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Weak convergence and weak compactness in the space of integrable functions with respect to a vector measure

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Abstract:

We consider weak convergence and weak compactness in the space $L^1(m)$ of real valued integrable functions with respect to a Banach space valued measure m equipped with its natural norm. We give necessary and sufficient conditions for a sequence in $L^1(m)$ to be weak Cauchy, and we give necessary and sufficient conditions for a subset of $L^1(m)$ to be conditionally sequentially weakly compact.

Keywords: Weak convergence; Weak compactness; Integrable functions; Measure and integration.

MSC (2010): 28B15; 28B20.

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1. Introduction

In this note we consider weak convergence and weak compactness in the space, $L^1(m)$, of integrable functions with respect to a Banach space valued, countably additive measure m. We use a characterization of the dual space $L^1(m)'$ due to Okada ([7]).

Let Σ be a $\sigma-$algebra of subsets of a set S , let X be a Banach space and $m : \Sigma \to X$ be countably additive. We consider the space $L^1(m)$ of functions $f : S \to \mathbf{R}$ which are integrable with respect to m. A measurable function $f : S \to \mathbf{R}$ is weakly m integrable if f is $x'm = x' \circ m$ integrable for every $x' \in X'$ and f is m integrable if f is weakly m integrable and for every $A \in \Sigma$ there exists $x_A \in X$ such that $\int_A f d\acute{x}m = x'(x_A)$ for $x' \in X'$; We set $x_A = \int_A f dm$. (See [5],[6] for the properties of the integral; another approach to the integral is given in [2] IV.10.) This space is a Banach space with respect to the norm

$$\|f\|_1 = \sup\{\int_S |f| \, d\,|x'm| : \|x'\| \le 1\},$$

where $|x'm|$ denotes the variation of the scalar measure $x'm$.

Notation 1. *If $g \in L^\infty(m)$, $A \in \Sigma$ and $x' \in X'$, denote the continuous linear functional*

$$f \to \int_S gf\chi_A dx'm = \int_A gf dx'm$$

on $L^1(m)$ by $g\chi_A x'$.

Okada has shown that if $l \in L^1(m)'$ there exist $g \in L^\infty(m)$, $\{A_j\} \subset \Sigma$ pairwise disjoint and $x'_j \in X'$, $\left\|x'_j\right\| \le 1$, such that

$$l(f) = \sum_{j=1}^{\infty} \int_{A_j} gf dx'_j m$$

for $f \in L^1(m)$, where the series converges; that is, $l = \sum_{j=1}^{\infty} g\chi_{A_j} x'_j$ ([7]). Note that if the series $\sum_{j=1}^{\infty} \int_{A_j} gf dx'_j m$ converges for every $f \in L^1(m)$ when $g \in L^\infty(m)$, $\{A_j\} \subset \Sigma$ pairwise disjoint and $x'_j \in X'$, $\left\|x'_j\right\| \le 1$, then $l(f) = \sum_{j=1}^{\infty} \int_{A_j} gf dx'_j m$ defines a continuous linear functional on $L^1(m)$ (for each n, $l_n(f) = \sum_{j=1}^{n} \int_{A_j} gf dx'_j m$ is linear and continuous and

$$l_n(f) \to \sum_{j=1}^{\infty} \int_{A_j} gf dx_j' m = l(f)$$

for every f so l is continuous by the Banach-Steinhaus Theorem).

Thus, the dual of $L^1(m)$ is the space of all $g \in L^\infty(m)$, $\{A_j\} \subset \Sigma$ pairwise disjoint and $x_j' \in X'$, $\left\|x_j'\right\| \leq 1$, such that the series $\sum_{j=1}^{\infty} \int_{A_j} gf dx_j' m$ converges for every $f \in L^1(m)$.

Not every $g \in L^\infty(m)$, $\{A_j\} \subset \Sigma$ pairwise disjoint and $x_j' \in X'$, $\left\|x_j'\right\| \leq 1$, may define elements of $L^1(m)'$; that is, there may exist such elements where the series $\sum_{j=1}^{\infty} \int_{A_j} gf dx_j' m$ may fail to converge. Indeed, we have

Theorem 2. *The measure m has bounded variation iff the series $\sum_{j=1}^{\infty} \int_{A_j} gf dx_j' m$ converges for every $f \in L^1(|m|)$, $g \in L^\infty(m)$, $\{A_j\} \subset \Sigma$ pairwise disjoint and $x_j' \in X'$, $\left\|x_j'\right\| \leq 1$.*

