On the upper geodetic global domination number of a graph

X. Lenin Xaviour¹ orcid.org/0000-0001-9263-6492

S. Robinson Chellathurai²

Manonmaniam Sundaranar University, Scott Christian College, Dept. of Mathematics, Nagercoil, TN, India.

¹ leninxaviour93@gmail.com ; ² robinchel@rediffmail.com

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Abstract:

A set S of vertices in a connected graph $G = (V, E)$ is called a geodetic set if every vertex not in S lies on a shortest path between two vertices from S . A set D of vertices in G is called a dominating set of G if every vertex not in D has at least one neighbor in D . A set D is called a global dominating set in G if S is a dominating set of both G and $\bar{G}$. A set S is called a geodetic global dominating set of G if S is both geodetic and global dominating set of G . A geodetic global dominating set S in G is called a minimal geodetic global dominating set if no proper subset of S is itself a geodetic global dominating set in G . The maximum cardinality of a minimal geodetic global dominating set in G is the upper geodetic global domination number $\bar{Y}_g^+(G)$ of G . In this paper, the upper geodetic global domination number of certain connected graphs are determined and some of the general properties are studied. It is proved that for all positive integers a, b, p where $3 \leq a \leq b < p$, there exists a connected graph G such that $\bar{Y}_g(G) = a$, $\bar{Y}_g^+(G) = b$ and $|V(G)| = p$.

Keywords: Geodetic set; Dominating set; Geodetic domination; Geodetic global domination; Upper geodetic global domination number.

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1. Introduction

Throughout this paper, graph $G=(V,E)$ we mean a simple, finite, connected, undirected graph with neither loops nor multiple edges. The *order* $|V|$ and *size* $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology we refer to West[9]. A graph G is said to be *totally disconnected* if $|E(G)| = 0$. The *open neighborhood* of any vertex v in G is $N(v) = \{x : xv \in E(G)\}$ and *closed neighborhood* of a vertex v in G is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex in the graph G is denoted by $deg(v)$ and the maximum degree (minimum degree) in the graph G is denoted by $\Delta(G)$ ($\delta(G)$). For a set $S \subseteq V(G)$ the open (closed) neighborhood $N(S)$ ($N[S]$) in G is defined as $N(S) = \bigcup_{v \in S} N(v)$ ($N[S] = \bigcup_{v \in S} N[v]$). We write K_p , P_p , C_p , W_p and F_p for the *complete graph*, a *path graph*, a *cycle graph*, a *wheel graph* and a *fan graph* of order p , respectively.

If G is a connected graph the *distance* $d(x, y)$ is the length of a shortest x - y path in G . The *diameter* is defined by $diam(G) = \max_{x,y \in V(G)} d(x, y)$. Two vertices u and v are said to be *antipodal vertices* if $d(u, v) = diam(G)$. If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a pendant edge, u a pendant vertex and v a support vertex. A vertex v of G is said to be an *extreme vertex* if the subgraph induced by its neighborhood is complete. The set of all extreme vertices is denoted by $Ext(G)$. A vertex v is said to be a *full vertex* if v is adjacent to all other vertices in G , that is, $deg(v) = p-1$. The set of all full vertices is denoted by $Fx(G)$. The *girth* of a graph G is the length of a shortest cycle contained in G and is denoted by $c(G)$. An acyclic connected graph is called a *tree*. An x - y path of length $d(x, y)$ is called *geodesic*. A vertex v is said to lie on a geodesic P if v is an internal vertex of P . The closed interval consists of x, y and all vertices lying on some x - y geodesic of G and for a non empty set $S \subseteq V(G)$, $I[S] = \bigcup_{x,y \in S} I[x, y]$.

A set $S \subseteq V(G)$ in a connected graph G is a geodetic set of G if $I[S] = V(G)$. The geodetic number of G , denoted by $g(G)$, is the minimum cardinality of a geodetic set of G . A geodetic set S in G is a minimal geodetic set in G if no proper subset of S is a geodetic set of G . The maximal cardinality of minimal geodetic set of G is the upper geodetic number $g^+(G)$ of G . Various concepts inspired by geodetic sets and minimal geodetic sets are introduced in [1, 2, 3].

A set $S \subseteq V(G)$ in a graph G is a dominating set of G if for every vertex v in $V - S$, there exists a vertex u in S such that v is adjacent to u . The

domination number of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [5].

The concept of geodetic dominating sets and minimal geodetic dominating sets are introduced and studied in [4, 6]. A set $S \subseteq V(G)$ is a geodetic dominating set of G if S is both geodetic and dominating set of G . The geodetic domination number, denoted by $\gamma_g(G)$, is the minimum cardinality of a geodetic dominating set of G [4]. A geodetic dominating set of G is called a minimal geodetic dominating set of G if no proper subset of S is a geodetic dominating set of G . The upper geodetic domination number, denoted by $\gamma_g^+(G)$, is the minimum cardinality of a minimal geodetic dominating set of G [6].

The complements $\overline{G} = (\overline{V}, \overline{E})$ of $G = (V, E)$ is the graph with vertex set $\overline{V} = \{\bar{v} : v \in V\}$ and $\overline{E} = \{(\bar{u}, \bar{v}) : \bar{u}, \bar{v} \in \overline{V}, u = v \text{ and } (u, v) \notin E\}$.

