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On Semi-open sets and Feebly open sets in generalized topological spaces

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Abstract:

In this paper, we introduce the notion of semi-open sets and feebly open sets in generalized topological spaces. Several properties of these notions are discussed. Also this paper considers (semi and feebly)-separation axioms for generalized topological spaces. We further investigate (semi-continuous, feebly-continuous, almost open)-functions in generalized topological spaces.

Keywords: Generalized topological spaces; Semi-open sets; Feeblyopen set; Semi-continuous mappings; Feebly-continuous mappings; Separation axioms..

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1. Introduction

In general topological spaces, the notion of semi-open sets was introduced by Levine [10] and Cameron [1]: A set A of a topological spaces (X, τ) is semi-open if there exists an open set U such that $U \subseteq A \subseteq cU$, where c is the closure operator. Since this concept has played role in several significant places in the study of topological spaces. In a topological space X, the union of open sets and the union of semi-open sets is the whole space X. Császár [3] adapted this notion of semi-open set to a generalized topological space: A pair (X, μ), $\mu \subseteq P(X)$, the power set of a set X, is called a generalized topological space (GTS) and μ is called generalized topology (GT) if μ is closed under arbitrary unions; the elements of μ are called μ-open sets; X may not be in μ; A set A is μ-semi-open if there exists a $U \in \mu$ such that $U \subseteq A \subseteq c_\mu U$, where c_μ is the closure operator in (X, μ). In this adaptation the union of μ-semi-open sets may not be equal to the union of μ-open sets and hence a contrasting behavior of the above notion with the classical definition of semi-open set is observed. To maintain the above equality, the notion of μ-semi-open set is modified. A set A in a GTS (X, μ) is called μ-semi-open if there exists a μ-open set U such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$, where M_μ is the union of all μ-open sets. The appropriateness of the definition introduced is shown by the fact that we are able to extend all the basic results on semi-open sets in [8, 9, 10, 14] to generalized topological spaces. Roughly the new definition does not permit the spread of μ-semi-open sets beyond the spread of μ-open sets. If $X \in \mu$, the two definitions of μ-semi-open set coincide.

Maheshwari and Tapi [11] introduced the notion of feebly open sets in Topological spaces which are closely related to semi-open sets. Greenwood and Rielly [7] studied feebly closed mappings. Following the same approach as in [17, 18], we further study μ-feebly open sets and separation axioms. The behaviour of the above notions in respect of several types of mappings is investigated.

The paper is organized as follows: Section 2 contains a summary of basic notions and results used in the paper. In Section 3, we introduced μ-semi-open set and μ-feebly open set. Several properties of these sets are discussed in this Section. Section 4, contains separation axioms: μ-semi-R_0, μ-semi-T_0, μ-semi-T_1, μ-semi-T_2, μ-feebly-R_0, μ-feebly-T_0, μ-feebly-T_1 and μ-feebly-T_2. In section 5, we discussed various type of functions like μ-semi-continuous, μ-feebly continuous functions, μ-closed functions.

2. Preliminaries

Let (X, μ) be GTS. Let $M_\mu = \cup\{U : U \in \mu\}$. In general X may not be a member of μ. If $X \in \mu$ then (X, μ) is called a strong GTS. A set $A \subseteq X$ is called μ-closed if its compliment in X is μ-open. The generalized closure of a set $A \subseteq X$, denoted by $c_\mu A$, is the intersection of all μ-closed sets containing A. The generalized interior of a set $A \subseteq X$, denoted by $i_\mu A$, is the union of all μ-open sets contained in A. The following properties are known and will be used without reference.

Theorem 2.1. *[2, 3, 4, 15, 16, 17] Let (X, μ) be a GTS and $A, B \subseteq X$. Then the following statements hold:*

1. $A \subseteq c_\mu A$ *and* $i_\mu A \subseteq A$.
2. $A \subseteq B$ *implies* $c_\mu A \subseteq c_\mu B$ *and* $i_\mu A \subseteq i_\mu B$.
3. $c_\mu c_\mu A = c_\mu A$ *and* $i_\mu i_\mu A = i_\mu A$.
4. $i_\mu A = X - c_\mu(X - A)$ *and* $c_\mu A = X - i_\mu(X - A)$.
5. $c_\mu A$ *is a* μ*-closed set.*
6. $c_\mu A = c_\mu(A \cap M_\mu)$ *and* $i_\mu A = i_\mu(A \cap M_\mu)$.
7. $x \in c_\mu A$ *if and only if* $x \in U \in \mu$ *implies* $U \cap A \neq \emptyset$.
8. *If* $U, V \in \mu$ *and* $U \cap V = \emptyset$ *then* $c_\mu U \cap V = \emptyset$ *and* $U \cap c_\mu V = \emptyset$.
9. $M_\mu - c_\mu A = X - c_\mu A$.
10. $i_\mu(c_\mu A - A) = \emptyset$.
11. $c_\mu i_\mu c_\mu i_\mu A = c_\mu i_\mu A$ *and* $i_\mu c_\mu i_\mu c_\mu A = i_\mu c_\mu A$.

The following lemma is immediate.

Lemma 2.2. *If μ and ν are GTs on a set X, then $\mu \subseteq \nu$ implies $c_\nu A \subseteq c_\mu A$, for all $A \subseteq X$.*

Definition 2.3. *Let (X, μ) be a GTS and Y be a subset of X. Let $\mu_Y = \{U \cap Y : U \in \mu\}$. Then (Y, μ_Y) is called a μ-subspace of (X, μ).*

Theorem 2.4. *[19] Let (X,μ) be GTS and (Y,μ_Y) be μ-subspace of (X,μ). Then $c_{\mu_Y}A = c_\mu A \cap Y$ for each subset A of Y.*

Definition 2.5. *A subset A of GTS (X,μ) is said to be μ-nowhere dense in X if $i_\mu c_\mu A = \emptyset$.*

The following Lemma from general topological spaces is also extended to GTS.

Lemma 2.6. *Let X be a topological spaces and $A \subseteq X$. If A is semi open and nowhere dense then $A = \emptyset$.*

Proof. If A is semi-open in X, then there exists an open set in X such that $U \subseteq A \subseteq cU$. Then $cU \subseteq cA \subseteq ccU = cU$. So $icU = icA = \emptyset$, where i is the interior operator of X, since A is nowhere dense. So $U \subseteq icU = \emptyset$. Thus $A = \emptyset$.

Definition 2.7. *Let (X,μ) be a GTS and $A \subseteq X$. A point $x \in X$ is said to be μ-accumulation point of A if $x \in U \in \mu$ implies $U \cap (A - \{x\}) \neq \emptyset$. The set of all accumulation points of A is called μ-derived set of A and denoted by A'.*

It may be remarked that for any set A, $X - M_\mu \subseteq A'$.

Lemma 2.8. *Let (X,μ) be a GTS and $A, B \subseteq X$ then*

(i) $c_\mu A = A \cup A'$.

(ii) $B' = X - M_\mu$ *and* $A \subseteq B$ *implies* $A' = X - M_\mu$.

Proof. (i) follows from Theorem 2.1(v). (ii) Let $x \in M_\mu$. Then there exists a $U \in \mu$ such that $x \in U$ and $U \cap (B - \{x\}) = \emptyset$. So $U \cap (A - \{x\}) = \emptyset$

Definition 2.9. *[2] Let (X,μ) and (Y,ν) be GTSs. A mapping $f : X \to Y$ is said to be (μ,ν)-continuous if $f^{-1}(G)$ is μ-open for each ν-open set G.*

It is remarked that if a mapping $f : (X,\mu) \to (Y,\nu)$ is such that $f(x) \in M_\nu$ for some $x \in X - M_\mu$, then f can not be (μ,ν)-continuous. Thus, for a (μ,ν)-continuous mapping f, it is necessary that $f(X - M_\mu) \subseteq Y - M_\nu$.

