



Minimal and maximal solutions to first-order differential equations with piecewise constant generalized delay

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Abstract:

In this paper we employ the method of maximal and minimal solutions coupled with comparison principles and the monotone iterative technique to obtain results of existence and approximation of solutions for differential equations with piecewise constant delay of generalized type (DEPCAG).

Keywords: Piecewise constant arguments; Monotone iterative method; Comparison results; Maximal and minimal solutions.

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1. Introduction

The study of differential equations with piecewise constant arguments has been treated widely in the literature. This type of equation, in which techniques of differential and difference equations are combined, models, among others, some biological phenomena (see [3, 15] and references therein), the stabilization of hybrid control systems with feedback discrete controller [16], or damped oscillators [7]. The first studies in this field have been given in [4, 20], after this, some papers related with stability, oscillation properties and existence of periodic solutions have been treated by several authors (see [5, 8, 9, 14, 22] for details).

It is well known that the theory of differential inequalities for initial-value problems has been very useful in the theory of differential equations [18, 21]. Existence of extremal solutions for a variety of nonlinear differential equations is studied by a combined approach of the method of maximal and minimal solutions and the monotone iterative technique [17]. It is natural to extend this useful method to retarded differential equations.

In this paper we continue the investigation of differential equations with piecewise constant argument of generalized type (DEPCAG) originated in [2, 19], using differential inequalities. These equations are closely related to impulse and loaded equations and, especially, to difference equations of a discrete argument. They have the structure of continuous dynamical systems within intervals of certain length. Continuity of a solution at a point joining any two consecutive intervals then implies recursion relations for the solution at such points. The results obtained in [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 19] show that all types of DEPCAG share similar characteristics.

First of all, it is natural to pose the initial-value problem for such equations not on an interval but at a number of individual points. Secondly, two-sided solutions exist for all types of DEPCAG. Finally, since DEPCAG combine the features of both differential and difference equations, their asymptotic behavior as $t \rightarrow \infty$ resembles the solutions growth of differential equations in some cases, while in others it inherits the properties of difference equations.

The paper is organized as follows, in Section 2 we present the main tools that we will use in the rest of the paper. Section 3 is devoted to obtain the unique solution of the associated linear problem, from which we derive comparison results.

In Section 4, we present results concerning the existence of extremal

solutions and the uniqueness of solution in the presence of lower and upper solutions.

2. Preliminaries

Let \mathbf{Z} and \mathbf{R} be the sets of integer and real numbers, respectively. Fix a real sequence t_i , $i \in \mathbf{Z}$, is a strictly ordered sequence of real number, such that $t_i < t_{i+1}$, $t_i \rightarrow \infty$ as $i \rightarrow \infty$. Let $\beta : \mathbf{R} \rightarrow \mathbf{R}$ be a step function given by $\beta(t) = t_i$ for $t \in I_i = [t_i, t_{i+1})$ and there exists a positive real number ϑ such that $t_{i+1} - t_i = \vartheta_i \leq \vartheta$, $i \in \mathbf{Z}$. For every $t \in \mathbf{R}$, let $i = i(t) \in \mathbf{Z}$ be the unique integer such that $t \in I_i = [t_i, t_{i+1})$. To simplify notations, we will denote by Θ that is a space of strictly ordered sequences $\{t_i\} \in \mathbf{R}$, $i \in \mathbf{Z}$, such that $|t_i| \rightarrow \infty$ if $|i| \rightarrow \infty$.

We study, in this paper, the existence of minimal and maximal solutions for the differential equation with piecewise constant generalized delay of the form

$$(2.1) \quad x'(t) = f(t, x(t), x(\beta(t))), \quad x(\tau) = c_0,$$

where $\beta(t) = t_i$ if $t \in I_i = [t_i, t_{i+1})$, $i \in \mathbf{Z}$, is the identification function and f is continuous on $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

We introduce the following definition.

Definition 1. A function x is a solution of the DEPCAG (2.1) in \mathbf{R} if i) x is continuous on \mathbf{R} .

ii) The derivative $x'(t)$ exists at each point $t \in \mathbf{R}$ with the possible exception of the points $t_i \in \mathbf{R}$, $i \in \mathbf{Z}$, where the one-side derivatives exist.

iii) The DEPCAG (2.1) is satisfied for x on each interval (t_i, t_{i+1}) , $i \in \mathbf{Z}$, and it holds for the right derivative at the points t_i , $i \in \mathbf{Z}$.