Proof. Suppose m has bounded variation, $|m|$. If $f \in L^1(|m|)$, $g \in L^\infty(m)$, $\{A_j\} \subset \Sigma$ pairwise disjoint and $x_j' \in X'$, $\left\|x_j'\right\| \leq 1$, then

$$\begin{aligned}
\left|\sum_{j=1}^{\infty} \int_{A_j} gf dx_j' m\right| &\leq \sum_{j=1}^{\infty} \left|\int_{A_j} gf dx_j' m\right| \leq \sum_{j=1}^{\infty} \int_{A_j} |gf|\, d\left|x_j' m\right| \\
&\leq \sum_{j=1}^{\infty} \int_{A_j} |gf|\, d|m| \leq \|g\|_\infty \sum_{j=1}^{\infty} \int_{A_j} |f|\, d|m| \\
&\leq \|g\|_\infty \int_S |f|\, d|m|
\end{aligned}$$

shows the series converges.

For the converse, set $f = g = 1$ and let $\{A_j\}$ be any pairwise disjoint sequence from Σ. Pick $x_j' \in X'$, $\left\|x_j'\right\| \leq 1$, such that $x_j'(m(A_j)) = \|m(A_j)\|$. Then

$$\sum_{j=1}^{\infty} \int_{A_j} gf dx_j' m = \sum_{j=1}^{\infty} x_j'(m(A_j)) = \sum_{j=1}^{\infty} \|m(A_j)\| < \infty.$$

This implies m has bounded variation ([12],[10] 3.51).

The computation above shows that if m has bounded variation and $L^1(m) = L^1(|m|)$, then the dual of $L^1(m)$ is the space of all $g \in L^\infty(m)$,

$\{A_j\} \subset \Sigma$ pairwise disjoint and $x'_j \in X'$, $\left\|x'_j\right\| \le 1$, such that the series $\sum_{j=1}^{\infty} \int_{A_j} gfdx'_jm$ converges for every $f \in L^1(m)$.

For later use we observe that if $l = \sum_{j=1}^{\infty} g\chi_{A_j} x'_j \in L^1(m)'$, then the series $\sum_{j=1}^{\infty} \int_{A_j} gfdx'_jm$ is subseries convergent for every $f \in L^1(m)$. Indeed, if $\sigma \subset \mathbf{N}$ and $f \in L^1(m)$ let $A = \cup_{j\in\sigma} A_j$ and set $f_\sigma = \chi_A f \in L^1(m)$. Then

$$l(f_\sigma) = \sum_{j=1}^{\infty} \int_{A_j} gf_\sigma dx'_jm = \sum_{j\in\sigma} \int_{A_j} gfdx'_jm.$$

We now consider weak convergence and weak sequential compactness in $L^1(m)$. First we consider necessary conditions for weak sequential compactness. A subset K of $L^1(m)$ is relatively sequentially weak compact if every sequence $\{f_k\}$ in K has a subsequence which is weakly convergent and K is conditionally sequentially weakly compact if every sequence $\{f_k\}$ in K has a subsequence which is weakly Cauchy.

Theorem 3. *Suppose $K \subset L^1(m)$ is conditionally sequentially weakly compact. Then*

(#) for every $\sum_{j=1}^{\infty} g\chi_{A_j} x'_j \in L^1(m)'$, the series $\{\sum_{j=1}^{\infty} \int_{A_j} gfdx'_jm : f \in K\}$ are uniformly subseries convergent.

Proof. If the conclusion fails to hold, there exist $\epsilon > 0$, $f_k \in K$ and an increasing sequence of intervals $\{I_k\}$ with

$$(*) \left| \sum_{j\in I_k} \int_{A_j} gf_k dx'_jm \right| > \epsilon.$$

We may assume that $\lim_k \langle l, f_k \rangle$ exists for every $l \in L^1(m)'$. For any $\sigma \subset \mathbf{N}$, define l_σ belonging to $L^1(m)'$ by

$$\langle l_\sigma, f \rangle = \sum_{j\in\sigma} \int_{A_j} gfdx'_jm.$$

Then for every $\sigma \subset \mathbf{N}$,

$$\lim_k \langle l_\sigma, f_k \rangle = \lim_k \sum_{j\in\sigma} \int_{A_j} gf_k dx'_jm$$

exists. Set $a_j = \lim_k \int_{A_j} gf_k dx'_jm$. By the Hahn-Schur Theorem ([10]7.18), the series

$$\{\sum_{j=1}^{\infty}\int_{A_j} gf_k dx_j' m : k\}$$

are uniformly subseries convergent and

$$\lim_k \sum_{j\in\sigma}\int_{A_j} gf dx_j' m = \sum_{j\in\sigma} a_j$$

uniformly for $\sigma \subset \mathbf{N}$. But, this contradicts (*).