Definition 2.10. *[5] Let X be a set and $\mathcal{B} \subseteq P(X)$ then $\mathcal{B}$ generates a GT μ on X: A set $A \in \mu$ if for each $x \in A$, there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq A$. $\mathcal{B}$ is called a generalized basis for this GT μ.*

Definition 2.11. *[5] Let $\{(X_i, \mu_i)\}$ be a family of GTSs. Let $\mathcal{B}$ be the collection: $\mathcal{B} = \{\prod A_i : A_i \in \mu_i\}$, where with the exception of a finite number of indices $A_i = M_{\mu_i}$. Then the GT μ on $X = \prod X_i$ generated by the generalized basis $\mathcal{B}$ is called generalized product topology denoted by $\mu = \prod \mu_i$. The pair (X, μ) is called a generalized product GTS.*

In the notations of above definition, let $A_i \subseteq X_i$ and $A = \prod A_i$, then the following lemma holds.

Lemma 2.12. *[5] $c_\mu A = \prod c_{\mu_i} A_i$.*

Definition 2.13. *Two sets A and B in a GT space (X, μ) are called μ-separated if $A \cap c_\mu B = \emptyset$ and $c_\mu A \cap B = \emptyset$.*

Definition 2.14. *[15] A set $S \subseteq X$ is said to be μ-connected if $S \cap M_\mu = U \cup V$, U and V are μ-separated, implies $U = \emptyset$ or $V = \emptyset$.*

Lemma 2.15. *Let (X, μ) be a GTS. $A, B \subseteq X$, A is μ-open and $A \subseteq B$, then $c_\mu A \subseteq c_\mu i_\mu c_\mu B$.*

Lemma 2.16. *If A and B are subsets of GTS (X, μ) and $A \cap B = \emptyset$, then $i_\mu c_\mu i_\mu A \cap i_\mu c_\mu i_\mu B = \emptyset$.*

Proof. If $A \cap B = \emptyset$, then $i_\mu A \cap i_\mu B = \emptyset$ so that $i_\mu A \cap c_\mu i_\mu B = \emptyset$. Therefore, $i_\mu A \cap i_\mu c_\mu i_\mu B = \emptyset$ which implies that $c_\mu i_\mu A \cap i_\mu c_\mu i_\mu B = \emptyset$. Hence, $i_\mu c_\mu i_\mu A \cap i_\mu c_\mu i_\mu B = \emptyset$.

Lemma 2.17. *If U and V are subsets of a GTS (X, μ), $U \in \mu$ and $U \subseteq V$, then $c_\mu U \subseteq c_\mu i_\mu c_\mu V$.*

Definition 2.18. *A subset A of a GTS (X, μ) is called*

1. *[2] μ-regular open (or μr-open) if $i_\mu c_\mu A = A$.*
2. *[6] μ-preopen (or p_μ-open) if $A \subseteq i_\mu c_\mu A$.*
3. *[6] μ-α-open (or α_μ-open) if $A \subseteq i_\mu c_\mu i_\mu A$.*

4. *[15] μ-β-open if $A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$.*
5. *μ-regular semi open (or rs_μ-open) if there exists a μ-regular open set U such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$.*

The collections of all μ-( ) sets in (i) to (v) of the above definitions are denoted by μr, p_μ, α_μ, β_μ, rs_μ respectively. The complements of the sets in the above definitions are named similarly by replacing the word "open" by "closed", for example μ-pre-closed (or p_μ-closed) for the complement of a p_μ-open set and vice-versa. It follows using Theorem 2.1, a subset A of GTS (X, μ) is a regular μ-closed (or μr-closed) if and only if $c_\mu i_\mu A = A$; A is p_μ-closed if and only if $c_\mu i_\mu A \subseteq A$; A is α_μ-closed if and only if $c_\mu i_\mu c_\mu A \subseteq A$; A is β_μ-closed if $i_\mu c_\mu i_\mu A \subseteq A$ and $X - M_\mu \subseteq A$. For any set A, $c_\mu i_\mu c_\mu A$ is α_μ-closed. Also if $A \in rs_\mu$ then $A \in s_\mu$ but not conversely.

Lemma 2.19. *If A is μr-closed subset of GTS (X, μ), then $i_\mu A$ is μr-open.*

The collection of μr-open sets in a GTS (X, μ) generates a GT on X, called semi-regularization GT on X and denoted by τ_s or τ_{s_μ}.

Lemma 2.20. *[17] If $\{x\}$ is μ-nowhere dense in a GTS (X, μ), then $\{x\} \cup (X - M_\mu)$ is α_μ-closed.*

3. μ-semi-open set and μ-feebly-open set

Definition 3.1. *A subset A of a GTS (X, μ) is called μ-semi-open if there exists a μ-open set U such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$.*

Note that the empty set is μ-semi-open in any GTS (X, μ). If (X, μ) is strong, then the above definition is reduced to the one in [2]. A subset in a GTS (X, μ) is called μ-semi-closed if its compliment in X is μ-semi-open.

Theorem 3.2. *A subset A of a GTS (X, μ) is μ-semi-open if and only if $A \subseteq c_\mu i_\mu A \cap M_\mu$.*

Proof. Let A be μ-semi-open. Then there exists a $U \in \mu$ such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$. Since $U \subseteq i_\mu A$, $c_\mu U \cap M_\mu \subseteq c_\mu i_\mu A \cap M_\mu$. Therefore, $A \subseteq c_\mu i_\mu A \cap M_\mu$. Conversely, $A \subseteq c_\mu i_\mu A \cap M_\mu$ implies $i_\mu A \subseteq A \subseteq c_\mu i_\mu A \cap M_\mu$.

Theorem 3.3. *The collection of μ-semi-open subsets in a GTS (X, μ), form a GT on X.*

Proof. Let $\{A_\alpha\}$ be a collection of μ-semi-open subsets in a GTS (X, μ). For each α, there exists $O_\alpha \in \mu$ such that $O_\alpha \subseteq A_\alpha \subseteq c_\mu O_\alpha \cap M_\mu$. Then $\cup_\alpha O_\alpha \subseteq \cup_\alpha A_\alpha \subseteq \cup_\alpha c_\mu O_\alpha \cap M_\mu \subseteq c_\mu(\cup_\alpha O_\alpha) \cap M_\mu$.

Let us denote the GT of μ-semi-open sets in a GTS (X, μ) by s_μ.

Theorem 3.4. *Every μ-open subset of a GTS (X, μ) is μ-semi-open.*

Proof. If $U \in \mu$, then $U \subseteq U \subseteq c_\mu U \cap M_\mu$.

Corollary 3.5. *For a GTS (X, μ), $\mu \subseteq s_\mu$.*

Theorem 3.6. *Let (X, μ) be a GTS and $A \in s_\mu$. If $A \subseteq B \subseteq c_\mu A \cap M_\mu$, then B is μ-semi-open.*

Proof. There is a μ-open set U such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$. Then $U \subseteq B$ and $c_\mu A \subseteq c_\mu(c_\mu U \cap M_\mu) \subseteq c_\mu(c_\mu U) = c_\mu U$. Therefore, $c_\mu A \cap M_\mu \subseteq c_\mu U \cap M_\mu$. Thus, $U \subseteq B \subseteq c_\mu U \cap M_\mu$.

Theorem 3.7. *Let $B = \{B_\alpha\}$ be a collection of subsets in GTS (X, μ) such that*

(i) $\mu \subseteq B$,

(ii) *If $B \in B$ and $B \subseteq D \subseteq c_\mu B \cap M_\mu$ then $D \in B$. Then $s_\mu \subseteq B$ and thus, s_μ is the smallest class of subsets in X satisfying (i) and (ii).*

Proof. Let $A \in s_\mu$. Then there exists a $U \in \mu$ such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$. $U \in B$ by (i) so that $A \in B$ by (ii).