A set $S \subseteq V(G)$ is called a global dominating set of G if it is a dominating set of both G and $\overline{G}$ [8]. The global domination number of G , denoted by $\bar{\gamma}(G)$, is the minimum cardinality of a geodetic global dominating set of G .

In [7], authors studied geodetic global domination number in graphs which is defined as follows:

Definition 1.1 Let $G = (V, E)$ be connected with at least two vertices. A subset $S \subseteq V(G)$ is called a geodetic global dominating set of G if S is both geodetic and global dominating set of G . The geodetic global domination number denoted by $\bar{\gamma}_g(G)$ is the minimum cardinality of a geodetic global dominating set of G and the geodetic global dominating set with cardinality $\bar{\gamma}_g(G)$ is called the $\bar{\gamma}_g$ -set of G or $\bar{\gamma}_g(G)$ -set.

2. Some useful known results

Theorem 2.1. [7] Let G be connected graph. Then $Ext(G) \subseteq S$ for all geodetic global dominating sets S in G .

Theorem 2.2. [7] Let G be a connected graph. Then $Fix(G) \subseteq S$ for all geodetic global dominating sets S in G .

Theorem 2.3. [7] Let G be a connected graph of order p then

- (i) $\bar{\gamma}_g(G) = 2$ if and only if $G = K_2$ or there exists a geodetic set $S = \{u, v\}$ such that $d(u, v) = 3$.
- (ii) $\bar{\gamma}_g(G) = p$ if and only if G contains only the extreme and full vertices.

Theorem 2.4. [7] Let G be a connected graph with $Fx(G) = \phi$. Then, $\bar{\gamma}_g(G) = p - 1$ if and only if $G \simeq C_4$.

Theorem 2.5. [7] For the complete bipartite graph $K_{p,q}$ ($p, q \geq 2$),

$$\bar{\gamma}_g(K_{p,q}) = \begin{cases} \min\{p, q\} + 1 & \text{if } 2 \leq p, q \leq 3 \\ 4 & \text{if } p, q \geq 4 \end{cases}$$

3. Upper Geodetic Global Domination Number of a Graph

Definition 3.1. Let $G = (V, E)$ be a connected graph of order $p \geq 2$. A geodetic global dominating set S in G is called a minimal geodetic global dominating set in G if no proper subset of S itself a geodetic global dominating set in G . The upper geodetic global domination number $\bar{\gamma}_g^+(G)$ is the maximum cardinality of a minimal geodetic global dominating set in G . A minimal geodetic global dominating set with cardinality $\bar{\gamma}_g^+(G)$ is called a $\bar{\gamma}_g^+$ -set of G or $\bar{\gamma}_g^+(G)$ -set.

We illustrate this definition by an example.

Example 3.2. For the graph G in Figure 3.1, it is easy to see that $S = \{v_1, v_6\}$ is a minimum geodetic dominating set of G and so $\gamma_g(G) = 2$. Since, $\text{diam}(G) = 2$, it is clear that no 2-element subset of vertices is a geodetic global dominating set of G . It is easily verified that $S_1 = \{v_1, v_2, v_6\}$ and $S_2 = \{v_1, v_3, v_6\}$ are the only two minimum geodetic global dominating sets and so $\bar{\gamma}_g(G) = 3$. It is also easily verified that $S'_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and $S'_2 = \{v_2, v_3, v_4, v_5, v_6\}$ are the only two minimal geodetic global dominating sets of order 5. Since there is no minimal geodetic global dominating set of order greater than 5, $\bar{\gamma}_g^+(G) = 5$.

Figure 3.1

Remark 3.3. Every minimum geodetic global dominating set is a minimal geodetic global dominating set of G , and the converse need not be true. For the graph G in Figure 3.1, $S'_1 = \{v_1, v_2, v_3, v_4, v_5\}$ is a minimal geodetic global dominating set of G and not a minimum geodetic global dominating set of G . Thus, the geodetic global domination number and the upper geodetic global domination number are different.

Theorem 3.4. Let G be a connected graph of order p . Then $2 \leq \bar{\gamma}_g(G) \leq \bar{\gamma}_g^+(G) \leq p$.

Proof. Any geodetic global dominating set has at least two vertices and so $\bar{\gamma}_g(G) \geq 2$. The inequality $\bar{\gamma}_g(G) \leq \bar{\gamma}_g^+(G)$ follows from Remark 3.3. Also, all vertices of G form a geodetic global dominating set of G and therefore, $\bar{\gamma}_g^+(G) \leq p$. Thus, $2 \leq \bar{\gamma}_g(G) \leq \bar{\gamma}_g^+(G) \leq p$. $\square$

Remark 3.5. The bound in Theorem 3.4 are sharp. For the path graph P_4 , $\bar{\gamma}_g(P_4) = 2$. For any bistar graph G of order at least 2, it is clear that the set of all end vertices is the unique minimum geodetic global dominating set of G and so $\bar{\gamma}_g(G) = \bar{\gamma}_g^+(G)$. All the inequalities in Theorem 3.4 can be strict. For the graph G in Figure 3.1, $\bar{\gamma}_g(G) = 3$, $\bar{\gamma}_g^+(G) = 5$ and $p = 6$. Thus, $2 < \bar{\gamma}_g(G) < \bar{\gamma}_g^+(G) < p$.