Remark 4. *Note that the conclusion (#) in Theorem 3 implies that the indefinite integrals $\{\int_{\cdot} gf dm : f \in K\}$ are uniformly countably additive. For if this is not the case , there exist $\epsilon > 0$, pairwise disjoint $\{A_j\} \subset \Sigma$, $f_k \in K$ and an increasing sequence of intervals $\{I_k\}$ such that $\left\|\sum_{j\in I_k}\int_{A_j} gf_k dm\right\| > \epsilon$. Set $B_k = \cup_{j\in I_k} A_j$ and pick $x_k' \in X'$, $\|x_k'\| \leq 1$, such that*

$$\left| x_k' \sum_{j\in I_k}\int_{A_j} gf_k dm\right| = \left|\int_{B_k} gf_k dx_k' m\right| = \left\|\sum_{j\in I_k}\int_{A_j} gf_k dm\right\| > \epsilon.$$

Using $g, \{B_k\}, \{x_k'\}$ this contradicts the condition (#).

Theorem 5. *Suppose $K \subset L^1(m)$ is conditionally (relatively) sequentially weakly compact. Then*

(##) for every $g \in L^\infty(m)$, $A \in \Sigma$ the set $\{\int_A gf dm : f \in K\}$ is conditionally (relatively) sequentially weakly compact.

Proof. The integration map $I_g : L^1(m) \to X$, $I_g(f) = \int_A gf dm$, is norm continuous

$$\begin{aligned}(\|I_g f\| &= \sup\{\left|\int_S gf dx' m\right| : \|x'\| \leq 1\} \\ &\leq \|g\|_\infty \sup\{\int_S |f|\, d\,|x' m| : \|x'\| \leq 1\} = \|g\|_\infty \|f\|_1)\end{aligned}$$

and, therefore, weak-weak continuous so the result is immediate.

We next consider sufficient conditions for weak sequential compactness. First an observation about weak convergence.

Lemma 6. *Let $\{f_k\} \subset L^1(m)$. Then $\{f_k\}$ is weakly Cauchy iff for every* $\sum_{j=1}^{\infty} g\chi_{A_j} x'_j \in L^1(m)'$

(i) $\lim_k \int_{A_j} g f_k dx'_j m = a_j$ exists for every j
and
(ii) the series $\{\sum_{j=1}^{\infty} \int_{A_j} g f_k dx'_j m : k \in \mathbf{N}\}$ converge uniformly.

Proof. $\Rightarrow$: For (i) consider the linear functional $l_j = g\chi_{A_j} x'_j \in L^1(m)'$. For (ii) consider the linear functional

$$l_\sigma = \sum_{j=1}^{\infty} g\chi_{A_j} x'_j \in L^1(m)', \sigma \subset \mathbf{N}.$$

Since $\lim_k l_\sigma(f) = \lim_k \sum_{j\in\sigma} \int_{A_j} g f_k dx'_j m$ exists, the Hahn-Schur Theorem ([10] 7.18) gives (ii) and also implies $\lim_k \sum_{j\in\sigma} \int_{A_j} g f_k dx'_j m = \sum_{j\in\sigma} a_j$ uniformly for $\sigma \subset \mathbf{N}$.
$\Leftarrow$: Let $\epsilon > 0$, $l = \sum_{j=1}^{\infty} g\chi_{A_j} x'_j \in L^1(m)'$. By (ii) there exists N such that $\left|\sum_{j=N+1}^{\infty} \int_{A_j} g f_k dx'_j m\right| < \epsilon$ for all k. By (i) there exists k_0 such that $i, k \geq k_0$ implies

$$\left|\sum_{j=1}^{N} \int_{A_j} g f_k dx'_j m - \sum_{j=1}^{N} \int_{A_j} g f_i dx'_j m\right| < \epsilon.$$

Then
$|l(f_k) - l(f_i)| \leq \left|\sum_{j=1}^{N} \int_{A_j} g f_k dx'_j m - \sum_{j=1}^{N} \int_{A_j} g f_i dx'_j m\right|$
$+ \left|\sum_{j=N+1}^{\infty} \int_{A_j} g f_k dx'_j m\right| + \left|\sum_{j=N+1}^{\infty} \int_{A_j} g f_i dx'_j m\right| < 3\epsilon$
for $i, k \geq k_0$.

