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Some hyperstability results of a *p*-radical functional equation related to Drygas mappings in non-Archimedean Banach spaces

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Abstract:

The aim of this paper is to introduce and solve the following p-radical functional equation related to Drygas mappings

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)+f(y)+f(-y), \quad x,y\in\mathbf{R},$$

where f is a mapping from $\mathbf{R}$ into a vector space X and $p\geq 3$ is an odd natural number. Using an analogue version of Brzdęk's fixed point theorem [12], we establish some hyperstability results for the considered equation in non-Archimedean Banach spaces. Also, we give some hyperstability results for the inhomogeneous p-radical functional equation related to Drygas mappings

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)=2f(x)+f(y)+f(-y)+G(x,y).$$

Keywords: Hyperstability; non-Archimedean Banach spaces; Radical functional equations; Drygas functional equations.

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1. Introduction

A classical question in the theory of functional equation is the following: *"When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of this equation."*
If the answer is affirmative, then we say that equation is stable. In 1940, S. M. Ulam [37]) asked the following question concerning the stability of group homomorphisms
Let $(G_1, *_1)$ be a group and let $(G_2, *_2)$ be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d\Big(h(x *_1 y), h(x) *_2 h(y)\Big) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with

$$d\Big(h(x), H(x)\Big) < \varepsilon$$

for all $x \in G_1$?
This question seems to be the starting point of studying the stability of functional equations. Since then, this question has attracted the attention of many researchers. The first partial answer was raised by D. H. Hyers [*24*] in 1941 under the assumption that G_1 and G_2 are Banach spaces for the the additive functional equation as follows:

Theorem 1.1.*[24] Let E_1 and E_2 be two Banach spaces and $f : E_1 \to E_2$ be a function such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Later, T. Aoki [8] and D. G. Bourgin [9] considered the problem of stability with unbounded Cauchy differences. In 1978, Th. M. Rassias [31] attempted to weaken the condition for the bound of the norm of Cauchy difference $\|f(x+y)-f(x)-f(y)\|$ and proved a generalization of Theorem1.1 by using a direct method (cf. Theorem 1.2):

Theorem 1.2. *[31] Let E_1 and E_2 be two Banach spaces. If $f: E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta\Big(\|x\|^p + \|y\|^p\Big)$$

for some $\theta \geq 0$, for some $p \in \mathbf{R}$ with $0 \leq p < 1$, and for all $x, y \in E_1$, then there exists a unique additive function $A: E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$$

for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

After then, Th. M. Rassias [32],[33] motivated Theorem 1.2 as follows:

Theorem 1.3. *[32],[33] Let E_1 be a normed space, E_2 be a Banach space,and $f: E_1 \to E_2$ be a function. If f satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta\Big(\|x\|^p + \|y\|^p\Big) \tag{1.1}$$

for some $\theta \geq 0$, for some $p \in \mathbf{R}$ with $p = 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $A: E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2-2^p|}\|x\|^p \tag{1.2}$$

for each $x \in E_1 - \{0_{E_1}\}$.

Note that Theorem 1.3 reduces to Theorem 1.1 when $p = 0$. For $p = 1$, the analogous result is not valid. Also, J. Brzdęk [11] showed the estimation(1.2) is optimal for $p \geq 0$ in the general case.

In 1994, P. Găvruţa[23] provided a further generalization of Rassias theorem in which he replaced the bound $\theta\left(\|x\|^p + \|y\|^p\right)$ in (1.1) by a general control function $\varphi(x, y)$ for the existence of a unique linear mapping.

Recently, J. Brzdęk [15] showed that Theorem 1.3 can be significantly improved. Namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1.1) must actually be additive. This result is called the hyperstability of Cauchy functional equation. However, the term of hyperstability was introduced for the first time probably in [28] and it was developed with the fixed point theorem of Brzdęk in [12]. There after, the hyperstability of a several functional equations have been studied by many authors (see, for example, [5, 7, 2, 15, 28]).

In 2013, Brzdęk [14] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular Theorem 1.3). Over the last few years, many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [16], [17] and references therein).

Characterizing quasi-inner product spaces, H. Drygas considers in [19] the functional equation

$$(1.3)\quad f(x)+f(y)=f(x-y)+\left\{f(\frac{x+y}{2})-f(\frac{x-y}{2})\right\},\quad x,y\in \mathbf{R},$$

which can be reduced to the following equation [34, Remark 9.2, p. 131]

$$(1.4)\qquad f(x+y)+f(x-y)=2f(x)+f(y)+f(-y),\quad x,y\in \mathbf{R}.$$

This equation is known in the literature as Drygas equation and is a generalization of the quadratic functional equation f(x+y)+f(x-y)=2f(x)+2f(y), x,y∈ $\mathbf{R}$. The general solution of Drygas equation was given by B. R. Ebanks, P. L. Kannappan and P. K. Sahoo in[20]. It has the form f(x)=A(x)+Q(x), x∈ $\mathbf{R}$, where $A\colon \mathbf{R} \to \mathbf{R}$ is an additive function and $Q\colon \mathbf{R} \to \mathbf{R}$ is a quadratic function, see also [25]. A set-valued version of Drygas equa- tion was considered by W. Smajdor in [36].
Recently, the hyperstability of the Drygas functional equation has been studied in [29], [35] and [6].

During the 16^{th} International Conference on Functional Equations and Inequalities (Będlewo, Poland, May 17-23, 2015), W. Sintunavarat presented a talk concerning the Ulam type stability (for information and further references concerning this notion see, e.g., [10]) of the so-called radical functional equation

$$f\left(\sqrt{x^2+y^2}\right)=f(x)+f(y)$$

in the class of real functions. A question of J. Schwaiger about the general solution of the equation was answered a bit later by the author of this paper (see [30], p. 196). In this regard, many papers concerning the solutions and stability of radical functional equations have been established (the readercan refer, for example, to [1, 2, 4, 21, 22, 26, 27]).