Corollary 3.6. Let G be a connected graph of order p . If $\bar{\gamma}_g^+(G) = 2$, then $\bar{\gamma}_g(G) = 2$.

Proof. This follows from Theorem 3.4 $\square$

Remark 3.7. The converse of the Corollary 3.6 need not be true. For the graph G given in Figure 3.2, we note that $\bar{\gamma}_g(G) = 2$ and $\bar{\gamma}_g^+(G) = 4$. Thus, $\bar{\gamma}_g^+(G) \neq 2$.

Corollary 3.8. Let G be a connected graph of order p . If $\bar{\gamma}_g(G) = p$, then $\bar{\gamma}_g^+(G) = p$.

Proof. This follows from Theorem 3.4. $\square$

Figure 3.2

The following theorem characterizes the graphs for which the upper geodetic global domination number is 2.

Theorem 3.9. Let G be a connected graph of order $p \geq 2$. Then $\bar{\gamma}_g^+(G) = 2$ if and only if $G = K_2$ or G has a geodetic set $S = \{x, y\}$ with $x, y \in \text{Ext}(G)$ and $d(x, y) = 3$.

Proof. Suppose $\bar{\gamma}_g^+(G) = 2$. Then by Corollary 3.6, $\bar{\gamma}_g(G) = 2$. By Theorem 2.3, we have $G = K_2$ or there is a geodetic set $S = \{x, y\}$ with $d(x, y) = 3$. If $G = K_2$, then the proof is over. Suppose that $d(x, y) = 3$. Then for each u in $V(G) - S$, u is adjacent to either x or y . Without loss of generality, we assume $ux \in E(G)$. Suppose on the contrary $x \notin \text{Ext}(G)$. Then there exists a vertex v such that $u, v \in N(x)$ with $d(u, v) = 2$. Consider, $D = N(x) \cup \{y\}$. Clearly, $|D| \geq 3$. Let $z \in V(G) - D$. If $z = x$, then z lies on a $u - v$ geodesic in G and so that $z \in I[D]$ and $z \in N[D]$. Also, $d(z, y) = 3, z \in N_{\bar{G}}[D]$, implies D is a geodetic global dominating set in G . Suppose $z = x$. Since, $z \in I[x, y]$, there exists w, t in $V(G)$ such that z lies on the $x - y$ geodesic $\{x, w, t, y\}$. It follows that $z \in I[D]$ and $z \in I[D]$. If $N(z) \subseteq D$, then there exists a set $T \subseteq D$ such

that T is a minimal geodetic global dominating set of cardinality greater than or equal to 3, a contradiction. Therefore, $N(z) \cap D \neq \emptyset$ for all $z \notin D$, implies $z \in N_{\overline{G}}[D]$. Thus both the cases, D is a geodetic global dominating set in G . Let $W \subseteq D$ be a minimal geodetic global dominating set in G . Since, $y \notin N[N(x)]$, $y \in W$. Also if $|W \cap N(x)| = 1$, then $x \notin I[W]$, a contradiction. Thus, $|W \cap N(x)| \geq 2$ implies $\bar{\gamma}_g^+(G) \geq |W| \geq 3$, a contradiction. Hence $\langle N[x] \rangle$ is complete and $x \in Ext(G)$. Similarly, we can prove that $y \in Ext(G)$. Conversely, if $G = K_2$, then $\bar{\gamma}_g^+(G) = 2$. Now, let G have a geodetic set $S = \{x, y\}$ with $d(x, y) = 3$ and here the induced subgraphs $\langle N[x] \rangle$ and $\langle N[y] \rangle$ are complete. Then $Ext(G) = \{x, y\}$ is the unique minimal geodetic global dominating set in G and so $\bar{\gamma}_g^+(G) = 2$. $\square$

Theorem 3.10. Let G be a connected graph of order p . Then,

- (i) Every minimal geodetic global dominating sets of G contains all the extreme vertices in G .
- (ii) Every minimal geodetic global dominating sets of G contains its full vertices.
- (iii) If G has a unique minimal geodetic global dominating set, then $\bar{\gamma}_g(G) = \bar{\gamma}_g^+(G)$.

Proof. (i) and (ii) from Theorem 2.1 and Theorem 2.2

- (iii) Since any minimal geodetic global dominating sets are nothing but the minimum geodetic global dominating sets, $\bar{\gamma}_g(G) = \bar{\gamma}_g^+(G)$. $\square$

The following Theorem characterizes the graphs for which the upper geodetic global domination number is p .

Theorem 3.11. Let G be a connected graph of order p . Then $\bar{\gamma}_g(G) = p$ if and only if $\bar{\gamma}_g^+(G) = p$.

Proof. If $\bar{\gamma}_g(G) = p$, then by Corollary 3.8, $\bar{\gamma}_g^+(G) = p$. Conversely, if $\bar{\gamma}_g^+(G) = p$. Then, the set of all vertices of G is the unique minimal geodetic global dominating set of G . It follows that G contains no proper geodetic global dominating sets and so the set of all vertices is the minimum geodetic global dominating set of G . Hence, $\bar{\gamma}_g(G) = p$. $\square$

Corollary 3.12. For the complete graph K_p ($p \geq 2$), $\bar{\gamma}_g(K_p) = \bar{\gamma}_g^+(K_p) = p$.