Theorem 3.8. *Let $A \subseteq Y \subseteq X$, where (X, μ) is a GTS and (Y, μ_Y) is the μ-subspace of (X, μ). If A is μ-semi-open in X, then A is μ_Y-semi-open in Y.*

Proof. For some $U \in \mu$, $U \subseteq A \subseteq c_\mu U \cap M_\mu$. By Theorem 2.4, now $A \subseteq Y$, $U = U \cap Y \subseteq A \cap Y = A \subseteq c_\mu U \cap M_\mu \cap Y = c_{\mu_Y} U \cap M_{\mu_Y}$.

Lemma 3.9. *If U is μ-open subset of a GTS (X, μ), then $c_\mu U - U$ is nowhere μ-dense in X.*

Proof. $X - c_\mu(c_\mu U - U) = X - c_\mu(c_\mu U \cap U^c) = i_\mu[(c_\mu U \cap U^c)^c] = i_\mu[U \cup (X - c_\mu U)] = U \cup (X - c_\mu U)$ since U is μ-open. So $c_\mu[X - c_\mu(c_\mu U - U)] = c_\mu[U \cup (X - c_\mu U)] = X$, that is, $X - c_\mu[X - c_\mu(c_\mu U - U)] = \emptyset$. Therefore, $i_\mu c_\mu(c_\mu U - U) = \emptyset$ and thus, $c_\mu U - U$ is nowhere μ-dense in X.

Theorem 3.10. *Let (X, μ) be a GTS and $A \in s_\mu$. Then $A = U \cup B$, where (i) $U \in \mu$, (ii) $U \cap B = \emptyset$ and (iii) B is nowhere μ-dense in X.*

Proof. Let $U \subseteq A \subseteq c_\mu U \cap M_\mu$ for some $U \in \mu$. But $A = U \cup (A - U)$. Let $B = A - U$. Then $B \subseteq c_\mu U - U$ and B is thus nowhere μ-dense by Lemma 3.9.

Theorem 3.11. *Let (X, μ) be a GTS and $A = O \cup B$ be a subset of M_μ such that (i) $\emptyset \neq O \in \mu$ (ii). A is μ-connected, and (iii). $B' = X - M_\mu$. Then $A \in s_\mu$.*

Proof. It is sufficient to show that $B \subseteq c_\mu O \cap M_\mu$. Deny. Then $B = B_1 \cup B_2$, where $B_1 \subseteq c_\mu O \cap M_\mu$ and $\emptyset \neq B_2 \subseteq M_\mu - c_\mu O$. Now $A = O \cup B_1 \cup B_2$ and $O \cup B_1 \neq \emptyset$ by (i). Then $O \cup B_1 \subseteq c_\mu O \cap M_\mu$ and $B_2 \cap c_\mu O = \emptyset$. Thus, $(O \cup B_1) \cap c_\mu B_2 = (O \cup B_1) \cap (B_2 \cup B_2{}') = (O \cup B_1) \cap (B_2 \cup (X - M_\mu)) = (O \cup B_1) \cap B_2 = \emptyset$. Also $c_\mu(O \cup B_1) \subseteq c_\mu(c_\mu \cap M_\mu) \subseteq c_\mu c_\mu O = c_\mu O$. Therefore, $B_2 \cap c_\mu(O \cup B_1) = \emptyset$. Thus, $O \cup B_1$ and B_2 are μ-separated sets. Therefore, A is not μ-connected, a contradiction to (ii).

Definition 3.12. *Let $B = \{B_\alpha\}$ be a collection of subsets in GTS (X, μ). Then we define Int $B = \{i_\mu B_\alpha : B_\alpha \in B\}$*

Lemma 3.13. *For a GTS (X, μ), $\mu =$Int s_μ.*

Proof. $\mu \subseteq$ Int s_μ since for each $U \in \mu$, $i_\mu U = U$, and $U \in s_\mu$ by 3.9. Conversely, let $U \in$ Int s_μ. Then $U = i_\mu A$ for some $A \in s_\mu$. Then $U = i_\mu A \in \mu$.

Theorem 3.14. *Let (X_1, μ_1) and (X_2, μ_2) be GTS, $X = X_1 \times X_2$ and $\mu = \mu_1 \times \mu_2$ be the generalized product topology on X. If A_1 is μ_1-semi-open and A_2 is μ_2-semi-open. Then $A_1 \times A_2$ is μ-semi-open.*

Proof. Let $A_i = O_i \cup B_i$, $O_i \in \mu_i$ and $B_i \subseteq C_{\mu_i} O_i \cap M_{\mu_i}$ for $i = 1, 2$. Then $A_1 \times A_2 = (O_1 \times O_2) \cup (B_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times B_2)$ then $O_1 \times O_2$ is μ-open and $(B_1 \times O_2) \cup (O_1 \times B_2) \cup (B_1 \times B_2) \subseteq (c_{\mu_1} O_1 \cap M_{\mu_1}) \times (c_{\mu_2} O_2 \cap M_{\mu_2}) = (c_{\mu_1} O_1 \times c_{\mu_2} O_2) \cap (M_{\mu_1} \times M_{\mu_2}) = c_{\mu_1 \times \mu_2}(O_1 \times O_2) \cap M_{\mu_1 \times \mu_2}$, by Lemma 2.12. It follows that $A_1 \times A_2$ is μ-semi-open in X.

Lemma 3.15. *If A is a μ-semi-open and μ-nowhere dense subset of a GTS (X, μ), then $A = \emptyset$.*

Proof. Let A be μ-semi open. Then there exists a $U \in \mu$ such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$. Thus, $c_\mu U \subseteq c_\mu A \subseteq c_\mu c_\mu U = c_\mu U$. Therefore, $c_\mu U = c_\mu A$. So $U \subseteq i_\mu c_\mu U = i_\mu c_\mu A = \emptyset$. So that $U = \emptyset$ and $c_\mu U = c_\mu \emptyset = X - M_\mu$. Therefore, $c_\mu U \cap M_\mu = \emptyset$. Thus, $A = \emptyset$.

Lemma 3.16. *Let A be a subset of a GTS (X, μ), then $i_{s_\mu} c_\mu A = c_\mu i_\mu c_\mu A \cap M_\mu$.*

Proof. Since $c_\mu i_\mu c_\mu A \cap M_\mu = c_\mu i_\mu c_\mu i_\mu c_\mu A \cap M_\mu = c_\mu i_\mu (c_\mu i_\mu c_\mu A \cap M_\mu) \cap M_\mu$, $c_\mu i_\mu c_\mu A \cap M_\mu$ is a μ-semi-open set. Since $c_\mu i_\mu c_\mu A \subseteq c_\mu A$, then $c_\mu i_\mu c_\mu A \cap M_\mu \subseteq c_\mu A$. So $c_\mu i_\mu c_\mu A \cap M_\mu \subseteq i_{s_\mu} c_\mu A$. On the other hand, if U is any μ-semi-open set contained in $c_\mu A \cap M_\mu$, then $U \subseteq c_\mu i_\mu (c_\mu A \cap M_\mu) \cap M_\mu = c_\mu i_\mu c_\mu A \cap M_\mu$. Therefore, $i_{s_\mu} c_\mu A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$.

Lemma 3.17. *For a subset A of a GTS (X, μ), $c_{s_\mu} A = A \cup i_\mu c_\mu A \cup (X - M_\mu)$.*

Proof. Since $c_{s_\mu} A$ is s_μ-closed, $i_\mu c_\mu (c_{s_\mu} A) \subseteq c_{s_\mu} A$. Therefore, $i_\mu c_\mu A \subseteq c_{s_\mu} A$. So $A \cup i_\mu c_\mu A \subseteq c_{s_\mu} A$. Also $X - M_\mu \subseteq c_{s_\mu} A$. Hence, $A \cup i_\mu c_\mu A \cup (X - M_\mu) \subseteq c_{s_\mu} A$. On the other hand, since $i_\mu c_\mu (A \cup i_\mu c_\mu A \cup (X - M_\mu)) \subseteq i_\mu c_\mu (A \cup c_\mu A \cup (X - M_\mu)) = i_\mu c_\mu A$. Therefore, $i_\mu c_\mu (A \cup i_\mu c_\mu A \cup (X - M_\mu)) \subseteq A \cup i_\mu c_\mu A \cup (X - M_\mu)$. Hence, $A \cup i_\mu c_\mu A \cup (X - M_\mu)$ is s_μ-closed, and therefore, $c_{s_\mu} A \subseteq A \cup i_\mu c_\mu A \cup (X - M_\mu)$.