Remark 7. *Note the proof above implies that $f_k \to 0$ weakly iff (i) holds for $a_j = 0$ and (ii).*

We consider conditional sequential weak compactness.

Theorem 8. *If $K \subset L^1(m)$ satisfies conditions (#) and (##), then K is conditionally sequentially weakly compact.*

Proof. Let $\{f_k\} \subset K$. There exists a countable algebra $\mathcal{A}$ such that each f_k is measurable with respect to the σ algebra Σ_1 generated by $\mathcal{A}$. By replacing Σ by Σ_1 we may assume that Σ is generated by a countable algebra $\mathcal{A}$. By (##) and the diagonalization method, there is a subsequence of $\{f_k\}$, still denoted by $\{f_k\}$, such that

$$weak - \lim_k \int_A f_k dm = F(A)$$

exists for every $A \in \mathcal{A}$ ([4], page 238).

We claim that $weak - \lim_k \int_A f_k dm = F(A)$ exists for every $A \in \Sigma$. For this, put

$$\Sigma_1 = \{A \in \Sigma : weak - \lim_k \int_A f_k dm = F(A) \ exists\}.$$

Note $\mathcal{A}$ is contained in Σ_1. We claim that Σ_1 is a monotone class. Suppose $B_j \in \Sigma_1$ with $B_j \uparrow B$. For every j,

$$weak - \lim_k \int_{B_j} f_k dm = F(B_j) \in X$$

exists. By (#)

$$weak - \lim_j \int_{B_j} f_k dm = \int_B f_k dm$$

uniformly for $k \in \mathbf{N}$. Let $x' \in X'$. Then

$$\lim_k x' \int_{B_j} f_k dm = \lim_k \int_{B_j} f_k dx'm = x'F(B_j)$$

for all j and

$$\lim_j x' \int_{B_j} f_k dm = x'F(B_j) = \lim_j \int_{B_j} f_k dx'm = x' \int_B f_k dm = \int_B f_k dx'm$$

uniformly for $k \in \mathbf{N}$. By the Iterated Limit Theorem ([2] I.7.6),

$$\lim_j \lim_k \int_{B_j} f_k dx'm = \lim_k \lim_j \int_{B_j} f_k dx'm = \lim_j x'F(B_j) = \lim_k \int_B f_k dx'm.$$

Therefore, $\{\int_B f_k dm\}_k$ is weak Cauchy. But (##) implies $\{\int_B f_k dm\}_k$ is relatively weak compact so $weak - \lim_k \int_B f_k dm = F(B)$ exists. Hence,

$B \in \Sigma_1$ and Σ_1 is a monotone class. By the Monotone Class Theorem ([3] I.6, [9] 2.1.6) $\Sigma_1 = \Sigma$ and the claim is established.

Thus, $weak - \lim_k \int_S gf_k dm$ exists for every simple function g.

We next claim that $\{\int_S gf_k dm\}$ is weak Cauchy for every $g \in L^\infty(m)$. Let $\epsilon > 0$. Pick a simple function h such that $\|h - g\|_\infty < \epsilon$. Fix $x' \in X'$, $\|x'\| \leq 1$. Then

$$
\begin{aligned}
(\&) \quad \left| x'(\int_S g(f_k - f_j)dm \right| &= \left| \int_S g(f_k - f_j)dx'm \right| \\
&\leq \left| \int_S (g-h)(f_k - f_j)dm \right| + \left| \int_S h(f_k - f_j)dm \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|g-h\|_\infty \int_S |f_k - f_j| \, d\,|x'm| + \left| \int_S h(f_k - f_j)dm \right| \\
&\leq \epsilon \|f_k - f_j\|_1 + \left| \int_S h(f_k - f_j)dm \right|.
\end{aligned}
$$

The first term on the right hand side of (&) is bounded by some $M\epsilon > 0$ and the last term is small for j, k large by the result for simple functions established above and the claim is established.

By (##) $\{\int_S gf_k dm\}$ is relatively sequentially weak compact so

$$(*) \; weak - \lim_k \int_S gf_k dm$$

exists.

Let $l = \sum_{j=1}^\infty g\chi_{B_j} x'_j \in L^1(m)'$. We show $\lim_k l(f_k)$ exists and this will establish the result. By (*)

$$\lim_k \int_{B_j} gf_k dx'_j m = \lim_k x'_j \int_{B_j} gf_k dm$$

exists for every j and by (#) the series $\{\sum_{j=1}^\infty \int_{B_j} gf_k dx'_j m\}_k$ converge uniformly for $k \in \mathbf{N}$. Lemma 6 shows $\lim_k l(f_k)$ exists.