Proof. This follows from Theorem 3.11 $\square$

Figure 3.3

Remark 3.13. The converse of Corollary 3.12 need not be true. For the graph G in Figure 3.3, the set $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is the unique minimum geodetic global dominating set of G and so $\bar{\gamma}_g(G) = 6$. Hence, by Theorem 3.11, $\bar{\gamma}_g^+(G) = 6$.

This examples show that there are non-complete graphs G of order p with $\bar{\gamma}_g^+(G) = \bar{\gamma}_g(G) = p$.

Theorem 3.14. If G is a connected graph of order p with $\bar{\gamma}_g(G) = p - 1$, then $\bar{\gamma}_g^+(G) = p - 1$.

Proof. Let $\bar{\gamma}_g(G) = p - 1$. Then either $\bar{\gamma}_g^+(G) = p$ or $\bar{\gamma}_g^+(G) = p - 1$ by applying Theorem 3.4. If $\bar{\gamma}_g^+(G) = p$, then by Theorem 3.11, $\bar{\gamma}_g(G) = p$, which contradicts our hypothesis. Hence, $\bar{\gamma}_g^+(G) = p - 1$. $\square$

Remark 3.15. The converse of Theorem 3.14 need not be true. For the graph in Figure 3.4, the set $S = \{v_1, v_2, v_3\}$ is a minimum geodetic global dominating set of G and so $\bar{\gamma}_g(G) = 3$. Also, it is easily seen that the set $S' = \{v_1, v_3, v_4, v_5\}$ is a minimal geodetic global dominating set and so $\bar{\gamma}_g^+(G) = 4 = p - 1$. But $\bar{\gamma}_g(G) = 3 = p - 1$.

Figure 3.4

Theorem 3.16. *Let G be a connected graph with cut vertices and let S be a minimal geodetic global dominating set of G . If v is a cut vertex of G then every component of $G - v$ contains an element of S .*

Proof. Suppose that there is a component H of $G - v$ such that H contains no vertex of S . Let u be a vertex in H . Since, S is a minimal geodetic global dominating set of G , there exist vertices $x, y \in S$ such that u lies on some $x - y$ geodesic path $P : x = u_0, u_1, \dots, u, \dots, u_l = y$ in G . Let P_1 be the $x - u$ subpath of P and P_2 be the $u - y$ subpath of P . Since, v is a cut vertex of G , both P_1 and P_2 contains v so that P is not a path, which is a contradiction. Thus, every component of $G - v$ contains an element of S . $\square$

beginincor Let G be a connected graph with cut vertices and let S be a minimal geodetic global dominating set of G . Then every branch of G contains an element of S .

Theorem 3.17. *For the complete bipartite graph $K_{p,q}$ ($p, q \geq 2$), $\bar{\gamma}_g^+(K_{p,q}) = \max\{p, q\} + 1$.*

Proof. Let $G = K_{p,q}$ where $p, q \geq 2$. Let $X = \{x_1, x_2, \dots, x_p\}$ and $Y = \{y_1, y_2, \dots, y_q\}$ be the bipartition of G . Without loss of generality, we assume $q \leq p$.

case (i) $q < p$, then, $|X| > |Y|$. Since, each $y_i \in I[x_j, x_k] \subseteq I[X]$ and $y_i \in N[x_j] \subseteq N[X]$, X is a geodetic dominating set of G . In G , $\langle X \rangle$ and $\langle Y \rangle$ are disjoint complete graphs. Therefore, X is not a dominating set of $\overline{G}$. Now, let $S = X \cup \{y_i\}, i = 1, 2, \dots, q$. Clearly, S is a dominating set of $\overline{G}$. Since, $I[S] = V(G)$ and $N[S] = V(G)$, that S is a geodetic global dominating set of G . Now, we show that S is a minimal geodetic global dominating set of G . Suppose there exists a geodetic global dominating set S_1 of G such that $S_1 \subset S$. Suppose there exists a vertex $x_j \in S$ such that $x_j \notin S_1$. Clearly, the vertex x_j does not lie on any geodesic path joining a pair of vertices in S_1 . That is, $I[S_1] = V(G)$. Thus, S_1 is not a geodetic global dominating set of G . If $y_i \notin S_1$, then $N[S_1] = V(\overline{G})$, implies S_1 is not a geodetic global dominating set of G . Both ways, we obtained a contradiction. Therefore, S is a minimal geodetic global dominating set of G . Hence, $\bar{\gamma}_g^+(G) \geq |S| = p + 1$. Now, we want to show that $\bar{\gamma}_g^+(G) = p + 1$. Suppose, there exists a minimal geodetic global dominating set S_2 of G such that $|S_2| > p + 1$. Since, $X \cup \{y_i\}$ is a minimal geodetic global dominating set of G and $|S_2| > |X \cup \{y_i\}|$, S_2 contains atleast two vertices from Y , say y_i, y_k . Then, clearly $S_2 - \{y_i\}$ is a geodetic global dominating set of G , which is a contradiction. Therefore, $\bar{\gamma}_g^+(G) = p + 1$.

case (ii) If $p = q$, then $|X| = |Y|$. Using a similar argument as in case (i), it can be show that $\bar{\gamma}_g^+(G) = q + 1 = p + 1$. Hence, $\bar{\gamma}_g^+(G) = \max\{p, q\} + 1$. $\square$

Theorem 3.18. For the path graph $P_p, (p \geq 4)$, $\bar{\gamma}_g^+(P_p) = \lceil \frac{p}{2} \rceil$.