Lemma 3.18. *[17] Let x be a point in a GTS (X, μ). Then $\{x\}$ is μ-nowhere dense or p_μ-open.*

Lemma 3.19. *Let x be a point in a GTS (X, μ). Then $\{x\}$ is μ-nowhere dense or $\{x\} \subseteq i_\mu c_\mu \{x\} = c_{s_\mu} \{x\} \cap M_\mu$.*

Proof. Suppose that $\{x\}$ is not μ-nowhere dense. Then $x \in i_\mu c_\mu\{x\}$. By Lemma 3.17, $c_{s_\mu}\{x\} = \{x\} \cup i_\mu c_\mu\{x\} \cup (X - M_\mu) = i_\mu c_\mu\{x\} \cup (X - M_\mu)$. Then $\{x\} \subseteq i_\mu c_\mu\{x\} = c_{s_\mu}\{x\} \cap M_\mu$.

Lemma 3.20. *Let A be a subset of GTS (X, μ). Then $i_{s_\mu}A = A \cap c_\mu i_\mu A \cap M_\mu$*

Proof. Since $c_{s_\mu}A = A \cup i_\mu c_\mu A \cup (X - M_\mu)$. Then $(X - c_{s_\mu}A) = X - (A \cup i_\mu c_\mu A \cup (X - M_\mu))$. So $i_{s_\mu}(X - A) = (X - A) \cap (X - i_\mu c_\mu A) \cap (X - (X - M_\mu))$, Therefore, $i_{s_\mu}(X - A) = (X - A) \cap c_\mu i_\mu(X - A) \cap M_\mu$. Let $X - A = B$, then $i_{s_\mu}B = B \cap c_\mu i_\mu B \cap M_\mu$.

Lemma 3.21. *Let A be a subset of GTS (X, μ). Then $c_{s_\mu}i_{s_\mu}A = i_{s_\mu}A \cup i_\mu c_\mu i_\mu A \cup (X - M_\mu)$.*

Proof. By Lemma 3.17, $c_{s_\mu}i_{s_\mu}A = i_{s_\mu}A \cup i_\mu c_\mu i_{s_\mu}A \cup (X - M_\mu)$. So $c_{s_\mu}i_{s_\mu}A = i_{s_\mu}A \cup i_\mu c_\mu(A \cap c_\mu i_\mu A \cap M_\mu) \cup (X - M_\mu) \subseteq i_{s_\mu}A \cup i_\mu$ $(c_\mu A \cap c_\mu i_\mu A \cap c_\mu M_\mu) \cup (X - M_\mu) \subseteq i_{s_\mu}A \cup (i_\mu c_\mu A \cap i_\mu c_\mu i_\mu A \cap i_\mu c_\mu M_\mu) \cup (X - M_\mu) \subseteq i_{s_\mu}A \cup i_\mu c_\mu i_\mu A \cup (X - M_\mu)$. To establish the opposite inclusion, we observe that $c_{s_\mu}i_{s_\mu}A = i_{s_\mu}A \cup i_\mu c_\mu i_{s_\mu}A \cup (X - M_\mu) \supseteq i_{s_\mu}A \cup i_\mu c_\mu i_\mu A \cup (X - M_\mu)$.

Lemma 3.22. *Let (X, μ) be a GTS. If $A \in s_\mu$, then $c_\mu i_\mu A = c_\mu A$.*

Proof. If $A \in s_\mu$, then $A \subseteq c_\mu i_\mu A \cap M_\mu \subseteq c_\mu i_\mu A$. Therefore, $i_\mu A \subseteq A \subseteq c_\mu i_\mu A$. Hence, $c_\mu i_\mu A \subseteq c_\mu A \subseteq c_\mu i_\mu A$.

Definition 3.23. *[16] A GTS (X, μ) is called extremally μ-disconnected if $c_\mu U \cap M_\mu \in \mu$ for every $U \in \mu$.*

Lemma 3.24. *If a GTS (X, μ) is extremally μ-disconnected, then $c_\mu A = c_{s_\mu}A$ for every $A \in s_\mu$.*

Proof. $c_{s_\mu}A \subseteq c_\mu A$ for any subset A of X. On the other hand, let $\emptyset \neq A \in s_\mu$ and $x \notin c_{s_\mu}A$, then there exists $B \in s_\mu$ such that $x \in B$ and $B \cap A = \emptyset$. Thus, $i_\mu B \cap i_\mu A = \emptyset$. Since X is extremally μ-disconnected, $(c_\mu i_\mu B \cap M_\mu) \cap c_\mu i_\mu A = \emptyset$. Since B is s_μ-open, $B \subseteq c_\mu i_\mu B \cap M_\mu$ so that $B \cap c_\mu i_\mu A = \emptyset$. Therefore, $x \notin c_\mu i_\mu A = c_\mu A$, by Lemma 3.22.

Definition 3.25. *Let (X, μ) be a GTS. A subset $A \subseteq X$ is said to be μ-feebly-open if there is a μ-open set U such that $U \subseteq A \subseteq c_{s_\mu}U \cap M_\mu$.*

The set of all μ-feebly-open sets is denoted by f_μ. The empty set is trivially μ-feebly-open. A set is μ-feebly-closed if its complement is μ-feebly-open.

Theorem 3.26. *A set A of (X, μ) is μ-feebly-open if and only if $A \subseteq c_{s_\mu} i_\mu A \cap M_\mu$.*

Proof. Let A be μ-feebly-open. Then there exists a $U \in \mu$ such that $U \subseteq A \subseteq c_{s_\mu} U \cap M_\mu$. Since $U \subseteq i_\mu A$, $c_{s_\mu} U \cap M_\mu \subseteq c_{s_\mu} i_\mu A \cap M_\mu$. Therefore, $A \subseteq c_{s_\mu} i_\mu A \cap M_\mu$. Conversely, $A \subseteq c_{s_\mu} i_\mu A \cap M_\mu$ implies $i_\mu A \subseteq A \subseteq c_{s_\mu} i_\mu A \cap M_\mu$.

Theorem 3.27. *The collection f_μ is a GT on X.*

Proof. Let $\{A_\alpha\}$ be a collection of μ-feebly-open sets in a GTS (X, μ). For each α there exists $O_\alpha \in \mu$ such that $O_\alpha \subseteq A_\alpha \subseteq c_{s_\mu} O_\alpha \cap M_\mu$. Then $\cup_\alpha O_\alpha \subseteq \cup_\alpha A_\alpha \subseteq \cup_\alpha c_{s_\mu} O_\alpha \cap M_\mu \subseteq c_{s_\mu}(\cup_\alpha O_\alpha) \cap M_\mu$.

Theorem 3.28. *For a GTS (X, μ), every μ-open set is μ-feebly-open.*

Proof. $U \in \mu$ implies $U \subseteq U \subseteq c_{s_\mu} U \cap M_\mu$.

Theorem 3.29. *For a GTS (X, μ), every f_μ-open set is s_μ-open.*

Proof. Let A is μ-feebly open. Then there exist μ-open set U such that $U \subseteq A \subseteq c_{s_\mu} U \cap M_\mu$. Since $c_{s_\mu} U \subseteq c_\mu U$, then $U \subseteq A \subseteq c_\mu U \cap M_\mu$.