Let us recall (see, for instance, [26]) some basic definitions and factsconcerning non-Archimedean normed spaces.

Definition 1.4. *By a non-Archimedean field we mean a field* $\mathbf{K}$ *equipped with a function (valuation)* $|\cdot| : \mathbf{K} \to [0, \infty)$ *such that for all* $r, s \in \mathbf{K}$*, the following conditions hold:*

1. $|r| = 0$ *if and only if* $r = 0$,
2. $|rs| = |r||s|$,
3. $|r + s| \leq \max\Big\{|r|, |s|\Big\}$.

The pair $(\mathbf{K}, |.|)$ *is called a valued field.*

In any non-Archimedean field we have $|1| = |-1| = 1$ and $|n| \leq 1$ for $n \in \mathbf{N}_0$. In any field $\mathbf{K}$ the function $|\cdot| : \mathbf{K} \to \mathbf{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x = 0, \end{cases}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings.

Definition 1.5. *Let* X *be a vector space over a scalar field* $\mathbf{K}$ *with a non-Archimedean non-trivial valuation* $|\cdot|$. *A function* $||\cdot||_* : X \to \mathbf{R}$ *is a non-Archimedean norm (valuation) if it satisfies the following conditions:*

1. $\|x\|_* = 0$ *if and only if* $x = 0$,
2. $\|rx\|_* = |r|\ \|x\|_*\ \ (r \in \mathbf{K}, x \in X)$,
3. *The strong triangle inequality (ultrametric); namely :* $\|x + y\|_* \leq \max\Big\{\|x\|_*, \|y\|_*\Big\}\ \ x, y \in X$.

Then $(X, \|\cdot\|_*)$ is called a non-Archimedean normed space or an ultrametric normed space.

Definition 1.6. *Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.*

1. *A sequence$\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a Cauchy sequence if the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero;*
2. *The sequence $\{x_n\}$ is said to be convergent if, there exists $x \in X$ such that, for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x\|_* \leq \varepsilon$, for all $n \geq N$. Then the point $x \in X$ is called the limit of the sequence $\{x_n\}$, which is denoted by $lim_{n\to\infty} x_n = x$;*
3. *If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space or an non-Archimedean Banach space.*

Definition 1.7. *(Linear space)[3] Let $\mathcal{K}$ be an arbitrary field (such as the field of real numbers or of complex numbers, for concrete examples). A nonempty set V of elements $x, y, z, \ldots$ of an arbitrary nature, together with an operation, called vector addition, or simply addition, associating with any two elements $x, y \in V$ an element $z \in V$, called the sum of x and y, and denoted by $z = x + y$, as well as an operation associating with any $x \in V$ and $a \in \mathcal{K}$ an element $w \in V$, called the product or scalar product of a and x, and denoted by $w = a \cdot x$, is called a linear space if*

1. *For all $x, y, z \in V$, the operation of vector addition satisfies:*
   - *Associativity, i.e., $(x + y) + z = x + (y + z)$.*
   - *Commutativity, i.e., $x + y = y + x$.*
   - *Existence of a neutral element, i.e., there exists an element $0_V \in V$, for which $x + 0_V = x = 0_V + x$.*
   - *Existence of additive inverses, i.e., there exists an element $x' \in V$ such that $x + x' = 0_V = x' + x$.*
2. *For all $x \in V$ and $\alpha, \beta \in \mathcal{K}$, the scalar product operation satisfies:*
   - *Associativity, i.e., $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$.*
   - *Neutrality of $1_V \in \mathcal{K}$, i.e., $11_V \cdot x = x$.*

3. *For all $x, y \in V$ and $\alpha, \beta \in \mathcal{K}$, the scalar product and vector addition operations are compatible in the sense that*
   - *Scalar product distributes over vector addition* $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$.
   - *Scalar product distributes over scalar addition* $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.

A linear space is also called a vector space, its elements are called vectors, and the elements of the field $\mathcal{K}$ are called scalars.

Throughout this paper, we will denote the set of natural numbers by $\mathbf{N}$, $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, the set of real numbers by $\mathbf{R}$, $\mathbf{R}_+ = [0, \infty)$ the set of non negative real numbers and $\mathbf{R}_0 = \mathbf{R} \backslash \{0\}$. By $\mathbf{N}_{m0}$, $m_0 \in \mathbf{N}$, we will denote the set of all natural numbers greater than or equal to m_0.

Let X be a linear space and let $p \in \mathbf{N}_3$ be an odd natural number. We introduce the following functional equation

$$(1.5)\ f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = 2f(x) + f(y) + f(-y) \quad x, y \in \mathbf{R},$$

where $f : \mathbf{R} \to X$ which is called p-radical functional equation related to Drygas equation (1.4).

The main purpose of this paper is to achieve the general solution of the functional equation (1.5) and establish some hyperstability results for the considered equation in non-Archimedean Banach space. We also provide some corollaries and outcomes concerning the hyperstability results for the inhomogeneous of p-radical functional equation.

Before proceeding to the main results, we state Theorem 1.8 which is useful for our purpose. To present it, we introduce the following three hypotheses:

(**H1**) X is a nonempty set, Y is an non-Archimedean Banach space over a non-Archimedean field, $f_1, ..., f_k : X \longrightarrow X$ and $L_1, ..., L_k : X \longrightarrow \mathbf{R}_+$ are given.