Proof. Let $P_p : v_1, v_2, \dots, v_p$ be a path of order $p \geq 4$ in which $v_i v_j \in E(P_p)$ if and only if $2 \leq j = i + 1 \leq p$. We prove this theorem by considering two cases.

Case (i) p is odd. Let $S = \{v_1, v_3, v_5, \dots, v_{p-2}, v_p\}$. Clearly S is a geodetic global dominating set in P_p with maximum cardinality. If we remove any vertex from S , then $I[S] = V(G)$ implies, S is not a geodetic global dominating set of P_p . Therefore, $\bar{\gamma}_g^+(P_p) = |S| = \lceil \frac{p}{2} \rceil$.

Case (ii) p is even. Consider $S' = \{v_1, v_3, v_5, \dots, v_{p-3}, v_p\}$. It is straight forward that S' is a minimal geodetic global dominating set in P_p , with maximum cardinality. Thus, $\bar{\gamma}_g^+(P_p) = |S| = \lceil \frac{p}{2} \rceil$. Hence, both the cases, $\bar{\gamma}_g^+(P_p) = \lceil \frac{p}{2} \rceil$. $\square$

Theorem 3.19. For the cycle C_p ($p \geq 3$), $\bar{\gamma}_g^+(C_p) = \begin{cases} 3 & \text{if } p=3,4,5 \\ \lfloor \frac{p}{2} \rfloor & \text{if } p \geq 6 \end{cases}$

Proof. Let $C_p : v_1, v_2, \dots, v_p, v_1$ be a cycle of order $p \geq 3$ in which $v_i v_j \in E(C_p)$ if and only if $j \equiv i + 1 \pmod{p}$. For $p = 3$, $C_3 \simeq K_3$. Then by Corollary 3.12, $\bar{\gamma}_g^+(K_3) = \bar{\gamma}_g^+(C_3) = 3$. If $p = 4$, then by Theorem 2.4, $\bar{\gamma}_g(C_4) = 4 - 1$. Therefore, by Theorem 3.14, $\bar{\gamma}_g^+(C_4) = 4 - 1 = 3$.

Now let $p \geq 5$ and let $V(C_p) = \{v_1, v_2, v_3, \dots, v_p\}$. For $p = 5$, take $S = \{v_1, v_3, v_5\}$. Clearly, S is a minimal geodetic global dominating set of C_5 and no minimal geodetic global dominating set exists with cardinality greater than $|S|$. Therefore, $\bar{\gamma}_g^+(C_5) = 3$. For $p \geq 6$, we consider two cases:

Case (i) p is even. Let $p = 2n$. It is clear that $S = \{v_1, v_3, v_5, \dots, v_{2n-3}, v_{2n-1}\}$ is a minimal geodetic global dominating set of C_{2n} and so $\bar{\gamma}_g^+(C_{2n}) \geq |S| = \frac{p}{2} = \lfloor \frac{p}{2} \rfloor$. Now we show that $\bar{\gamma}_g^+(C_{2n}) \leq \lfloor \frac{p}{2} \rfloor$. Suppose there exists a minimal geodetic global dominating set S_1 such that $|S_1| > \frac{p}{2}$. Then, S_1 contains two adjacent vertices. Let v_i and v_j be two adjacent vertices in S_1 . Clearly, $S_1 - \{v_i\}$ or $S_2 = \{v_j\}$ is a geodetic global dominating set of C_{2n} , which is a contradiction. Therefore, $\bar{\gamma}_g^+(C_{2n}) \leq \lfloor \frac{p}{2} \rfloor$. Hence, $\bar{\gamma}_g^+(C_{2n}) = \lfloor \frac{p}{2} \rfloor$.

Case (ii) p is odd. Let $p = 2n + 1$. It is easy to verified that $S = \{v_1, v_3, v_5, \dots, v_{2n-3}, v_{2n-1}\}$ is a minimal geodetic dominating set of C_{2n+1} such that $|S| = n = \frac{p-1}{2} = \lfloor \frac{p}{2} \rfloor$. By the similar argument as in case (i), we show that S is of maximal cardinality in C_{2n+1} . Therefore, $\bar{\gamma}_g^+(C_{2n+1}) = \lfloor \frac{p}{2} \rfloor$. Hence, both the cases, $\bar{\gamma}_g^+(C_p) = \lfloor \frac{p}{2} \rfloor, p \geq 6$. $\square$

Theorem 3.20. For the wheel graph W_p ($p \geq 6$), $\bar{\gamma}_g^+(W_p) = \lceil \frac{2p+1}{3} \rceil$.