Corollary 3.30. *For a GTS (X, μ), $\mu \subseteq f_\mu \subseteq s_\mu$.*

Lemma 3.31. *A subset A of (X, μ) containing $(X - M_\mu)$ is μ-feebly closed if and only if $i_{s_\mu} c_\mu A \subseteq A$.*

Proof. Let A be μ-feebly closed. Then there exist a $U \in \mu$ such that $U \subseteq (X - A) \subseteq c_{s_\mu} U \cap M_\mu$. Then $X - c_{s_\mu} i_\mu U \subseteq A$, that is, $i_{s_\mu} c_\mu (X - U) \subseteq A$. Then $i_{s_\mu} c_\mu A \subseteq A$. Conversely, let $i_{s_\mu} c_\mu A \subseteq A$. Then $X - A \subseteq c_{s_\mu}(X - c_\mu A)$. Thus, A is μ-feebly-closed.

Theorem 3.32. *Let (X, μ) be a GTS. Then $A \subseteq X$ is μ-feebly-open if and only if A is α_μ-open.*

Proof. If A is μ-feebly-open, then there is a μ-open set U such that $U \subseteq A \subseteq c_{s_\mu} U \cap M_\mu$. By Lemma 3.17, $c_{s_\mu} U = U \cup i_\mu c_\mu U \cup (X - M_\mu) = i_\mu c_\mu U \cup (X - M_\mu)$. So $U \subseteq A \subseteq (i_\mu c_\mu U \cup (X - M_\mu)) \cap M_\mu = i_\mu c_\mu U$. Consequently, $i_\mu c_\mu i_\mu A = i_\mu c_\mu U$. Thus, we have that $A \subseteq i_\mu c_\mu i_\mu A$. Conversely, let $A \subseteq X$ be a α_μ-open. Then $i_\mu A \subseteq A \subseteq i_\mu c_\mu i_\mu A$. Now let $U = i_\mu A$. Then $U \subseteq A \subseteq i_\mu c_\mu U$. Therefore, $U \subseteq A \subseteq (U \cup i_\mu c_\mu U \cup (X - M_\mu)) \cap M_\mu$. So, by Lemma 3.17, $U \subseteq A \subseteq c_{s_\mu} U \cap M_\mu$.

4. (μ-semi and μ-feebly)-separation axioms

Definition 4.1. *A GTS (X, μ) is called*

1. *μ-R_0 [18] if $x \in U \in \mu$ implies $c_\mu\{x\} \cap M_\mu \subseteq U$.*
2. *μ-T_0 [17] if for any pair of distinct points $x, y \in M_\mu$ there exists μ-open set containing precisely one of x and y.*
3. *μ-T_1 [13] if $x, y \in M_\mu$, $x \neq y$ implies the existence of μ-open sets U_1 and U_2 such that $x \in U_1$ and $y \notin U_1$ and $y \in U_2$ and $x \notin U_2$.*
4. *μ-T_2 [13] if $x, y \in M_\mu$, $x \neq y$ implies the existence of disjoint μ-open sets U_1 and U_2 containing x and y, respectively.*
5. *μ-regular (G-regular) [12, 13] if for each μ-closed set F and a point $x \notin F$ there are disjoint μ-open sets U and V such that $x \in U$ and $F \cap M_\mu \subseteq V$.*
6. *μ-normal (G-normal) [12, 13] if for any μ-closed sets A and B such that $A \cap B \cap M_\mu = \emptyset$ there exist disjoint μ-open sets U and V such that $A \cap M_\mu \subseteq U$ and $B \cap M_\mu \subseteq V$.*

Definition 4.2. *A GTS (X, μ) is said to be*

1. *μ-semi-R_0 if $x \in U \in s_\mu$ implies $c_{s_\mu}\{x\} \cap M_\mu \subseteq U$.*
2. *μ-feebly-R_0 if $x \in U \in f_\mu$ implies $c_{f_\mu}\{x\} \cap M_\mu \subseteq U$.*

Theorem 4.3. *A GTS (X, μ) is μ-semi-R_0 if and only if $x \in U \in s_\mu$, implies $i_\mu c_\mu\{x\} \subseteq U$.*

Proof. Let (X,μ) be μ-semi-R_0 and if $x \in U \in s_\mu$. Then $c_{s_\mu}\{x\} \cap M_\mu \subseteq U$. By Lemma 3.17, $(\{x\} \cup i_\mu c_\mu\{x\} \cup (X - M_\mu)) \cap M_\mu \subseteq U$. Therefore, $\{x\} \cup i_\mu c_\mu\{x\} \subseteq U$. Conversely, if for if $x \in U \in s_\mu$, implies $i_\mu c_\mu\{x\} \subseteq U$. Then $\{x\} \cup i_\mu c_\mu\{x\} \subseteq U$. Therefore, $(\{x\} \cup i_\mu c_\mu\{x\} \cup (X - M_\mu)) \cap M_\mu \subseteq U$. Hence, by Lemma 3.17, $c_{s_\mu}\{x\} \cap M_\mu \subseteq U$.

Theorem 4.4. *If a GTS (X,μ) is μ-R_0, then it is μ-semi-R_0.*

Proof. Let $x \in V \in s_\mu$. There is a μ-open set U such that $U \subseteq V \subseteq c_\mu U \cap M_\mu$. Suppose that $x \in U$. Since (X,μ) is μ-R_0, then $c_\mu\{x\} \cap M_\mu \subseteq U$. Therefore, $c_{s_\mu}\{x\} \cap M_\mu \subseteq U \subseteq V$. Now suppose that $x \in V - U \subseteq (c_\mu U \cap M_\mu) - U$. Then $i_\mu c_\mu\{x\} = \emptyset$ and $c_{s_\mu}\{x\} \cap M_\mu \subseteq V$.

Theorem 4.5. *A GTS (X,μ) is μ-feebly-R_0 if and only if (X,α_μ) is α_μ-R_0.*

Proof. The proof follows from Theorem 3.32.

Theorem 4.6. *If (X,μ) is μ-feebly-R_0, then (X,α_μ) is α_μ-semi-R_0.*

Proof. The proof follows from Theorem 4.4 and Theorem 4.5. The converse of the above Theorem is not true.

Example 4.7. *Let $X = \{a,b,c\}$ and $\mu = \{\{\emptyset\},\{a\},\{b\},\{a,b\},X\}$ be GT on X. Then, $s_\mu = \{\{\emptyset\},\{a\},\{b\},\{a,b\},\{b,c\},\{a,c\},X\}$, $f_\mu = \alpha_\mu = \mu$, and $s_{\alpha_\mu} = s_\mu$. Cleary, (X,α_μ) is α_μ-semi-R_0 but (X,μ) is not μ-feebly-R_0.*

Theorem 4.8. *If (X,μ) is μ-R_0, then it is μ-feebly-R_0.*

Proof. Let $x \in U \in \alpha_\mu$. By Lemma 3.19, $\{x\}$ is μ-nowhere dense or $\{x\} \subseteq i_\mu c_\mu\{x\} = c_{s_\mu}\{x\} \cap M_\mu$. If $\{x\}$ is μ-nowhere dense, $c_{\alpha_\mu}\{x\} \cap M_\mu = \{x\} \subseteq U$. If $\{x\} \subseteq i_\mu c_\mu\{x\} = c_{s_\mu}\{x\} \cap M_\mu$, then $c_\mu\{x\} \cap M_\mu \subseteq i_\mu c_\mu\{x\}$ since (X,μ) is μ-R_0. $c_{\alpha_\mu}\{x\} \cap M_\mu \subseteq c_{s_\mu}\{x\} \cap M_\mu$. By Theorem 4.4, (X,μ) is μ-semi-R_0 and $U \in s_\mu$. Then $c_{s_\mu}\{x\} \cap M_\mu \subseteq U$. Hence, $c_{\alpha_\mu}\{x\} \cap M_\mu \subseteq U$. Thus (X,α_μ) is α_μ-R_0, So that by, Theorem 3.16, (X,μ) is μ-feebly-R_0.