(**H2**) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\left\| \mathcal{T}\xi(x) - \mathcal{T}\mu(x) \right\|_* \leq \max_{1 \leq i \leq k} \left\{ L_i(x) \left\| \xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big) \right\|_* \right\},$$

$$\xi, \mu \in Y^X, \quad x \in X.$$

(**H3**) $\Lambda : \mathbf{R}_+^X \longrightarrow \mathbf{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \max_{1 \leq i \leq k} \Big\{ L_i(x)\delta\Big(f_i(x)\Big)\Big\}, \qquad \delta \in \mathbf{R}_+^X, \quad x \in X.$$

Thanks to a result due to J. Brzdęk and K. Ciepliński [13, Remark 2],we state an analogue of the fixed point theorem [13, Theorem 1] in non-Archimedean Banach space. We use it to assert the existence of a uniquefixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

Theorem 1.8. *Let hypotheses (***H1***)-(***H3***) be valid and functions* $\varepsilon : X \longrightarrow \mathbf{R}_+$ *and* $\varphi : X \longrightarrow Y$ *fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\|_* \leq \varepsilon(x), \qquad x \in X,$$

$$\lim_{n\to\infty} \Lambda^n \varepsilon(x) = 0, \qquad x \in X.$$

Then there exists a unique fixed point $\psi \in Y^X$ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\|_* \leq \sup_{n\in\mathbf{N}_0} \Lambda^n \varepsilon(x), \qquad x \in X.$$

Moreover

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x), \qquad x \in X.$$

2. Main results

The next theorem can be derived from ([18], Corollary 2.3 and Proposition2.4(a)). However, for the convenience of readers we present it with a directproof.

Theorem 2.1. *Let* Y *be a linear space and* $p \in \mathbf{N}_3$ *be an odd natural number. A function* $f : \mathbf{R} \to Y$ *satisfies the functional equation (1.5) if and only if*

$$f(x) = D(x^p), \quad x \in \mathbf{R} \tag{2.1}$$

where $D : \mathbf{R} \to Y$ *is a Drygas function.*

Proof. First, if $f:\mathbf{R}\to Y$ satisfies (2.1) for all $x\in\mathbf{R}$, then, for each $x,y\in\mathbf{R}$, we get that

$$\begin{aligned} f(\sqrt[p]{x^p+y^p})+f(\sqrt[p]{x^p-y^p}) &= D(x^p+y^p)+D(x^p-y^p)\\ &= 2D(x^p)+D(y^p)+D((-y)^p) \quad \text{which means}\\ &= 2f(x)+f(y)+f(-y)\end{aligned}$$

that f is a solution of equation (1.5). Also, if $f:\mathbf{R}\to Y$ is a solution of (1.5), then we can write $D(x)=f(\sqrt[p]{x})$ for all $x\in\mathbf{R}$ to find that

$$\begin{aligned} D(x+y)+D(x-y) &= f(\sqrt[p]{x+y})+f(\sqrt[p]{x-y})\\ &= 2f(\sqrt[p]{x})+f(\sqrt[p]{y})+f(\sqrt[p]{-y})\\ &= 2D(x)+D(y)+D(-y),\end{aligned}$$

for all $x,y\in\mathbf{R}$. □

Next, we examine the hyperstability of the equation (1.5) in non-Archimedean Banach space by using, as a basic tool, the fixed point Theorem 1.8.

Theorem 2.2. *Let p be an odd natural number, $(X\ ,\ \|.\|_*)$ be a non-Archimedean Banach space and let $h_1,h_2:\mathbf{R}_0\to\mathbf{R}_+$ be two functions such that*

$$\mathcal{U}: = \Big\{n\in\mathbf{N}:\alpha_n=\max\{\lambda_1(n+1)\lambda_2(n+1)\ ,\ \lambda_1(2n+1)\lambda_2(2n+1)$$

$$\lambda_1(n)\lambda_2(n)\lambda_1(-n)\lambda_2(-n)\ \}<1\Big\}\neq\phi,$$

where

$$\lambda_i(m):=\inf\{t\in\mathbf{R}_+ : h_i(mx)\le t\,h_i(x),\ \ x\in\mathbf{R}_0\},$$

for all $m\in\mathbf{N}$, where $i=1,2$ such that

$$\lim_{m\to\infty}\lambda_1(m+1)\lambda_2(m)=0. \tag{2.2}$$

Assume that $f:\mathbf{R}\to X$ satisfies the inequality

$$\left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)\right\|_*\le h_1(x^p)h_2(y^p), \tag{2.3}$$

for all $x,y\in\mathbf{R}_0$ such that $x\neq y$ and $x\neq -y$. Then f is a solution of the equation (1.5) on $\mathbf{R}_0$.

Proof. Replacing x by $\sqrt[p]{m+1}\,x$ and y by $\sqrt[p]{m}\,x$ in the inequality (2.3), we get

$$\left\|2f\left(\sqrt[p]{m+1}\,x\right)-f\left(\sqrt[p]{2m+1}\,x\right)+f\left(\sqrt[p]{m}\,x\right)+f\left(\sqrt[p]{-m}\,x\right)-f(x)\right\|_*$$
$$\leq h_1((m+1)x^p)h_2((m)x^p), \tag{2.4}$$

for all $x \in \mathbf{R}_0$. For each $m \in \mathbf{N}$, we define the operator $\mathcal{T}_m : X^{\mathbf{R}_0} \to X^{\mathbf{R}_0}$ by