Proof. Let $p \geq 6$ and $V(W_p) = \{v, v_1, v_2, \dots, v_{p-1}\}$, where $\deg(v_i) = 3$ for $1 \leq i \leq p-1$ and $\deg(v) = p-1$. By theorem 2.2, every geodetic global dominating set contains $\{v\}$. If $p \equiv 0 \pmod{3}$ or $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$, $S = \{v, v_1, v_2, v_4, v_5, \dots, v_{p-5}, v_{p-4}, v_{p-2}\}$ or $S = \{v, v_1, v_2, v_4, v_5, \dots, v_{p-7}, v_{p-5}, v_{p-3}, v_{p-2}\}$ or $S = \{v, v_1, v_2, v_4, v_5, \dots, v_{p-7}, v_{p-6}, v_{p-4}, v_{p-2}\}$ is a geodetic global dominating set of W_p , respectively. And so, $|S| = \lceil \frac{2p+1}{3} \rceil$. Let $S' = S - \{x\}$, where $x \in S$. If $x = v$, then S' does not dominates $\bar{W}_p$ and so S' is not a global dominating set of W_p . If $x = v_i$, for $1 \leq i \leq p-1$, then $I[S'] = V(W_p)$ and so S' is not a geodetic set of W_p . Therefore, S is a minimal geodetic global dominating set of W_p , and so $\bar{\gamma}_g^+(W_p) \geq |S|$. Moreover, there does

not exists a minimal geodetic global dominating set S_1 with $|S_1| > |S|$, we conclude that $\bar{\gamma}_g^+(W_p) = |S| = \lceil \frac{2p+1}{3} \rceil$ $\square$

Theorem 3.21. For the fan graph F_p ($p \geq 5$), $\bar{\gamma}_g^+(F_p) = \lceil \frac{2p+1}{3} \rceil$.

Proof. Let $p \geq 5$ and $V(F_p) = \{v, v_1, v_2, \dots, v_{p-1}\}$, where $\deg(v_1) = \deg(v_{p-1}) = 2$, $\deg(v) = p - 1$ and $\deg(v_i) = 3$ for $2 \leq i \leq p - 2$. It is clear that $\{v_1, v_{p-1}\}$ be the set of extreme vertices of F_p and $\{v\}$ is a set of full vertex in F_p . Therefore, by Theorem 2.1 and Theorem 2.2, every geodetic global dominating set contains $\{v_1, v_{p-1}, v\}$. If $p \equiv 0(\text{mod } 3)$ or $p \equiv 1(\text{mod } 3)$ or $p \equiv 2(\text{mod } 3)$, then $S = \{v, v_1, v_2, v_4, v_5, \dots, v_{p-5}, v_{p-4}, v_{p-2}, v_{p-1}\}$ or $S = \{v, v_1, v_2, v_4, v_5, \dots, v_{p-5}, v_{p-4}, v_{p-3}, v_{p-1}\}$ or $S = \{v, v_1, v_2, v_4, v_5, \dots, v_{p-4}, v_{p-3}, v_{p-1}\}$ is a geodetic global dominating set of F_p , respectively and so $|S| = \lceil \frac{2p+1}{3} \rceil$. As a similar discussion in Theorem 3.21 shows that S is a $\bar{\gamma}_g^+$ -set of F_p and so $\bar{\gamma}_g^+(F_p) = \lceil \frac{2p+1}{3} \rceil$. $\square$

Observation 3.22. Let $G = (V, E)$ be a connected graph of order $p \geq 2$ and let $S \subseteq V(G)$.

- (i) If S is a minimal geodetic set of G which is also a global dominating set of G , then S is a minimal geodetic global dominating set of G .
- (ii) If S is a minimal global dominating set of G which is also a geodetic set of G , then S is a minimal geodetic global dominating set of G .

Observation 3.23. For any connected graph G , $\gamma_g^+(G) \leq \bar{\gamma}_g^+(G)$.

Theorem 3.24. Let S be a γ_g^+ -set of a connected graph G . Then S is a $\bar{\gamma}_g^+$ -set of G if and only if S is a dominating set in $\overline{G}$.

Proof. This follows directly from the definition. $\square$

To aid in our discussion throughout this paper, we define a set L_S as follows.

Definition 3.25. Let G be a connected graph with a subset $S \subset V$ in G then $L_S(G)$ be the set of all vertices v in $V - S$ such that v dominates S in G ; that is, $L_S(G) = \{v \in V - S | S \subseteq N(v)\}$. In otherwords, each vertex in L_S is adjacent to every vertex in S .

Theorem 3.26. Let G be a connected graph and S be a γ_g^+ -set of G . If $L_S = \phi$ and S' be a minimal dominating set in $\langle \overline{L_S} \rangle$, then $S \cup S'$ is a minimal geodetic global dominating set of G .

Proof. Let $L_S = \phi$ and let S' be a minimal dominating set in $\langle \overline{L_S} \rangle$. Since S is a γ_g^+ -set of G , $S \cup S'$ dominates $\overline{G}$. It follows that $S \cup S'$ is a geodetic global dominating set of G . We show that $S \cup S'$ is a minimal geodetic global dominating set of G . Suppose there exists a proper subset W of $S \cup S'$ such that W is a geodetic global dominating set of G . Then we have the following cases:

Case (i) $W \cap S = \phi$ or $W \cap S' = \phi$. If $W \cap S = \phi$, then $W \subset S'$. Since W is a global dominating set of G , W dominates $\langle \overline{L_S} \rangle$. It follows that S' is not a minimal dominating set in $\langle \overline{L_S} \rangle$. Suppose $W \cap S' = \phi$ then $W \subset S$. Since W is both geodetic and global dominating set of G , W is a geodetic global dominating set of G . Thus S is not a minimal geodetic global dominating set of G .