Definition 4.9. *A GTS (X,μ) is μ-semi-T_0 (μ-feebly-T_0) if (X,s_μ) (resp. (X,f_μ)) is s_μ-T_0 (resp. f_μ-T_0)*

Obviously, if a GTS (X,μ) is μ-T_0 then it is μ-semi-T_0. The converse is not true.

Example 4.10. *Let $X = \{a, b, c, d\}$ and $\mu = \{\{\emptyset\}, \{b\}, \{d\}, \{b, d\}, \{a, b, c\}, X\}$ be GT on X. Then, $s_\mu = \{\{\emptyset\}, \{b\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, X\}$. Clearly, (X, μ) is μ-semi-T_0 but not μ-T_0.*

Theorem 4.11. *A GTS (X, μ) is μ-feebly-T_0 if and only if (X, α_μ) is α_μ-T_0.*

Proof. The proof follows from Theorem 3.32.

Theorem 4.12. *If a GTS (X, μ) is μ-semi-T_0, then (X, α_μ) is $\alpha_\mu - T_0$.*

Proof. Let (X, μ) be μ-semi-T_0 and $x, y \in M_\mu$ and $x \neq y$. Let $U \in s_\mu$ be such that $y \in U$ and $x \notin U$. Then $y \notin c_{s_\mu}\{x\}$. By Lemma 3.19, $\{x\}$ is μ-nowhere dense or $\{x\} \subseteq i_\mu c_\mu\{x\} = c_{s_\mu}\{x\} \cap M_\mu$. If $\{x\}$ is μ-nowhere dense, then by Lemma 2.20, $\{x\} \cup (X - M_\mu)$ is α_μ-closed. So there is α_μ-open set containing y but not x. If $\{x\} \subseteq i_\mu c_\mu\{x\} = c_{s_\mu}\{x\} \cap M_\mu$, then $y \notin i_\mu c_\mu\{x\}$. So there is μ-regular open set containing x but not y. Thus, (X, α_μ) is $\alpha_\mu - T_0$.

Definition 4.13. *A GTS (X, μ) is said to be*

1. *μ-T_D if $c_\mu\{x\} - \{x\}$ is μ-closed for each $x \in M_\mu$.*
2. *μ-semi-T_D if $c_\mu\{x\} - \{x\}$ is s_μ-closed for each $x \in M_\mu$.*
3. *s_μ-T_D if $c_{s_\mu}\{x\} - \{x\}$ is μ-closed for each $x \in M_\mu$.*
4. *s_μ-semi-T_D if (X, s_μ) is s_μ-T_D.*

It is obvious that if a GTS is μ-T_D, then it is μ-semi-T_D, and if (X, μ) is s_μ-T_D, then it is s_μ-semi-T_D.

Theorem 4.14. *If a GTS (X, μ) is μ-semi-T_D then it is s_μ-semi-T_D.*

Proof. For $x \in M_\mu$, $c_{s_\mu}\{x\} - \{x\} = (c_\mu\{x\} - \{x\}) \cap c_{s_\mu}\{x\}$.

Theorem 4.15. *In a GTS (X, μ), if for each $x \in M_\mu$, $\{x\}$ is either μ-nowhere dense or μ-open, then (X, α_μ) is α_μ-T_D.*

Proof. For $x \in M_\mu$, by the assumption, $\{x\}$ is μ-nowhere dense or μ-open. If $\{x\}$ is μ-nowhere dense, then $c_{\alpha_\mu}\{x\} = \{x\} \cup (X - M_\mu)$. So that $c_{\alpha_\mu}\{x\} - \{x\} = X - M_\mu$ is a α_μ-closed set. If $\{x\}$ is μ-open, then $c_{\alpha_\mu}\{x\} - \{x\} = c_{\alpha_\mu}\{x\} \cap (X - \{x\})$ is α_μ-closed. Thus, (X, α_μ) is α_μ is T_D.

Theorem 4.16. *If (X, μ) is μ-semi-T_D, then it is s_μ-semi-T_D.*

Proof. Let $x \in M_\mu$, then $c_{s_\mu}\{x\} - \{x\} = (c_\mu\{x\} - \{x\}) \cap s_\mu\{x\}$. Since $c_\mu\{x\} - \{x\}$ is s_μ-closed, $c_{s_\mu}\{x\} - \{x\}$ is s_μ-closed.

Definition 4.17. *A GTS (X, μ) is μ-semi-T_1 (μ-feebly-T_1) if for each pair of distinct points $x, y \in M_\mu$, there is a $U \in s_\mu$ (resp. $U \in f_\mu$) set containing x but not y.*

Obviously, a GTS (X, μ) is μ-semi-T_1, then it is μ-semi-T_0 and for each $x \in M_\mu$, $\{x\} \cup (X - M_\mu)$ is s_μ-closed if and only if (X, μ) is μ-semi-T_1.

Theorem 4.18. *A GTS (X, μ) is μ-semi-T_1 if and only if each singleton $\{x\} \subseteq M_\mu$ is μ-nowhere dense or μ-regular open.*

Proof. Let (X, μ) be μ-semi-T_1 and $x \in M_\mu$. Then $\{x\} \cup (X - M_\mu)$ is s_μ-closed so that $c_{s_\mu}\{x\} \cap M_\mu = \{x\}$. By Lemma 3.19, either $\{x\}$ is μ-nowhere dense, or $i_\mu c_\mu\{x\} = \{x\}$ so that $\{x\}$ is μ-regular-open.

Conversely, let $x \in M_\mu$. If $\{x\}$ is μ-nowhere dense, then $\{x\} \cup (X - M_\mu)$ is s_μ-closed and if $\{x\}$ is μr-open, then $i_\mu c_\mu(\{x\} \cup (X - M_\mu)) = i_\mu c_\mu\{x\} = \{x\} \subseteq \{x\} \cup (X - M_\mu)$ and so $\{x\} \cup (X - M_\mu)$ is s_μ-closed.

Definition 4.19. *Let (X, μ) be a GTS. A set $\{x\} \subseteq M_\mu$ is μ-clopen if $\{x\}$ is μ-open and $\{x\} \cup (X - M_\mu)$ is μ-closed.*

Theorem 4.20. *A GTS (X, μ) is μ-feebly-T_1 if and only if (X, α_μ) is α_μ-T_1.*

Proof. The proof follows from Theorem 3.32.

Theorem 4.21. *A GTS (X, μ) is μ-feebly-T_1 if and only if for each $x \in M_\mu$, $\{x\}$ is μ-nowhere dense or $\{x\}$ is μ-clopen.*

Proof. Let (X, α_μ) be α_μ-T_1. Then $c_{\alpha_\mu}\{x\} = \{x\} \cup (X - M_\mu)$, so that $c_\mu i_\mu c_\mu(\{x\} \cup (X - M_\mu)) \subseteq \{x\} \cup (X - M_\mu)$. By Lemma 3.19, $\{x\}$ is μ-nowhere dense or $\{x\} \subseteq i_\mu c_\mu\{x\} \subseteq c_{s_\mu}\{x\} \cap M_\mu$. If $\{x\} \subseteq i_\mu c_\mu\{x\}$, then $\{x\} \subseteq i_\mu c_\mu\{x\} \subseteq c_\mu i_\mu c_\mu(\{x\} \cup (X - M_\mu)) \cap M_\mu = \{x\}$. So that $\{x\}$ is μ-open and $\{x\} \cup (X - M_\mu)$ is μ-closed. Thus, $\{x\}$ is μ-clopen. Conversely, since for each $x \in M_\mu$, μ-nowhere dense set $\{x\}$ and μ-clopen set $\{x\}$, $\{x\} \cup (X - M_\mu)$ is α_μ, (X, α_μ) is α_μ-T_1.

Corollary 4.22. *If a GTS (X, μ) is μ-feebly-T_1 then it is μ-semi-T_1.*

The converse of the above corollary is not true (see [9]).