We will use, in the discussion of the problem, several properties of the function $\lambda_{a,b}(t) = \lambda(t) : [\tau, \infty) \rightarrow \mathbf{R}$ defined by

$$(2.2) \quad \lambda_{a,b}(t) = \lambda(t) = e^{at} + \frac{b}{a}(e^{at} - 1).$$

It is easy to verify that $\lambda'_{a,b}(t) = (a+b)e^{at}$ for all $t \in [\tau, \infty)$ and, as an immediate consequence, is strictly monotone increasing on \mathbf{R} for $a+b > 0$ and strictly monotone decreasing on \mathbf{R} if $a+b < 0$. Obviously, when $a = -b$, is a constant function equal to 1. Clearly, for all $a, b \in \mathbf{R}$, we have that $\lambda(0) = 1$.

3. The linear problem: comparison results

In this section we analyze the following linear differential equation with a general piecewise constant delay

$$(3.1) \quad y'(t) = ay(t) + by(\beta(t)) + g(t), \quad y(\tau) = c_0,$$

where $\tau \in \Theta$, a and b are real constants and $g(t)$ is continuous function on $[\tau, \infty)$. At a first moment we prove that this problem has a unique solution.

Theorem 1. *The linear initial value problem with the DEPCAG (3.1) has on $[\tau, \infty)$ a unique solution given by*

$$(3.2) \quad y(t) = \mu(t)(c_0\mu_{i(\tau)}^{i(t)-1} + S) + \int_{\beta(t)}^t e^{a \cdot (t-s)} g(s) ds,$$

where

$$S = \sum_{j=i(\tau)}^{i(t)-1} \mu_j^{i(t)-1} \int_{t_j}^{t_{j+1}} e^{a \cdot (t_{j+1}-s)} g(s) ds,$$

$g(t)$ is continuous on $[\tau, \infty)$, $\tau \in \Theta$, and

$$\mu(t) = \lambda(t - \beta(t)), \quad \mu_j^i = \prod_{k=j}^i \lambda(t_{k+1} - t_k) = \prod_{k=j}^i \lambda(\vartheta_k), \quad \text{and} \quad i \geq j > i(\tau).$$

Proof. Let $y_n(t)$ be solutions of the DEPCAG (3.1) in the interval $[t_n, t_{n+1})$, satisfying the condition $y_n(t_n) = y_n$, for $n \in \{i(\tau) + j\}_{j \in \mathbf{N}}$. Then we have

$$(3.3) \quad y_n(t) = y_n \lambda(t - t_n) + \int_{t_n}^t e^{a \cdot (t-s)} g(s) ds,$$

where, λ is given in (2.2). From (3.3) and $t \rightarrow t_{n+1}$, we obtain

$$(3.4) \quad y_{n+1} = y_n \lambda(\vartheta_n) + \int_{t_n}^{t_{n+1}} e^{a \cdot (t_{n+1}-s)} g(s) ds.$$

Applying (3.4) repeatedly for y_{n-1} , y_{n-2} , ..., $y_{i(\tau)+1}$ yields

$$(3.5) \quad \begin{aligned} y_{n+1} &= \left(\prod_{j=i(\tau)+1}^n \lambda(\vartheta_j) \right) y_{i(\tau)+1} \\ &+ \sum_{j=i(\tau)+1}^n \prod_{k=j+1}^n \lambda(\vartheta_k) \times \int_{t_j}^{t_{j+1}} e^{a \cdot (t_{j+1}-s)} g(s) ds \\ &+ \int_{t_n}^{t_{n+1}} e^{a \cdot (t_{n+1}-s)} g(s) ds. \end{aligned}$$

From (3.3), one gets

$$(3.6) \quad y(t_{i(\tau)+1}) = y(\tau)\lambda(t_{i(\tau)+1} - \tau) + \int_{\tau}^{t_{i(\tau)+1}} e^{a \cdot (t_{i(\tau)+1} - s)} g(s) ds.$$