$\mathrm{T}_m\xi(x) := 2\xi\left(\sqrt[p]{m+1}\,x\right)-\xi\left(\sqrt[p]{2m+1}\,x\right)+\xi\left(\sqrt[p]{m}\,x\right)+\xi\left(\sqrt[p]{-m}\,x\right),$
for all $\xi \in X^{\mathbf{R}_0}$, $x \in \mathbf{R}_0$ and the function $\varepsilon_m : \mathbf{R}_0 \to \mathbf{R}_+$ by

$$\varepsilon_m(x) := h_1((m+1)x^p)h_2((m)x^p), \quad m \in \mathbf{N}, \quad x \in \mathbf{R}_0.$$

We observe that

$$\varepsilon_m(x) \leq \lambda_1(m+1)\lambda_2(m)h_1(x^p)h_2(x^p), \tag{2.5}$$

for all $x \in \mathbf{R}_0$ and all $m \in \mathcal{U}$. Then the inequality (2.4) become as

$$\left\|\mathcal{T}_m f(x) - f(x)\right\|_* \leq \varepsilon_m(x), \quad x \in \mathbf{R}_0.$$

Furthermore, the operator $\Lambda_m : \mathbf{R}_+^{\mathbf{R}_0} \to \mathbf{R}_+^{\mathbf{R}_0}$ defined by

$$\Lambda_m\delta(x) := \max_{1\leq i\leq 4}\{L_i(x)\delta\left(f_i(x)\right\},$$

for all $x \in \mathbf{R}_0$ and all $\delta \in \mathbf{R}_+^{\mathbf{R}_0}$ where $f_1(x) = \sqrt[p]{m+1}$
$f_2(x) = \sqrt[p]{2m+1}\,x$, $f_3(x) = \sqrt[p]{m}\,x$, $f_4(x) = \sqrt[p]{-m}\,x$, and
$L_1(x) = L_2(x) = L_3(x) = L_4(x) = 1.$

Moreover, for every $x \in \mathbf{R}_0$, $\xi, \mu \in X^{\mathbf{R}_0}$, we obtain

$$\begin{aligned}
&\Big\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x)\Big\|_*\\
&=\Big\|2\left(\xi\left(\sqrt[p]{m+1}\,x\right)-\mu\left(\sqrt[p]{m+1}\,x\right)\right)-\left(\xi\left(\sqrt[p]{2m+1}\,x\right)\right.\\
&\quad\left.-\mu\left(\sqrt[p]{2m+1}\,x\right)+\left(\xi\left(\sqrt[p]{m}\,x\right)-\mu\left(\sqrt[p]{m}\,x\right)\right)+\left(\xi\left(\sqrt[p]{-m}\,x\right)-\mu\left(\sqrt[p]{-m}\,x\right)\right)\right\|_*\\
&\le\max\Big\{2\Big\|\xi\left(\sqrt[p]{2m+1}\,x\right)-\mu\left(\sqrt[p]{2m+1}\,x\right)\Big\|_*,\\
&\quad\Big\|\xi\left(\sqrt[p]{m+1}\,x\right)-\mu\left(\sqrt[p]{m+1}\,x\right)\Big\|_*,\Big\|\xi\left(\sqrt[p]{m}\,x\right)-\mu\left(\sqrt[p]{m}\,x\right)\Big\|_*\\
&\quad\Big\|\xi\left(\sqrt[p]{-m}\,x\right)-\mu\left(\sqrt[p]{-m}\,x\right)\Big\|_*\Big\}\\
&\le\max\Big\{\Big\|\xi\left(\sqrt[p]{2m+1}\,x\right)-\mu\left(\sqrt[p]{2m+1}\,x\right)\Big\|_*,\ \Big\|\xi\left(\sqrt[p]{m+1}\,x\right)\\
&\quad-\mu\left(\sqrt[p]{m+1}\,x\right)\Big\|_*,\Big\|\xi\left(\sqrt[p]{m}\,x\right)-\mu\left(\sqrt[p]{m}\,x\right)\Big\|_*,\ \Big\|\xi\left(\sqrt[p]{-m}\,x\right)-\mu\left(\sqrt[p]{-m}\,x\right)\Big\|_*\Big\}\\
&=\max\nolimits_{1\le i\le 4}\Big\{L_i(x)\Big\|\xi(f_i(x))-\mu(f_i(x))\Big\|_*\Big\},
\end{aligned}$$

which means that (**H2**) is valid. Now we will show, by induction on $n\in\mathbf{N}_0$, that

$$\Lambda^n\varepsilon_m(x)\le\lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x^p)h_2(x^p). \tag{2.6}$$

for all $x\in\mathbf{R}_0$ and all $m\in\mathcal{U}$ where

$$\alpha_m=\max\Big\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1)\ ,\ \lambda_1(m)\lambda_2(m)\ ,$$
$$\lambda_1(-m)\lambda_2(-m)\ \Big\}.$$

For $n=0$, the inequality (2.6) is exactly (2.5). Next we will assume that (2.6) holds for $n=k$, where $k\in\mathbf{N}$. Then

$$\begin{aligned}
&\Lambda_m^{k+1}\varepsilon_m(x)=\Lambda_m\left(\Lambda_m^k\varepsilon_m(x)\right)\\
&=\max\Big\{\Lambda_m^k\varepsilon_m\left(\sqrt[p]{m+1}\,x\right)\ ,\ \Lambda_m^k\varepsilon_m\left(\sqrt[p]{2m+1}\,x\right)\ ,\ \Lambda_m^k\varepsilon_m\left(\sqrt[p]{m}\,x\right),\\
&\quad\Lambda_m^k\varepsilon_m\left(\sqrt[p]{-m}\,x\right)\ \Big\}\\
&\le\lambda_1(m+1)\lambda_2(m)\alpha_m^k\max\Big\{h_1\Big((m+1)x^p\Big)h_2\Big((m+1)x^p\Big),\\
&\quad h_1\Big((2m+1)x^p\Big)h_2\Big((2m+1)x^p\Big),\ h_1\Big((m)x^p\Big)h_2\Big((m)x^p\Big),\\
&\quad h_1\Big((-m)x^p\Big)h_2\Big((-m)x^p\Big)\ \Big\}\\
&\le\lambda_1(m+1)\lambda_2(m)\alpha_m^k\max\Big\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1)\\
&\quad\lambda_1(m)\lambda_2(m),\lambda_1(-m)\lambda_2(-m)\ \Big\}h_1(x^p)h_2(x^p)\\
&=\lambda_1(m+1)\lambda_2(m)\alpha_m^{k+1}h_1(x^p)h_2(x^p),
\end{aligned}$$