Definition 4.23. *A subset A of a GTS (X, μ) is said to be μ-regular semi-open if there is a μr-open set U of X such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$. The set of all μ-regular semi-open sets is denoted by rs_μ.*

Definition 4.24. *A GTS (X, μ) is called*

1. *μ-semi-T_2 if (X, s_μ) is s_μ-T_2.*
2. *μ-feebly-T_2 if (X, f_μ) is f_μ-T_2.*

Theorem 4.25. *Let (X, μ) be GTS. Then (X, μ) is μ-T_2 if and only if (X, α_μ) is α_μ-T_2.*

Proof. Let (X, α_μ) is α_μ-T_2. For distinct points x and y in M_μ, there exist $U, V \in \alpha_\mu$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore, by Lemma 2.16, $i_\mu c_\mu i_\mu U \cap i_\mu c_\mu i_\mu V = \emptyset$. Conversely is obvious.

Lemma 4.26. *A subset A of a GTS (X, μ) is μ-regular semi-open if and only if A is s_μ-open and $A \cup (X - M_\mu)$ is s_μ-closed.*

Proof. Let $A \in rs_\mu$. Then there exists a μ-regular open set U of X such that $U \subseteq A \subseteq c_\mu U \cap M_\mu$. Then $U = i_\mu c_\mu U = i_\mu c_\mu A$. Therefore, $i_\mu c_\mu A \subseteq A$ so that $A \cup (X - M_\mu)$ is s_μ-closed. Also $A \subseteq c_\mu U \cap M_\mu = c_\mu i_\mu U \cap M_\mu \subseteq c_\mu i_\mu A \cap M_\mu$, so that $A \in s_\mu$. Conversely, let $A \in s_\mu$ and $A \cup (X - M_\mu)$ be s_μ-closed. Then $i_\mu c_\mu A \subseteq A \subseteq c_\mu i_\mu A \cap M_\mu \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$ and $i_\mu c_\mu A$ is μ-regular open. Then A is μ-regular semi-open.

From Lemma 3.17, it is easily observed that if $A \in s_\mu$, then $c_{s_\mu} A \cap M_\mu \in s_\mu$ and so, by Lemma 4.26, $c_{s_\mu} A \cap M_\mu \in rs_\mu$. Thus, the following theorem follows.

Theorem 4.27. *A GTS (X, μ) is μ-semi-T_2 if and only if for each pair of distinct points $x, y \in M_\mu$, there is a μ-regular semi-open set U containing x but not y or containing y but not x.*

Definition 4.28. *A GTS (X, μ) is called*

1. *μ-regular semi-T_0 if for each pair of distinct points $x, y \in M_\mu$, there is a μ-regular semi-open set U containing one of the points.*
2. *μ-regular semi-T_1 if $x, y \in M_\mu$, $x \neq y$ implies the existence of μ-regular semi-open sets U_1 and U_2 such that $x \in U_1$ and $y \notin U_1$ and $y \in U_2$ and $x \notin U_2$.*
3. *μ-regular semi-T_2 if $x, y \in M_\mu$, $x \neq y$ implies the existence of disjoint μ-regular semi-open sets U_1 and U_2 containing x and y, respectively.*

Theorem 4.29. *If a GTS (X, μ) is μ-regular semi-T_2, then it is μ-semi-T_2.*

Proof. The proof follows from Lemma 4.26.

From Theorem 4.27, the following implications follow:

$$\begin{array}{ccccc} \mu\text{-regular semi-}T_2 & \Rightarrow & \mu\text{-regular semi-}T_1 & \Rightarrow & \mu\text{-regular semi-}T_0 \\ & & & & \Updownarrow \\ \mu\text{-semi-}T_0 & \Leftarrow & \mu\text{-semi-}T_1 & \Leftarrow & \mu\text{-semi-}T_2 \end{array}$$

The following corollary is corollary 2.23 [20].

Corollary 4.30. *If U is μ-open, then $c_{\tau_s}U = c_\mu U$.*

Lemma 4.31. $rs_\mu = rs_{\tau_s}$.

Proof. From Corollary 4.30, it follows that if a set A is τ_s-semi-open, then it is μ-semi-open and hence from Lemma 4.26, $rs_{\tau_s} \subseteq rs_\mu$. On the other hand, let $A \in rs_\mu$. Then there exists a μr-open set U such that $U \subseteq A \subseteq c_\mu U$. Since $ro_\mu = ro_{\tau_s}$, by corollary 4.30, $A \in rs_{\tau_s}$.

A property P in (X, μ) is said to be μ-semi-regular if provided (X, μ) has P if and only if (X, τ_s) has P.

Theorem 4.32. *μ-semi-T_2 is μ-semi-regular property.*

Proof. The proof follows from Theorem 4.27 and Lemma 4.31.

5. Mappings

Definition 5.1. *A mapping $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-semi-continuous at a point $x \in X$ if for each ν-open set V containing $f(x)$, there exists a μ-semi-open set U containing x such that $f(U) \subseteq V$. If f is (μ, ν)-semi-continuous at each point of X then f is called (μ, ν)-semi-continuous on X.*

Remark 5.2. *Note that if $f : (X, \mu) \to (Y, \nu)$ is a mapping and $f(x) \in Y - M_\nu$ then f is trivially (μ, ν)-semi-continuous at x. If $x \in X - M_\mu$ and $f(x) \in M_\nu$, then f is not (μ, ν)-semi-continuous at x since there is no μ-semi-open set U containing x. Thus, for f to be (μ, ν)-semi-continuous it is necessary that $f(X - M_\mu) \subseteq Y - M_\nu$.*

Theorem 5.3. *For a mapping $f : (X, \mu) \to (Y, \nu)$, the following statements are equivalent:*

1. *f is (μ, ν)-semi-continuous.*
2. *$f^{-1}(V)$ is μ-semi-open for each ν-open set V.*
3. *$f^{-1}(F)$ is μ-semi-closed for each ν-closed set F.*
4. *$f(c_{s_\mu} A) \subseteq c_\nu(f(A))$ for any subset A of X.*
5. *$c_{s_\mu}(f^{-1}(B)) \subseteq f^{-1}(c_\nu B)$ for any subset B of Y.*
6. *$f^{-1}(i_\nu(B)) \subseteq i_{s_\mu}(f^{-1}(B))$ for any subset B of Y.*

Proof. The implications, (i) implies (ii), and (ii) if and only if (iii) are obvious.
$(iii) \Rightarrow (iv)$. By $f^{-1}(c_\nu(f(A)))$ is s_μ-closed and $A \subseteq f^{-1}(c_\nu(f(A)))$. Therefore, $c_{s_\mu} A \subseteq f^{-1}(c_\nu(f(A)))$. Thus, $f(c_{s_\mu} A) \subseteq c_\nu(f(A))$.
$(iv) \Rightarrow (v)$. Let $A = f^{-1}(B)$. Then $f(c_{s_\mu}(f^{-1}(B))) \subseteq c_\nu B$. Thus, $c_{s_\mu}(f^{-1}(B)) \subseteq f^{-1}(c_\nu B)$.
$(v) \Rightarrow (vi)$. By (iv), $c_{s_\mu}(f^{-1}(Y - B)) \subseteq f^{-1}(c_\nu(Y - B))$. Then $f^{-1}(i_\nu(B)) \subseteq i_{s_\mu}(f^{-1}(B))$.
$(vi) \Rightarrow (i)$. Let $U \in \nu$. Then, by (vi), $f^{-1}(U) \subseteq i_{s_\mu}(f^{-1}(U))$. Therefore, $f^{-1}(U) = i_{s_\mu}(f^{-1}(U))$.