Use (3.3), (3.5), (3.6) and $\tau \in \Theta$ to obtain finally

$$(3.7) \quad \begin{aligned} y_n(t) &= \lambda(t - t_n) \left(\prod_{j=i(\tau)}^{n-1} \lambda(\vartheta_j) \right) y(\tau) \\ &+ \lambda(t - t_n) \left(\prod_{j=i(\tau)}^{n-1} \lambda(\vartheta_j) \right) \int_{\tau}^{t_{i(\tau)}} e^{a \cdot (\tau - s)} g(s) ds \\ &+ \lambda(t - t_n) \sum_{j=i(\tau)}^{n-1} \prod_{k=j+1}^{n-1} \lambda(\vartheta_k) \int_{t_j}^{t_{j+1}} e^{a \cdot (t_{j+1} - s)} g(s) ds \\ &+ \int_{t_n}^t e^{a \cdot (t_n - s)} g(s) ds, \end{aligned}$$

from where (3.2) follows. \square

Note that (3.2) is obtained with the implicit assumption $a \neq 0$. If $a = 0$, then the unique solution of (3.1) is given by

$$(3.8) \quad y(t) = (1 + b(t - \beta(t))) \left\{ c_0 \prod_{j=i(\tau)}^{i(t)-1} (1 + b\vartheta_j) + \hat{S} \right\} + \int_{\beta(t)}^t g(s) ds,$$

where

$$\hat{S} = \sum_{j=i(\tau)}^{i(t)-1} \left(\prod_{k=j}^{i(t)-1} (1 + b\vartheta_k) \right) \int_{t_j}^{t_{j+1}} g(s) ds,$$

which is the limiting case of (3.2) as $a \rightarrow 0$.

Remark 1. Theorem 1 generalizes the linear differential equation with a piecewise constant delay considered by Aftabizadeh and Wiener [1, pp.190] with $\beta(t) = [t]$.

In order to obtain existence results for the DEPCAG (2.1) we will use monotone iterative techniques. It is very well known that a fundamental tool to treat this kind of problems consists in maximum principles of the linear operator studied above. From the form of the solution of the linear DEPCAG (3.1) we can deduce the following.

Lemma 1. Suppose that $y \in C[[\tau, \infty), \mathbf{R}]$, and the derivative $y'(t)$ exists at each point $t \in [\tau, \infty)$, with the possible exception of the points $t_i \in [\tau, \infty)$, $i \in \mathbf{Z}$, where one-sided derivatives exist. Assume that

$$(3.9) \quad y'(t) \leq ay(t) + by(\beta(t)), \quad y(\tau) \leq 0,$$

where a and b are constants such that

$$(3.10) \quad b \geq -\frac{ae^{a\vartheta_j}}{e^{a\vartheta_j} - 1}, \quad j \geq i(\tau).$$

Then $y(t) \leq 0$ on $[\tau, \infty)$.

Proof. For $t \in [t_n, t_{n+1})$, for $n \geq i(\tau)$, consider

$$y'_n(t) \leq ay_n(t) + by_n(t_n), \quad y_n(t_n) \leq 0.$$

From this and (3.10) we find that $y_n(t) \leq 0$, $t \in [t_n, t_{n+1})$. This implies, because of the continuity, $y(t) \leq 0$ for $t \geq \tau$. \square

Remark 2. Lemma 1 generalizes the corresponding inequality considered by Aftabizadeh and Wiener [1, Lemma 3.2] with $\beta(t) = [t]$.

Note that if $a = 0$, from the proof of Lemma 1, we can obtain the following Corollary.

Corollary 1. Suppose that $y \in C[[\tau, \infty), \mathbf{R}]$, and the derivative $y'(t)$ exists at each point $t \in [\tau, \infty)$, with the possible exception of the points $t_i \in [\tau, \infty)$, $i \in \mathbf{Z}$, where one-sided derivatives exist. Assume that

$$(3.11) \quad y'(t) \leq by(\beta(t)), \quad y(\tau) \leq 0,$$

where b is a real constant such that

$$(3.12) \quad b \geq -\frac{1}{\vartheta_j}, \quad j \geq i(\tau).$$

Then $y(t) \leq 0$ on $[\tau, \infty)$.

