of X which forms a base of (X, τ). Let $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$. If $x, y \in B$ for some $B \in \mathcal{B}$, then $\overline{\{x\}} \subseteq \overline{B} = B$ and $\overline{\{y\}} \subseteq \overline{B} = B$. Since B is a minimal open set of (X, τ) then by Theorem 4.10, $\overline{\{x\}} = \overline{\{y\}} = B$. Therefore, there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1, y \in B_2$ and $B_1 \cap B_2 = \emptyset$. This ends the proof that (X, τ) is R_1.

Theorem 4.12. Let (X, τ) be locally indiscrete. Then (X, τ) is ω-T_2 if and only if it is a discrete topological space.

Proof. Suppose that (X, τ) is locally indiscrete and ω-T_2. Then by Corollary 4.9 it is T_1. Thus, the singletons are closed in (X, τ). Since (X, τ) is locally indiscrete, then the singletons are open in (X, τ). This shows that (X, τ) is a discrete topological space. Conversely, if (X, τ) is a discrete topological space, then (X, τ) is obviously ω-T_2.

Theorem 4.13. Every ω-locally indiscrete topological space is ω-$R_1\,(II)$.

Proof. Follows from Theorem 1.23 (a).

Theorem 4.14. Let (X, τ) be an anti-locally countable topological space. Then the following are equivalent:

a. (X, τ) is R_1.
b. (X, τ) is ω-$R_1\,(I)$.
c. (X, τ) is ω-$R_1\,(II)$.

Proof. (a) $\Longrightarrow$ (b): Theorem 4.4.
(b) $\Longrightarrow$ (c): Theorem 4.5.

(c) $\Longrightarrow$ (a): We apply Theorem 1.24. Let $x \in X$. By (c), we have $Cl_{\theta_\omega}(\{x\}) = \overline{\{x\}}$. Since (X,τ) is anti-locally countable, then by Theorem 1.7, $Cl_\theta(\{x\}) = Cl_{\theta_\omega}(\{x\})$. Therefore, we have $Cl_\theta(\{x\}) = \overline{\{x\}}$ Thus by Theorem 1.24, it follows that (X,τ) is R_1.

The following example shows that the converse of Theorem 4.4 is not true in general:

Example 4.15. Consider $(\mathbf{N},\tau)$ where τ is the cofinite topology. It is given in [4] as an example of an ω-T_2 topological space that is not T_2. By Theorem 4.10, $(\mathbf{N},\tau)$ is ω-$R_1\,(I)$ and by Corollary 4.9 it is T_1. On the other hand, if it is R_1, then by Theorem 1.21, it is T_2. Therefore, it is not R_1.

The following example shows that the converse of Theorem 4.5 is not true in general:

Example 4.16. Consider $(\mathbf{R},\tau)$ where $\tau = \{\emptyset, \mathbf{R}, \mathbf{Q}\}$. Clearly that $(\mathbf{R},\tau)$ is ω-locally indiscrete and by Theorem 4.13 it is ω-$R_1\,(II)$. To see that $(\mathbf{R},\tau)$ is not ω-$R_1\,(I)$, suppose to the contrary that it is ω-$R_1\,(I)$. Since $\overline{\left\{\sqrt{2}\right\}} = \mathbf{R} - \mathbf{Q}$ and $\overline{\{1\}} = \mathbf{R}$, there are $U \in \tau_\omega$ and $V \in \tau$ such that $1 \in U$, $\sqrt{2} \in V$ and $U \cap V = \emptyset$. Since the only open set which contains $\sqrt{2}$ is $\mathbf{R}$, then $V = \mathbf{R}$ and so $U \cap V = U \neq \emptyset$, a contradiction.

The following example shows that the converse of Theorem 4.7 is not true in general:

Example 4.17. Let (X,τ) be any locally indiscrete topological space that is not discrete. By Theorem 4.11 it is R_1, so by Theorem 4.4 it is ω-$R_1\,(I)$. On the other hand, by Theorem 4.12 it is not ω-T_2.

Question 4.18. Is every ω-$R_1\,(III)$ topological space ω-$R_1\,(II)$?

Definition 4.19. A topological space (X,τ) is said to be weakly ω-R_1 if for all $x, y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$, there exist $U, V \in \tau_\omega$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem 4.20. Let (X,τ) be a topological space. Then (X,τ_ω) is R_1 if and only if (X,τ_ω) is T_2.

Proof. Since (X,τ_ω) is T_1, then by Theorem 1.21 (X,τ_ω) is R_1 if and only if (X,τ_ω) is T_2.