for all $x \in \mathbf{R}_0$ and all $m \in \mathcal{U}$. It shows that (2.6) holds for $n = k+1$. We conclude that the inequality (2.6) holds for all $n \in \mathbf{N}_0$.
Since $\alpha_m < 1$ for all $m \in \mathcal{U}$, we get

$$\lim_{n\to\infty} \Lambda^n \varepsilon_m(x) = 0,$$

for all $x \in \mathbf{R}_0$. According to Theorem 1.8, there exists, for each $m \in \mathcal{U}$, a fixed point $\mathcal{F}_m : \mathbf{R}_0 \to X$ of the operator $\mathcal{T}_m$ such that

$$\begin{aligned} \left\| f(x) - \mathcal{F}_m(x) \right\|_* &\le \sup_{n\in\mathbf{N}} \left\{ \Lambda_m^n \varepsilon_m(x) \right\} \\ &\le \sup_{n\in\mathbf{N}} \left\{ \lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x^p)h_2(x^p) \right\}, \quad x \in \mathbf{R}_0. \end{aligned}$$

Moreover,

$$\mathcal{F}_m(x) = \lim_{n\to\infty} \left(\mathcal{T}_m^n f\right)(x), \qquad x \in \mathbf{R}_0.$$

Next, we should prove the following inequality

$$\left\| \mathcal{T}_m^n f\left(\sqrt[p]{x^p+y^p}\right) + \mathcal{T}_m^n f\left(\sqrt[p]{x^p-y^p}\right) - 2\mathcal{T}_m^n f(x) - \mathcal{T}_m^n f(y) - \mathcal{T}_m^n f(-y) \right\|_*$$

$$\le \alpha_m^n h_1(x^p) h_2(y^p), \tag{2.7}$$

for all $m \in \mathcal{U}$, all $x, y \in \mathbf{R}_0$ such that $x \neq y$, $x \neq -y$ and all $n \in \mathbf{N}$.

We proceed by induction that the case $n = 0$ gives us (2.3). Assume that (2.8) holds for $n = k$ where $k \in \mathbf{N}$. Then for each $m \in \mathcal{U}$ and every $x, y \in \mathbf{R}_0$ such that $x \neq y$ and $x \neq -y$, we have

$$
\begin{aligned}
&\left\|\mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p-y^p}\right)-2 \mathcal{T}_m^{k+1} f(x)-\mathcal{T}_m^{k+1} f(y)-\mathcal{T}_m^{k+1} f(-y)\right\|_* \\
&=\Big\| 2 \mathcal{T}_m^k f\left(\sqrt[p]{m+1} \sqrt[p]{x^p+y^p}\right)-\mathcal{T}_m^k f\left(\sqrt[p]{2 m+1} \sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{m} \sqrt[p]{x^p+y^p}\right) \\
&+\mathcal{T}_m^k f\left(\sqrt[p]{-m} \sqrt[p]{x^p+y^p}\right)+2 \mathcal{T}_m^k f\left(\sqrt[p]{m+1} \sqrt[p]{x^p-y^p}\right)-\mathcal{T}_m^k f\left(\sqrt[p]{2 m+1} \sqrt[p]{x^p-y^p}\right) \\
&+\mathcal{T}_m^k f\left(\sqrt[p]{m} \sqrt[p]{x^p-y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{-m} \sqrt[p]{x^p-y^p}\right)-4 \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\, x\right) \\
&+2 \mathcal{T}_m^k f\left(\sqrt[p]{2 m+1}\, x\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{m}\, x\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{-m}\, x\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\, y\right) \\
&+\mathcal{T}_m^k f\left(\sqrt[p]{2 m+1}\, y\right)-\mathcal{T}_m^k f\left(\sqrt[p]{m}\, y\right)-\mathcal{T}_m^k f\left(\sqrt[p]{-m}\, y\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\,(-y)\right) \\
&+\mathcal{T}_m^k f\left(\sqrt[p]{2 m+1}\,(-y)\right)-\mathcal{T}_m^k f\left(\sqrt[p]{m}\,(-y)\right)-\mathcal{T}_m^k f\left(\sqrt[p]{-m}\,(-y)\right) \Big\|_* \\
&\leq \max \Big\{\Big\| \mathcal{T}_m^k f\left(\sqrt[p]{m+1} \sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{m+1} \sqrt[p]{x^p-y^p}\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{m+1}\, x\right) \\
&-\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\, y\right)-\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\,(-y)\right) \Big\|_*, \\
&\Big\| \mathcal{T}_m^k f\left(\sqrt[p]{2 m+1} \sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{2 m+1} \sqrt[p]{x^p-y^p}\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{2 m+1}\, x\right) \\
&-\mathcal{T}_m^k f\left(\sqrt[p]{2 m+1}\, y\right)-\mathcal{T}_m^k f\left(\sqrt[p]{2 m+1}\,(-y)\right) \Big\|_*, \\
&\Big\| \mathcal{T}_m^k f\left(\sqrt[p]{m} \sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{m} \sqrt[p]{x^p-y^p}\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{m}\, x\right) \\
&-\mathcal{T}_m^k f\left(\sqrt[p]{m}\, y\right)-\mathcal{T}_m^k f\left(\sqrt[p]{m}\,(-y)\right) \Big\|_*, \\
&\Big\| \mathcal{T}_m^k f\left(\sqrt[p]{-m} \sqrt[p]{x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{-m} \sqrt[p]{x^p-y^p}\right)-2 \mathcal{T}_m^k f\left(\sqrt[p]{-m}\, x\right) \\
&-\mathcal{T}_m^k f\left(\sqrt[p]{-m}\, y\right)-\mathcal{T}_m^k f\left(\sqrt[p]{-m}\,(-y)\right) \Big\|_* \Big\} \\
&\leq \max \Big\{\alpha_m^k h_1\Big((m+1) x^p\Big) h_2\Big((m+1) y^p\Big),\ \alpha_m^k h_1\Big((2 m+1) x^p\Big) h_2\Big((2 m+1) y^p\Big), \\
&\alpha_m^k h_1\Big((m) x^p\Big) h_2\Big((m) y^p\Big),\ \alpha_m^k h_1\Big((-m) x^p\Big) h_2\Big((-m) y^p\Big)\Big\} \\
&\leq \alpha_m^k h_1(x^p) h_2(y^p) \max \Big\{\lambda_1(m+1) \lambda_2(m+1),\ \lambda_1(2 m+1) \lambda_2(2 m+1), \\
&\lambda_1(m) \lambda_2(m),\ \lambda_1(-m) \lambda_2(-m)\Big\} \\
&=\alpha_m^{k+1} h_1(x^p) h_2(y^p).
\end{aligned}
$$