4. Minimal and maximal solutions

In this section we apply the monotone method to prove the existence of minimal and maximal solutions for DEPCAG, so we recall the DEPCAG (2.1):

$$(4.1) \quad x'(t) = f(t, x(t), x(\beta(t))), \quad x(\tau) = c_0,$$

where $\tau \in \Theta$ and $f \in C[[\tau, \infty) \times \mathbf{R} \times \mathbf{R}, \mathbf{R}]$. This constructive method yields monotone sequences that converge to solutions of (4.1). Since each member of these sequences happens to be the solution of a linear delay differential equation which can be computed explicitly, the advantage and importance of the technique need no special emphasis. For more details of the monotone iterative technique, the reader should refer to [4, 23].

We introduce the following definitions.

Definition 2. We say that $\underline{\alpha}(t)$ is a lower solution of the DEPCAG (4.1) if the following inequalities hold

$$\underline{\alpha}(t) \leq f(t, \underline{\alpha}(t), \underline{\alpha}(\beta(t))), \quad \underline{\alpha}(\tau) \leq c_0.$$

In an analogous way, we say that $\overline{\alpha}(t)$ is an upper solution of the DEPCAG (4.1) if the following inequalities hold

$$\overline{\alpha}(t) \geq f(t, \overline{\alpha}(t), \overline{\alpha}(\beta(t))), \quad \overline{\alpha}(\tau) \geq c_0.$$

Theorem 2. Let $\underline{\alpha}(t)$ and $\overline{\alpha}(t)$ be lower and upper solutions of the DEPCAG (4.1) such that $\underline{\alpha}(t) \leq \overline{\alpha}(t)$ on $I = [\tau, \infty)$. Suppose that

$$(4.2) \quad f(t, u, v) - f(t, x, y) \geq a \cdot (u - x) + b \cdot (v - y), \quad t \geq \tau,$$

for $\underline{\alpha}(t) \leq x(t) \leq u(t) \leq \overline{\alpha}(t)$, $\underline{\alpha}(t) \leq y(t) \leq v(t) \leq \overline{\alpha}(t)$, and

$$(4.3) \quad b \geq -\frac{ae^{a\vartheta_j}}{e^{a\vartheta_j} - 1}, \quad j \geq i(\tau).$$

Then there exist monotone sequences $\{\underline{\alpha}_m(t)\}$ and $\{\overline{\alpha}_m(t)\}$ with $\underline{\alpha}_0(t) = \underline{\alpha}(t)$, $\overline{\alpha}_0(t) = \overline{\alpha}(t)$ such that $\underline{\alpha}_m(t) \rightarrow \underline{\alpha}_*(t)$, $\overline{\alpha}_m(t) \rightarrow \overline{\alpha}_*(t)$ as $m \rightarrow \infty$ monotonically on I , and $\overline{\alpha}_*(t)$, $\underline{\alpha}_*(t)$ are minimal and maximal solutions of the DEPCAG (4.1).

Proof. For any $\eta \in C[[\tau, \infty), \mathbf{R}]$ such that $\underline{\alpha}(t) \leq \eta(t) \leq \overline{\alpha}(t)$, we consider the linear DEPCAG (3.1) with

$$g(t) = f(t, \eta(t), \eta(\beta(t))) - a\eta(t) - b\eta(\beta(t)),$$

i.e.,

$$(4.4) \quad y'(t) = ay(t) + by(\beta(t)) + f(t, \eta(t), \eta(\beta(t))) - a\eta(t) - b\eta(\beta(t)),$$

with $y(\tau) = c_0$. It is clear that for every such η , by Theorem 1, there exists a unique solution $y(t)$ of the linear DEPCAG (4.4) on $[\tau, \infty)$. Define a mapping \mathfrak{S} by $\mathfrak{S}\eta = y$, where y is the unique solution of the linear DEPCAG (4.4). This mapping will be used to define the sequences $\{\underline{\alpha}_m(t)\}$ and $\{\overline{\alpha}_m(t)\}$. We complete the proof by four steps:

Step 1. We claim that $\underline{\alpha} \leq \mathfrak{S}\underline{\alpha}$ and $\mathfrak{S}\overline{\alpha} \leq \overline{\alpha}$.