Theorem 5.4. *A mapping $f : (X, \mu) \to (Y, \nu)$ is (μ, ν)-semi-continuous if and only if $f : (X, s_\mu) \to (Y, \nu)$ is (s_μ, ν)-continuous.*

Definition 5.5. *Let (X, μ) and (Y, ν) be GTSs. A function $f : (X, \mu) \to (Y, \nu)$ is called*

1. *[4] closed if the image of μ-closed set is ν-closed.*
2. *α_μ-closed if $f : (X, \mu) \to (Y, \alpha_\nu)$ is closed.*
3. *semi-closed if $f : (X, \mu) \to (Y, s_\nu)$ is closed.*
4. *feebly-closed if $f : (X, \mu) \to (Y, f_\nu)$ is closed.*
5. *pre-closed if $f : (X, \mu) \to (Y, p_\nu)$ is closed.*
6. *[4] open if the image of μ-open set is ν-open.*
7. *semi-open if $f : (X, \mu) \to (Y, s_\nu)$ is open.*

Theorem 5.6. *The following statements are equivalent:*

1. *$f : (X, \mu) \to (Y, \nu)$ is feebly-closed.*
2. *$f : (X, \mu) \to (Y, \nu)$ is α_μ-closed.*

Theorem 5.7. *A function $f : (X, \mu) \to (Y, \nu)$ is α_μ-closed if and only if it is semi-closed and pre-closed.*

Theorem 5.8. *Let $f : (X, \mu) \to (Y, \nu)$ be a closed function, and $B, C \subseteq Y$.*

1. *If U is a μ-open set such that $f^{-1}(B) \subseteq U$, then there exists a ν-open set V containing B such that $f^{-1}(V) \subseteq U$.*
2. *If $f^{-1}(B) \subseteq R \in \mu$, $f^{-1}(C) \subseteq S \in \mu$ and $R \cap S = \emptyset$, then there exist disjoint μ-open sets U and V such that $B \subseteq U$ and $C \subseteq V$.*

Proof.

1. Let $V \subseteq Y$ be such that $(Y - V) = f(X - U)$. Since f is closed, then V is ν-open. Since $f^{-1}(B) \subseteq U$, $(Y - V) = f(X - U) \subseteq ff^{-1}(Y - B) \subseteq (Y - B)$. Therefore, $B \subseteq V$. Now $(X - U) \subseteq f^{-1}f(X - U) = f^{-1}(Y - V) = X - f^{-1}(V)$. Hence, $f^{-1}(V) \subseteq U$.
2. Let $f^{-1}(B) \subseteq R \in \mu$, $f^{-1}(C) \subseteq S \in \mu$ and $R \cap S = \emptyset$. Then, by (i), there exist μ-open sets U and V containing B and C, respectively, such that $f^{-1}(U) \subseteq R$ and $f^{-1}(V) \subseteq S$. Since, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, $U \cap V = \emptyset$.

Theorem 5.9. *Let $f:(X,\mu)\to(Y,\nu)$ be feebly-closed and $B,C\subseteq Y$. If there exist two disjoint μ-open sets R and S such that $f^{-1}(B)\subseteq R$ and $f^{-1}(C)\subseteq S$, then there exist two disjoint ν-open sets $R^{'}$ and $S^{'}$ such that $B\subseteq R^{'}$ and $C\subseteq S^{'}$.*

Proof. If $f:(X,\mu)\to(Y,\nu)$ is feebly closed then by Theorem 5.6 it is α_μ-closed. Therefore, $f:(X,\mu)\to(Y,\alpha_\nu)$ is closed. Thus, the result follows from Theorem 5.8.

Definition 5.10. *A function $f:(X,\mu)\to(Y,\nu)$ is said to be*

1. *(μ,ν)-weekly-open if $f(U)\subseteq i_\nu(f(c_\mu U))$ for every μ-open set U.*
2. *(μ,ν)-almost-open if for every μr-open set U of X, $f(U)$ is ν-open in Y.*
3. *(μ,ν)-irresolute if $f^{-1}(U)$ is s_μ-open for every s_ν-open set U.*

U of X, $f(U)$ is ν-open in Y.

Lemma 5.11. *If a function $f:(X,\mu)\to(Y,\nu)$ is (μ,ν)-almost-open, then it is (μ,ν)-weekly-open.*

Proof. Let U be a μ-open set. Since f is $(\mu$-$\nu)$-almost-open, $f(i_\mu c_\mu U)$ is ν-open. Hence, $f(U)\subseteq f(i_\mu c_\mu U)\subseteq i_\nu f(c_\mu U)$.

The converse of the above lemma is not true in general (See Example 1.5 in [14]).

Theorem 5.12. *If a function $f:(X,\mu)\to(Y,\nu)$ is (μ,ν)-almost-open, (μ,ν)-semi-continuous and $f(M_\mu)\subseteq M_\nu$, then it is (μ,ν)-irresolute.*

Proof. Let $V\in s_\nu$. Then there exists a ν-open set A such that $A\subseteq V\subseteq c_\nu A\cap M_\nu$, so that $f^{-1}(A)\subseteq f^{-1}(V)\subseteq f^{-1}(c_\nu A\cap M_\nu)$. Therefore, in view of Remark 5.2, $f^{-1}(A)\subseteq f^{-1}(V)\subseteq f^{-1}(c_\nu A)\cap M_\mu$. Since f is (μ,ν)-semi-continuous, $f^{-1}(A)\in s_\mu$ and thus, $f^{-1}(A)\subseteq c_\mu i_\mu(f^{-1}(A))\cap M_\mu$. Put $F=Y-f(X-c_\mu i_\mu(f^{-1}(A)))$. Then F is ν-closed because f is (μ,ν)-almost-open and $c_\mu i_\mu(f^{-1}(A))$ is μr-closed. Now $X-c_\mu i_\mu(f^{-1}(A))\subseteq X-f^{-1}(A)=f^{-1}(Y-A)$. Therefore, $f(X-c_\mu i_\mu(f^{-1}(A)))\subseteq f(f^{-1}(Y-A))\subseteq(Y-A)$, so that $A\subseteq F$. Thus $c_\nu A\subseteq F$. Similarly, $f^{-1}(F)\subseteq c_\mu i_\mu f^{-1}(A)$. Therefore, $f^{-1}(c_\nu A)\subseteq c_\mu(f^{-1}(A))$. Then $f^{-1}(c_\nu A)\cap M_\mu\subseteq c_\mu(f^{-1}(A))\cap M_\mu$. Hence, by Theorem 3.6, $f^{-1}(V)\in s_\mu$.

Theorem 5.13. *If $f : (X, \mu) \to (Y, \nu)$ is semi-open mapping, then $f^{-1}(c_{s_\nu} G) \subseteq c_\mu(f^{-1}(G))$ for any set $G \subseteq Y$.*

Proof. If $x \notin c_\mu(f^{-1}(G))$. Then there exists a μ-open set U containing x such that $U \cap f^{-1}(G) = \emptyset$. Then $f(x) \in f(U)$ and $f(U) \cap G = \emptyset$. Since f is semi-open, $f(U)$ is semi-open. Therefore, $f(x) \notin c_{s_\mu} G$.

Theorem 5.14. *If a function $f : (X, \mu) \to (Y, \nu)$ is semi-open, (μ, ν)-semi-continuous, $f(M_\mu) \subseteq M_\nu$ and (Y, ν) is extremally ν-disconnected, then f is irresolute.*

Proof. Let $V \in s_\nu$. Then there exists ν-open set U such that $U \subseteq V \subseteq c_\nu U \cap M_\nu$. Therefore, $f^{-1}(U) \subseteq f^{-1}(V) \subseteq f^{-1}(c_\nu U \cap M_\nu) = f^{-1}(c_\nu U) \cap M_\mu$. Since Y is extremally ν-disconnected, by lemma 3.24 $c_{s_\nu} U = c_\nu U$. Then by Theorem 5.13, $f^{-1}(U) \subseteq f^{-1}(V) \subseteq c_\mu(f^{-1}(U)) \cap M_\mu$. Since f is (μ, ν)-semi-continuous, $f^{-1}(U) \in s_\mu$. Hence, by Theorem 3.6 , $f^{-1}(V) \in s_\mu$.

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