Thus, we have shown that (2.8) holds for $n \in \mathbf{N}_0$, and all $m \in \mathcal{U}$. Letting $n \to \infty$ in (2.8), we obtain

$$\mathcal{F}_m\left(\sqrt[p]{x^p+y^p}\right)+\mathcal{F}_m\left(\sqrt[p]{x^p-y^p}\right)=2 \mathcal{F}_m(x)+\mathcal{F}_m(y)+\mathcal{F}_m(-y),$$

for all $x, y \in \mathbf{R}_0$ such that $x \neq y$, $x \neq -y$ and $m \in \mathcal{U}$. This implies that $\mathcal{F}_m : \mathbf{R} \to X$ is a solution of the equation (1.5).

Therefore, we construct a sequence $\{\mathcal{F}_m\}_{m\in\mathcal{U}}$ of the solutions of equation (1.5) on $\mathbf{R}_0$ such that

$$\begin{aligned}\left\|\mathcal{F}_m(x) - f(x)\right\|_* &\leq \sup_{n\in\mathbf{N}} \Lambda_m^n \varepsilon_m(x)\\ &\leq \sup_{n\in\mathbf{N}} \Big\{\lambda_1(m+1)\lambda_2(m)\alpha_m^n h_1(x^p)h_2(x^p)\Big\},\end{aligned}$$

for all $x \in \mathbf{R}_0$ and all $m \in \mathcal{U}$. Letting $n \to \infty$ in the previous inequality and using (2.2), we deduce that f is a solution of the equation (1.5) on $\mathbf{R}_0$ which means that the equation (1.5) is hyperstable on $\mathbf{R}_0$. □

In a similar way, we can prove the following theorem.

Theorem 2.3. *Let p be an odd natural number, $(X\ ,\ \|.\|_*)$ be a non-Archimedean Banach space and let $h : \mathbf{R}_0 \to \mathbf{R}_+$ be a mapping such that*

$$\mathcal{U} := \Big\{n \in \mathbf{N} : \alpha_n = \max\{\lambda(n+1)\ ,\ \lambda(2n+1)\ ,\ \lambda(n)\ ,\ \lambda(-n)\ \} < 1\Big\} \neq \phi,$$

where

$$\lambda(n) = \inf\left\{t \in \mathbf{R}_+ \colon h(nx) \leq t\, h(x),\ \ x \in \mathbf{R}_0\right\},$$

for all $n \in \mathbf{N}$, such that

$$\lim_{n\to\infty} \Big(\lambda(n+1) + \lambda(n)\Big) = 0.$$

Assume that $f : \mathbf{R} \to X$ satisfies the inequality

$$\left\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - f(y) - f(-y)\right\|_* \leq h(x^p) + h(y^p), \tag{2.8}$$

for all $x, y \in \mathbf{R}_0$ such that $x \neq y$ and $x \neq -y$. Then f is a solution of the equation (1.5) on $\mathbf{R}_0$.

Proof. We will suffice with the basic idea of the proof. Replacing x by $\sqrt[p]{m+1}\ x$ and y by $\sqrt[p]{m}\ x$ in the inequality (2.8) where $x \in \mathbf{R}_0$, $m \in \mathcal{U}$, we get

$$\begin{aligned}&\left\|2f\left(\sqrt[p]{m+1}\ x\right) - f\left(\sqrt[p]{2m+1}\ x\right) + f\left(\sqrt[p]{m}\ x\right) + f\left(\sqrt[p]{-m}\ x\right) - f(x)\right\|_*\\ &\leq h((m+1)x^p) + h((m)x^p)\\ &\leq \Big(\lambda(m+1) + \lambda(m)\Big)h(x^p),\end{aligned}$$