Case (ii) $S \cap W = \phi$ and $S' \cap W = \phi$. Since $S \cap S' = \phi$, $W = S$ and $W = S'$. If $W = S$, choose a vertex $v_i \in S$ such that $v_i \notin W$. Then the vertex v_i does not lie on any geodesic joining pair of vertices of W , implies that W is not a geodetic global dominating set of G . If $W = S'$, there exists a vertex $y_i \in S'$ such that $y_i \notin W$. Then $y_i \notin N[W]$ in $\overline{G}$ implies that W is not a geodetic global dominating set of G . Thus, both the ways we obtained a contradiction. Hence, $S \cup S'$ is a minimal geodetic global dominating set of G . $\square$

Theorem 3.27. Let G be a connected graph and S be a γ_g^+ -set of G . Then, $\bar{\gamma}_g^+(G) \leq \gamma_g^+(G) + |L_S|$.

Proof. Let S be a minimal geodetic dominating set of G such that $\gamma_g^+(G) = |S|$. If S dominates $\overline{G}$, then by Theorem 3.25, S is a $\bar{\gamma}_g^+$ -set of G and so $\bar{\gamma}_g^+(G) = |S| = \gamma_g^+(G)$. Now, assume that S does not dominate $\overline{G}$. Then, there exist vertices in $V - S$, which is adjacent to every vertex in S . This shows that $L_S = \phi$. Let S' be a minimal dominating set in $\langle \overline{L_S} \rangle$ with maximum cardinality. Then by Theorem 3.27, $S \cup S'$ is a minimal geodetic global dominating set in G . We note that S and S' is of maximum cardinality. Since $S \cap S' = \phi$, $S \cup S'$ is a $\bar{\gamma}_g^+$ -set of G and so $\bar{\gamma}_g^+(G) = |S \cup S'|$. Since $|S'| \leq |L_S|$, we conclude that $\bar{\gamma}_g^+(G) \leq \gamma_g^+(G) + |L_S|$. $\square$

Theorem 3.28. Let G be a connected graph. If S is a γ_g^+ -set of G , then

- (i) $\bar{\gamma}_g^+(G) = \gamma_g^+(G)$ if and only if $L_S = \phi$.
- (ii) $\bar{\gamma}_g^+(G) = \gamma_g^+(G) + 1$ if and only if $\langle L_S \rangle$ is totally disconnected.
- (iii) $\bar{\gamma}_g^+(G) = \gamma_g^+(G) + |L_S|$ if and only if $\langle L_S \rangle$ is complete.

Proof. Let G be a connected graph and S be a γ_g^+ -set of G .

(i) First assume $L_S = \phi$. Then by Theorem 3.28, $\bar{\gamma}_g^+(G) \leq \gamma_g^+(G)$. By Observation 3.24, we conclude that $\bar{\gamma}_g^+(G) = \gamma_g^+(G)$. Now assume $\bar{\gamma}_g^+(G) = \gamma_g^+(G)$. Let S be a γ_g^+ -set of G which is also a $\bar{\gamma}_g^+$ -set of G . Therefore by Theorem 3.25, S is a dominating set of $\overline{G}$, implies no vertex in $V - S$ is adjacent to every vertex in S . That is, $SN(v)$ for every v in $V - S$. Hence, $L_S = \phi$.

(ii) Assume $\langle L_S \rangle$ is totally disconnected. Let $x \in V(\langle L_S \rangle)$. Then, $S \cup \{x\}$ is a $\bar{\gamma}_g^+$ -set of G and so $\bar{\gamma}_g^+(G) = |S \cup \{x\}| = \gamma_g^+(G) + 1$. Conversely, assume $\bar{\gamma}_g^+(G) = \gamma_g^+(G) + 1$. By (i) $L_S = \phi$. To prove $\langle L_S \rangle$ is totally disconnected. Suppose $\langle L_S \rangle$ contains a vertex x such that x is adjacent to at least one vertex y in $V(\langle L_S \rangle)$. This shows every minimal geodesic global dominating set contains $S \cup \{x, y\}$, implies $\bar{\gamma}_g^+(G) \geq \gamma_g^+(G) + 2$, which is a contradiction. Hence $\langle L_S \rangle$ is totally disconnected.

(iii) Assume $\langle L_S \rangle$ is complete. Then $V(\langle L_S \rangle)$ is a minimum dominating set in $\langle \overline{L_S} \rangle$. Since $S \cap L_S = \phi$, $S \cup L_S$ is a minimal geodesic global dominating set of G and so $\bar{\gamma}_g^+(G) \geq |S \cup L_S| = \gamma_g^+(G) + |L_S|$. By Theorem 3.28, we conclude $\bar{\gamma}_g^+(G) = \gamma_g^+(G) + |L_S|$. Conversely, assume $\bar{\gamma}_g^+(G) = \gamma_g^+(G) + |L_S|$. To prove $\langle L_S \rangle$ is complete. Suppose $\langle L_S \rangle$ is non-complete. By (i) $L_S = \phi$. There exists a vertex x in L_S , which is non-adjacent to a vertex y in L_S . It follows that $S \cup (L_S - \{x\})$ is a $\bar{\gamma}_g^+$ -set of G and so $\bar{\gamma}_g^+(G) = |S \cup (L_S - \{x\})| < \bar{\gamma}_g^+(G) + |L_S|$, a contradiction to our hypothesis. Hence $\langle L_S \rangle$ is complete. $\square$

Theorem 3.29. For every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_g^+(G) = a$ and $\bar{\gamma}_g^+(G) = b$.