Let $\underline{\alpha}_1 = \mathfrak{S}\underline{\alpha}$, then $\underline{\alpha}_1$ satisfies

$$\underline{\alpha}'_1(t) - a\underline{\alpha}_1(t) - b\underline{\alpha}_1(\beta(t)) = f(t, \underline{\alpha}(t), \underline{\alpha}(\beta(t))) - a\underline{\alpha}(t) - b\underline{\alpha}(\beta(t)).$$

Let

$$p(t) = \underline{\alpha}(t) - \underline{\alpha}_1(t).$$

On each unit interval $[t_n, t_{n+1})$, $n \geq i(\tau)$, we have

$$p_n(t) = \underline{\alpha}_n(t) - \underline{\alpha}_{n1}(t),$$

where $\underline{\alpha}_{n1}(t)$ is the solution of the DEPCAG (4.4) on $[t_n, t_{n+1})$ with $\eta = \underline{\alpha}_n$ and $\underline{\alpha}_{n1}(t_n) = \hat{c}_n$. Then from $p'_n(t) = \underline{\alpha}'_n(t) - \underline{\alpha}'_{n1}(t)$ and the fact that $\underline{\alpha}_n(t)$ is the lower solution of the DEPCAG (4.1), i.e.,

$$\underline{\alpha}'_n(t) \leq f(t, \underline{\alpha}_n(t), \underline{\alpha}_n(t_n)),$$

we have

$$p'_n(t) \leq ap_n(t) + bp_n(t_n), \quad p_n(t_n) \leq 0.$$

From (4.3) and Lemma 1, we obtain $p_n(t) \leq 0$ for $[t_n, t_{n+1})$, $n \geq i(\tau)$, and thus $p(t) \leq 0$ on $[\tau, \infty)$, which shows that $\underline{\alpha} \leq \mathfrak{S}\underline{\alpha}$. Similarly, we can show that $\mathfrak{S}\overline{\alpha} \leq \overline{\alpha}$.

Step 2. We show that $\mathfrak{S}\eta_1 \leq \mathfrak{S}\eta_2$ if $\underline{\alpha} \leq \eta_1 \leq \eta_2 \leq \bar{\alpha}$.

Let $\eta_1^* = \mathfrak{S}\eta_1$, $\eta_2^* = \mathfrak{S}\eta_2$ and $p = \eta_1^* - \eta_2^*$, then for $[t_n, t_{n+1})$, and by (4.2), we obtain

$$\begin{aligned}
 p'_n(t) &= f(t, \eta_1(t), \eta_1(\beta(t))) - f(t, \eta_2(t), \eta_2(\beta(t))) \\
 &\quad + a \cdot (\eta_1^*(t) - \eta_1(t)) - a \cdot (\eta_2^*(t) - \eta_2(t)) \\
 &\quad + b \cdot (\eta_1^*(\beta(t)) - \eta_1(\beta(t))) - b \cdot (\eta_2^*(\beta(t)) - \eta_2(\beta(t))) \\
 &\leq -a \cdot (\eta_2(t) - \eta_1(t)) - b \cdot (\eta_2(\beta(t)) - \eta_1(\beta(t))) \\
 &\quad + a \cdot (\eta_1^*(t) - \eta_1(t)) - a \cdot (\eta_2^*(t) - \eta_2(t)) \\
 &\quad + b \cdot (\eta_1^*(\beta(t)) - \eta_1(\beta(t))) - b \cdot (\eta_2^*(\beta(t)) - \eta_2(\beta(t))) \\
 &\leq a \cdot (\eta_1^*(t) - \eta_2^*(t)) + b \cdot (\eta_1^*(\beta(t)) - \eta_2^*(\beta(t))) = ap_n(t) + bp_n(t_n),
 \end{aligned}
 \tag{4.5}$$

with $p_n(t_n) \leq 0$. Still by (4.3) and Lemma 1, we obtain $p_n(t) \leq 0$ for $[t_n, t_{n+1})$, $n \geq i(\tau)$, and thus $p(t) \leq 0$ on $[\tau, \infty)$, which implies $\mathfrak{S}\eta_1 \leq \mathfrak{S}\eta_2$.

Step 3. We prove that the DEPCAG (4.1) have solutions.