Theorem 4.21. Let (X, τ) be a topological space. If (X, τ_ω) is R_1, then (X, τ) is weakly ω-R_1.

Proof. Suppose (X, τ_ω) is R_1. Let $x, y \in X$ with $\overline{\{x\}} \neq \overline{\{y\}}$. Then $x \neq y$ and so $\overline{\{x\}}^{\omega} = \{x\} \neq \{y\} = \overline{\{y\}}^{\omega}$. Since (X, τ_ω) is R_1, then there exist $U, V \in \tau_\omega$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. This shows that (X, τ) is weakly ω-R_1.

The converse of Theorem 4.21 is not true in general as the following example clarifies:

Example 4.22. Consider $(\mathbf{R}, \tau)$, where $\tau = \{\emptyset, \mathbf{R}, (-\infty, 2], (2, \infty)\}$.

$(\mathbf{R}, \tau)$ is weakly ω-R_1: Since is locally indiscrete, then by Theorem 4.11 it is R_1.

$(\mathbf{R}, \tau_\omega)$ is not R_1: By Theorem 4.20 it is sufficient to see that $(\mathbf{R}, \tau_\omega)$ is not T_2. Suppose to the contrary that $(\mathbf{R}, \tau_\omega)$ is T_2. Then there are $U, V \in \tau_\omega$ such that $0 \in U, 1 \in V$ and $U \cap V = \emptyset$. Choose $H, S \in \tau$ and countable subsets $C, D \subseteq \mathbf{R}$ such that $0 \in H - C \subseteq U$ and $1 \in S - D \subseteq V$. Then we must have $H = S = (-\infty, 2]$ and $(H - C) \cap (S - D) \subseteq U \cap V = \emptyset$. So, $(-\infty, 2] \subseteq C \cup D$, a contradiction.

Theorem 4.23. If (X, τ) is locally countable, then (X, τ_ω) is R_1.

Proof. Suppose that (X, τ) is locally countable. Then (X, τ_ω) is a discrete topological space and so (X, τ_ω) is T_2. By Theorem 4.20, (X, τ_ω) is R_1.

Theorem 4.24. A locally indiscrete topological space (X, τ) is locally countable if and only if it has a base which consists of countable sets that form a partition of X.

Proof. *Necessity.* Suppose that (X, τ) is locally indiscrete and locally countable. Let $\mathcal{B}$ be a base for (X, τ) which forms a partition of X. Let $B \in \mathcal{B}$. Choose $x \in B$ and choose a countable set $U \in \tau$ such that $x \in U$. Since $B \cap U \in \tau - \{\emptyset\}$ and $B \cap U \subseteq B$, then $B \cap U = B$ and so $B \subseteq U$. Therefore, B is countable.

Sufficiency. Obvious.

Theorem 4.25. Let (X, τ) be a locally indiscrete topological space. Then (X, τ_ω) is R_1 if and only if (X, τ) is locally countable.

Proof. *Necessity.* Suppose that (X, τ) is locally indiscrete and (X, τ_ω) is R_1. Let $\mathcal{B}$ be a base for (X, τ) which forms a partition of X. Let $B \in \mathcal{B}$.

We are going to show that B is countable for every $B \in \mathcal{B}$. Suppose to the contrary that there is $B \in \mathcal{B}$ such that B is uncountable. Choose $x, y \in B$ with $x \neq y$. Since (X, τ_ω) is R_1, then by Theorem 4.20, (X, τ_ω) is T_2. So there are $U, V \in \tau_\omega$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Choose $H, S \in \tau$ and countable subsets $C, D \subseteq X$ such that $x \in H - C \subseteq U$ and $y \in S - D \subseteq V$. Since $x \in H \cap B \subseteq B$ and $y \in S \cap B \subseteq B$, then $B \subseteq H \cap S$. On the other hand, $(H - C) \cap (S - D) \subseteq U \cap V = \emptyset$. So, $B \subseteq C \cup D$, a contradiction. Theorefore, by Theorem 4.24, (X, τ) is locally countable.
Sufficiency. Suppose that (X, τ) is locally countable. Then (X, τ_ω) is a discrete topological space and so (X, τ_ω) is T_2. By Theorem 4.20, (X, τ_ω) is R_1.

Theorem 4.26. Every ω-$R_1\,(III)$ topological space is weakly ω-R_1.

Question 4.27. Is every weakly ω-R_1 topological space ω-$R_1\,(III)$?

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