for all $m \in \mathcal{U}$ and all $x \in \mathbf{R}_0$. We define operators $\mathcal{T}_m : X^{\mathbf{R}_0} \to X^{\mathbf{R}_0}$ and $\Lambda_m : \mathbf{R}_+^{\mathbf{R}_0} \to \mathbf{R}_+^{\mathbf{R}_0}$ by

$$\mathcal{T}_m\xi(x) := 2\xi\left(\sqrt[p]{m+1}\,x\right) - \xi\left(\sqrt[p]{2m+1}\,x\right) + \xi\left(\sqrt[p]{m}\,x\right) + \xi\left(\sqrt[p]{-m}\,x\right),$$

for all $\xi \in X^{\mathbf{R}_0}$ and all $x \in \mathbf{R}_0$ and

$$\Lambda_m\delta(x) := \max\left\{\delta\left(\sqrt[p]{m+1}\,x\right), \delta\left(\sqrt[p]{2m+1}\,x\right), \delta\left(\sqrt[p]{m}\,x\right), \delta\left(\sqrt[p]{-m}\,x\right)\right\}.$$

Moreover, we write

$$\varepsilon_m(x) = h\Big((m+1)x^p\Big) + h\Big((m)x^p\Big) \leq \Big(\lambda(m+1) + \lambda(m)\Big)h(x^p), \quad x \in \mathbf{R}_0.$$

As in Theorem 2.2, we observe that inequality (2.8) takes the following form

$$\Big\|f(x) - \mathcal{T}_m(x)|\Big\|_* \leq \varepsilon_m(x), \ \ x \in \mathbf{R}_0\ ,\ m \in \mathcal{U},$$

then we complete the proof by similar steps of the proof of Theorem 2.2. □

3. Consequences

In this section, we get, as particular cases of our main results, the hyperstability results in the sense of Hyers-Ulam-Rassiass. Also, we get the same results for the inhomogeneous general p-radical functional equation

$$(3.1)\ f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) = 2f(x) + f(y) + f(-y) + G(x, y).$$

Corollary 3.1. *Let p be an odd natural number, $(X\ ,\ \|.\|_*)$ be a non-Archimedean Banach space and let $c, r, s \in \mathbf{R}$ such that $r + s < 0$ and $c \geq 0$. Assume that a function $f : \mathbf{R} \to X$ satisfies the inequality*

$$\Big\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - f(y) - f(-y)\Big\|_* \leq c|Q_1(x^p)|^r|Q_2(y^p)|^s, \tag{3.2}$$

for all $x, y \in \mathbf{R}_0$ where $Q_1, Q_2 : \mathbf{R} \to \mathbf{R}_+$ are two quadratic mappings. Then f is a solution of the equation (1.5) on $\mathbf{R}_0$.

Proof. The proof follows from Theorem 2.2 by taking $h_1, h_2 : \mathbf{R}_0 \to \mathbf{R}_+$ as follows:

$$h_1(x^p) = c_1|Q_1(x^p)|^r$$

and

$$h_2(x^p) = c_2|Q_2(x^p)|^s$$

for all $x, y \in \mathbf{R}_0$ where $c_1, c_2 \in \mathbf{R}_+$ such that $c_1c_2 = c \geq 0$. For each $m \in \mathbf{N}$, we define $\lambda_1(m)$ as in Theorem 2.2 by

$$\begin{aligned}
\lambda_1(m) &= \inf\{t \in \mathbf{R}_+ : h_1(mx^p) \leq th_1(x^p)\} \\
&= \inf\left\{t \in \mathbf{R}_+ : c_1\left|Q_1(mx^p)\right|^r \leq tc_1\left|Q_1(x^p)\right|^r\right\} \\
&= \inf\left\{t \in \mathbf{R}_+ : m^{2r}\left|Q_1(x^p)\right|^r \leq t\left|Q_1(x^p)\right|^r\right. \\
&= m^{2r},
\end{aligned}$$

for all $x \in \mathbf{R}_0$. Also, for each $m \in \mathbf{N}$, we have $\lambda_2(m) = m^{2s}$. It is clear that there exists $m_0 \in \mathbf{N}$ such that, for each $m \geq m_0$, we get

$$\begin{aligned}
\alpha_m &= \max\Big\{\lambda_1(m+1)\lambda_2(m+1)\ ,\ \lambda_1(2m+1)\lambda_2(2m+1), \\
&\quad \lambda_1(m)\lambda_2(m)\ ,\ \lambda_1(m)\lambda_2(m)\ \Big\}, \\
&= \max\Big\{(m+1)^{2(r+s)}\ ,\ \ (2m+1)^{2(r+s)}\ ,\ m^{2(r+s)}\ ,\ (-m)^{2(r+s)}\ \Big\} < 1
\end{aligned}$$

According to Theorem 2.2, there exists a unique function $\mathcal{F}_m : \mathbf{R}_0 \to X$ satisfies the equation (1.5) such that

$$\begin{aligned}
\|\mathcal{F}_m - f(x)\|_* &\leq c\ \sup_{n\in\mathbf{N}}\Big\{\lambda_1(m+1)\lambda_2(m)\alpha_m^n|Q_1(x^p)|^r|Q_2(x^p)|^s\Big\} \\
&= c(m+1)^{2r}\ m^{4s}|Q_1(x^p)|^r|Q_2(x^p)|^s \sup_{n\in\mathbf{N}}\Big\{\alpha_m^n\Big\},
\end{aligned}$$

for all $x \in \mathbf{R}_0$. On the other hand, Since $r + s < 0$, one of r, s must be negative. Assume that $r < 0$. Then

$$\lim_{m\to\infty} \lambda_1(m+1)\lambda_2(m) = \lim_{m\to\infty} m^{2(r+s)} = 0 \tag{3.3}$$

We get the desired result. $\square$

Corollary 3.2. *Let p be an odd natural number, $(X\ ,\ \|.\|_*)$ be a non-Archimedean Banach space and let $c, r \in \mathbf{R}$ such that $c \geq 0$ and $r < 0$. Assume that a function $f : \mathbf{R} \to X$ satisfies the inequality*

$$\left\|f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - f(y) - f(-y)\right\|_* \leq c\left(|Q(x^p)|^r + |Q(y^p)|^r\right), \tag{3.4}$$

for all $x, y \in \mathbf{R}_0$ where $Q : \mathbf{R} \to \mathbf{R}_+$ is a quadratic mapping. Then f is a solution of the equation (1.5) on $\mathbf{R}_0$.