Let $\underline{\alpha}_i = \mathfrak{S}\underline{\alpha}_{i-1}$, $\bar{\alpha}_i = \mathfrak{S}\bar{\alpha}_{i-1}$, $n = 1, 2, 3, \dots$. Following the first two s, we have

$$\underline{\alpha}(t) = \underline{\alpha}_0(t) \leq \underline{\alpha}_1(t) \leq \dots \leq \underline{\alpha}_n(t) \leq \dots \leq \bar{\alpha}_n(t) \leq \dots \leq \bar{\alpha}_1(t) \leq \bar{\alpha}_0(t) = \bar{\alpha}(t).$$

Obviously, each $\underline{\alpha}_i$, $\bar{\alpha}_i$ ($i = 1, 2, 3, \dots$) satisfies

$$\underline{\alpha}'_i(t) - a\underline{\alpha}_i(t) - b\underline{\alpha}_i(\beta(t)) = f(t, \underline{\alpha}_{i-1}(t), \underline{\alpha}_{i-1}(\beta(t))) - a\underline{\alpha}_{i-1}(t) - b\underline{\alpha}_{i-1}(\beta(t)),$$

$$\bar{\alpha}'_i(t) - a\bar{\alpha}_i(t) - b\bar{\alpha}_i(\beta(t)) = f(t, \bar{\alpha}_{i-1}(t), \bar{\alpha}_{i-1}(\beta(t))) - a\bar{\alpha}_{i-1}(t) - b\bar{\alpha}_{i-1}(\beta(t)),$$

with $\underline{\alpha}(\tau) = \bar{\alpha}(\tau) = c_0$. Therefore there exist $\underline{\alpha}_*$ and $\bar{\alpha}_*$ such that

$$\lim_{i \rightarrow \infty} \underline{\alpha}_i = \underline{\alpha}_*(t), \quad \lim_{i \rightarrow \infty} \bar{\alpha}_i = \bar{\alpha}_*(t)$$

uniformly on $[\tau, \infty)$. Clearly, $\underline{\alpha}_*$ and $\bar{\alpha}_*$ satisfy the DEPCAG (4.1).

Step 4. We prove $\underline{\alpha}_*$ and $\bar{\alpha}_*$ are extreme solutions of the DEPCAG (4.1). Let $y(t)$ be any solution of the DEPCAG (4.1), which satisfies $\underline{\alpha}(t) \leq y(t) \leq \bar{\alpha}(t)$, $t \in [\tau, \infty)$. Also suppose there exists a positive integer n such that for $t \in [\tau, \infty)$, $\underline{\alpha}_n(t) \leq y(t) \leq \bar{\alpha}_n(t)$.

Setting $p(t) = \underline{\alpha}_{n+1}(t) - y(t)$, then for $t \in [\tau, \infty)$, and by (4.2), we obtain

$$\begin{aligned}
 p'(t) &= f(t, \underline{\alpha}_n(t), \underline{\alpha}_n(\beta(t))) + a \cdot (\underline{\alpha}_{n+1}(t) - \underline{\alpha}_n(t)) \\
 &\quad + b \cdot (\underline{\alpha}_{n+1}(\beta(t)) - \underline{\alpha}_n(\beta(t))) - f(t, y(t), y(\beta(t))) \\
 (4.6) \quad &= a \cdot (\underline{\alpha}_{n+1}(t) - y(t)) + b \cdot (\underline{\alpha}_{n+1}(\beta(t)) - y(\beta(t))) \\
 &\quad + f(t, \underline{\alpha}_n(t), \underline{\alpha}_n(\beta(t))) - f(t, y(t), y(\beta(t))) \\
 &\quad - a \cdot (\underline{\alpha}_n(t) - y(t)) - b \cdot (\underline{\alpha}_n(\beta(t)) - y(\beta(t))) \\
 &\leq ap(t) + bp(\beta(t)).
 \end{aligned}$$

By (4.3) and Lemma 1, we have for all $t \in [\tau, \infty)$, $p(t) \leq 0$, i.e., $\underline{\alpha}_{n+1} \leq y(t)$. Similarly, we can prove $y(t) \leq \bar{\alpha}_{n+1}$, $t \in [\tau, \infty)$. Thus $\underline{\alpha}_{n+1} \leq y(t) \leq \bar{\alpha}_{n+1}$, for all $t \in [\tau, \infty)$, which implies $\underline{\alpha}_*(t) \leq y(t) \leq \bar{\alpha}_*(t)$. We complete the proof. \square

Conclusions

In this paper, the monotone iterative technique is used to solve first-order differential equations with piecewise constant generalized delay (2.1) and some new comparison results are obtained. Two converging monotone sequences are obtained with the monotone iterative technique based on upper and lower solutions. Those two converging monotone sequences will converge to the extremal solution of the DEPCAG (2.1).

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