Another similar weak compactness result for a "weak type topology" on $L^1(m)$ is established in [11] 9.15.

If $L^1(m)$ is weakly sequentially complete, the conditions (#) and (##) above in Theorem 8 imply that the subset K is relatively sequentially compact. Conditions which guarantee that $L^1(m)$ is order isomorphic to an AL space and is weakly sequentially complete are given in [1] and conditions for which $L^1(m)$ is reflexive and is, therefore, weakly sequentially complete are given in [7]. See also [8].

We use the results above to show that $L^1(m)$ is weakly sequentially complete when m is an atomic measure and X is weakly sequentially complete.

Let $\mathcal{P}$ be the power set of $\mathbf{N}$ and let $m : \mathcal{P} \to X$ be countably additive. First, an observation.

Proposition 9. *The function $f : \mathbf{N} \to \mathbf{R}$ is m integrable $\Longleftrightarrow$ the series $\sum_{j=1}^{\infty} f(j)m(j)$ is subseries convergent.*

Proof. $\Longrightarrow$: Follows from the countable additivity of the integral $\int_{\cdot} f dm$.

$\Longleftarrow$: Let $x' \in X'$ and $A \in \mathcal{P}$. Then $\int_A f dx'm = \sum_{j\in A} f(j)x'm(j) = x' \sum_{j\in A} f(j)m(j)$ so f is m integrable with $\int_A f dm = \sum_{j\in A} f(j)m(j)$.

Theorem 10. *If X is weakly sequentially complete, then $L^1(m)$ is weakly sequentially complete.*

Proof. Let $\{f_j\}$ be weakly Cauchy in $L^1(m)$. For $x' \in X'$ and $k \in \mathbf{N}$, define $e^k x' \in L^1(m)'$ by $e^k x'(f) = f(k)x'm(k)$. Then $\lim_j e^k x'(f_j) = \lim_j x' f_j(k)m(k)$ exists. Pick $x_k'{}'$ such that $x_k'(m(k)) = \|m(k)\|$ and $\|x_k'\| \le 1$. Define

$$F(k) = \lim_j x_k' f_j(k)m(k) = \lim_j f_j(k) \|m(k)\|$$

and set $f(k) = F(k)/\|m(k)\|$ if $m(k) \neq 0$ and $f(k) = 0$ otherwise. Then

$$\lim_j f_j(k) = \lim_j \|m(k)\| f(k)/\|m(k)\| = F(k)/\|m(k)\| = f(k)$$

so $f_j \to f$ pointwise.

We claim $f \in L^1(m)$. For $\sigma \subset \mathbf{N}$ and $x' \in X'$, define $l_\sigma \in L^1(m)'$ by $l_\sigma(h) = \sum_{i\in\sigma} h(i)x'm(i)$. Then $\lim_j l_\sigma(f_j) = \lim_j \sum_{i\in\sigma} f_j(i)x'm(i)$ exists and $\lim_j f_j(i)x'm(i) = f(i)x'm(i)$. The Hahn-Schur Theorem ([10] 7.18) implies that the series $\sum_i f(i)x'm(i)$ is subseries convergent and

$$(*) \lim_j \sum_{i\in\sigma} f_j(i)x'm(i) = \sum_{i\in\sigma} f(i)x'm(i)$$

uniformly for $\sigma \subset \mathbf{N}$. Thus, the series $\sum_i f(i)m(i)$ is weakly unconditionally Cauchy and weakly subseries convergent since X is weakly sequentially complete. The Orlicz-Pettis Theorem gives that the series is norm subseries convergent so f is m integrable by the proposition above.

Next, we show $f_j \to f$ weakly. Let $l = \sum_{k=1}^{\infty} g\chi_{A_k} x'_k \in L^1(m)'$ with $\|x'_k\| \le 1$, $\{A_k\} \subset \mathbf{N}$ pairwise disjoint and $g \in L^\infty(m)$. We show $\lim_j l(f_j)$ exists by checking Lemma 6. Now $\{g(i)\} \in l^\infty$. Since X is weakly sequentially complete, Theorem 7.30 of [10] applies to condition (*) and implies that

$$\lim_j \int_{A_k} g f_j dx'_k = \lim_j \sum_{i \in A_k} g(i) f_j(i) x'_k m(i)$$

exists for every k so (i) of Lemma 6 holds. Now $\{f_j : j\}$ is conditionally sequentially weakly compact so condition (#) of Theorem 3 implies (ii) of Lemma 6. Hence, $f_j \to f$ weakly.

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