Proof. The proof is similar to the proof of Corollary 3.1 with taking $h : \mathbf{R}_0 \to \mathbf{R}_+$ defined by $h(x^p) = c \left|Q(x^p)\right|^r$ for all $x \in \mathbf{R}_0$ where $c \geq 0$ and $r < 0$. $\square$

In the following corollaries, we get the hyperstability results for the inhomogeneous general p-radical functional equation related to quadratic mappings.

Corollary 3.3. *Let p be an odd natural number, $c, r, s \in \mathbf{R}$ such that $c \geq 0$ and $r + s < 0$, $(X\ ,\ \|.\|_*)$ be a non-Archimedean Banach space, $G : \mathbf{R} \times \mathbf{R} \to X$ be a function such that $G(0,0) = 0$ and let $f : \mathbf{R} \to X$ be a function such that $f(0) = 0$. Assume that f and G satisfy the inequality*

$$\left\| f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - 6f(y) - f(-y) - G(x,y) \right\|_*$$

$$\leq c\, |Q_1(x^p)|^r |Q_2(y^p)|^s, \tag{3.5}$$

for all $x, y \in \mathbf{R}_0$, where $Q_1, Q_2 : \mathbf{R} \to \mathbf{R}_+$ are two quadratic mappings. If the functional equation

$$f\left(\sqrt[p]{x^p + y^p}\right) + f\left(\sqrt[p]{x^p - y^p}\right) - 2f(x) - f(y) - f(-y) - G(x,y) = 0 \tag{3.6}$$

has a solution $f_0 : \mathbf{R} \to X$ on $\mathbf{R}_0$, then f is a solution of the equation (3.6) on $\mathbf{R}_0$.

Proof. Let $\psi : \mathbf{R} \to X$ be a function defined by $\psi(x) := f(x) - f_0(x)$ for all $x \in \mathbf{R}$. Then we get that

$$\begin{aligned}
&\left\|\psi\left(\sqrt[p]{x^p+y^p}\right)+\psi\left(\sqrt[p]{x^p-y^p}\right)-2\psi(x)-\psi(y)-\psi(-y)\right\|_*\\
&=\Big\| f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)\\
&\quad -G(x,y)-f_0\left(\sqrt[p]{x^p+y^p}\right)-f_0\left(\sqrt[p]{x^p-y^p}\right)\\
&\quad +2f_0(x)+f_0(y)+f_0(-y)+G(x,y)\Big\|_*\\
&\le \max\Big\{\left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)\right\|_*,\\
&\quad \left\|f_0\left(\sqrt[p]{x^p+y^p}\right)+f_0\left(\sqrt[p]{x^p-y^p}\right)-2f_0(x)-f_0(y)-f_0(-y)-G(x,y)\right\|_*\Big\}\\
&\le \left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)\right\|_*\\
&\le c\left|Q_1(x^p)\right|^r\left|Q_2(y^p)\right|^s,
\end{aligned}$$

for all $x, y \in \mathbf{R}_0$. By using Corollary 3.1, we deduce that ψ is a solution of equation (1.5). Moreover, for all $x, y \in \mathbf{R}_0$, we have

$$\begin{aligned}
&f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)\\
&=\psi\left(\sqrt[p]{x^p+y^p}\right)+\psi\left(\sqrt[p]{x^p-y^p}\right)-2\psi(x)-\psi(y)-\psi(-y)\\
&+f_0\left(\sqrt[p]{x^p+y^p}\right)+f_0\left(\sqrt[p]{x^p-y^p}\right)-2f_0(x)-f_0(y)-f_0(-y)-G(x,y)=0,
\end{aligned}$$

which means that f is a solution of (3.6) on $\mathbf{R}_0$. □

With an analogous proof of Corollary 3.3, we can prove the following corollary.

Corollary 3.4. *Let p be an odd natural number, $c, r \in \mathbf{R}$ such that $c \ge 0$ and $r < 0$, $(X\ ,\ \|.\|_*)$ be a non-Archimedean Banach space and let $G : \mathbf{R}\times\mathbf{R} \to X$ be a function such that $G(0,0) = 0$ and $f : \mathbf{R} \to X$ be a function such that $f(0) = 0$. Assume that f and G satisfy the inequality*

$$\begin{aligned}
&\left\|f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)\right\|_*\\
&\le c\left(|Q(x^p)|^r+|Q(y^p)|^r\right),
\end{aligned}$$

for all $x, y \in \mathbf{R}_0$, where $Q : \mathbf{R} \to \mathbf{R}_+$ is a quadratic mapping. If the functional equation

$$f\left(\sqrt[p]{x^p+y^p}\right)+f\left(\sqrt[p]{x^p-y^p}\right)-2f(x)-f(y)-f(-y)-G(x,y)=0, \tag{3.7}$$

has a solution $f_0 : \mathbf{R} \to X$ on $\mathbf{R}_0$, then f is a solution of the equation (3.7) on $\mathbf{R}_0$.

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