Proof. If $a = b$, let G be a complete graph of order a . Then by Corollary 3.12, $\gamma_g^+(G) = \bar{\gamma}_g^+(G) = a$. Now, let $a < b$. We consider two cases:

Case (i) Let $b = a + 1$. Consider the graph G obtained from the path graph on three vertices $P_3 : x, y, z$ by adding new vertices $v_1, v_2, \dots, v_{a-1}$ and join $v_1, v_2, \dots, v_{a-1}$ to both x and z . This is shown in Figure 3.5. Let $S = \{y, v_1, v_2, \dots, v_{a-1}\}$. It is easy to observe that S or $\{x, z\}$ are the only minimal geodesic dominating sets of G and so $\gamma_g^+(G) = |S| = a$. Now, let $S_1 = S \cup \{x\}$ or $S_1 = S \cup \{z\}$. It is clear that S_1 is a geodesic global

dominating set of G . Now, we show that S_1 is a minimal geodetic global dominating set of G . Suppose on the contrary, that S_1 is not a minimal geodetic global dominating set of G . Then, there is a proper subset S' of S_1 such that S' is a geodetic global dominating set of G .

Figure 3.5

Let $v \in S_1$ and $v \notin S'$. If $v = x$ or $v = z$, then S' does not dominate $\overline{G}$. If $v = v_i$ for some $i = 1, 2, \dots, a - 1$, then v_i does not lie on a geodesic P joining any pair of vertices in S' and so that S' is not a geodetic global dominating set of G , which is a contradiction. Thus, S_1 is a minimal geodetic global dominating set of G and so $\bar{\gamma}_g^+(G) \geq |S_1| = b$. Since the order of G is $b + 1$ and either $x \notin S_1$ or $z \notin S_1$, we conclude that $\bar{\gamma}_g^+(G) = b$.

G
Figure 3.6

Case (ii) Let $b = a + k$, where $k \geq 2$. Consider the graph H obtained from the path $P_3 : x, y, z$ by adding new vertices $v_1, v_2, \dots, v_a$ and join each v_i ($1 \leq i \leq a$) with x, y, z . Now, we add $k - 2$ new vertices $u_1, u_2, \dots, u_{k-2}$ to the graph H and join each u_i ($1 \leq i \leq k - 2$) to every vertices in H , thereby producing the graph G in Figure 3.6. Then, as in case (i), $S = \{v_1, v_2, \dots, v_a\}$ is a minimal geodetic dominating set with maximum cardinality and $V(G) - \{x\}$ is a minimal geodetic global dominating set with maximum cardinality. Thus, $\gamma_g^+(G) = a$ and $\bar{\gamma}_g^+(G) = b$. $\square$

Corollary 3.30. For every pair of integers a and b with $2 \leq a < b$, the smallest possible order of a connected graph G for which $\gamma_g^+(G) = a$ and $\bar{\gamma}_g^+(G) = b$ is $b + 1$.

Theorem 3.31. For any three integers a, b, p with $3 \leq a \leq b < p$, there exists a connected graph G of order p , such that $\bar{\gamma}_g(G) = a$ and $\bar{\gamma}_g^+(G) = b$.

Proof. If $p = b + 1$, then let $a = 3 + k$ and $b = r + k$, where $k \geq 0$ and $r \geq 3$. Assume $k = 0$. Consider the graph $G = K_{2,r-1}$ with partition $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \dots, y_{r-1}\}$ as shown in Figure 3.7. By Theorem 2.5, $\bar{\gamma}_g(G) = \min\{X, Y\} + 1 = 3$. Also by Theorem 3.18, $\bar{\gamma}_g^+(G) = \max\{X, Y\} + 1 = r$. Therefore, in this case $\bar{\gamma}_g(G) = 3$ and $\bar{\gamma}_g^+(G) = r$.

Figure 3.7

Now, suppose that $k \geq 1$. Let $H = K_2 + \overline{K_k}$ with $V(K_2) = \{y, z\}$ and $V(\overline{K_k}) = \{z_1, z_2, \dots, z_k\}$. Let G be the connected graph obtained by taking join of $K_{1,r-2}$ (with partite sets $\{x\}$ and $\{y_1, y_2, \dots, y_{r-2}\}$) and H and joining each y_i ($1 \leq i \leq r-2$) with y and z . Also add a new edge xz in G and is shown in Figure 3.8.

Figure 3.8

It is clear that $\text{diam}(G) = 2$ and G is a connected graph of order $b + 1$. Also, $\text{Ext}(G) = \{x_1, x_2, \dots, x_k\}$ and $F_x(G) = \{z\}$. The set $S = \text{Ext}(G) \cup F_x(G)$ is not a geodetic global dominating set in G . We note that $S \cup \{x, y\}$ is a minimum geodetic global dominating set in G and so $\gamma_g(G) = |S \cup \{x, y\}| = 3 + k = a$. Since S is always contained in a geodetic

global dominating set in G , that $S \cup \{y_1, y_2, \dots, y_{r-2}\}$ is a minimal geodetic global dominating set of G and so $\bar{\gamma}_g^+(G) \geq b$. Since the order of G is $b+1$, by Theorem 3.11, it follows that $\bar{\gamma}_g^+(G) = b$. Suppose that $p = b+2$. Let $a = 3+k$ and $b = r+k$, where $r \geq 3$ and $k \geq 0$. Let G be the graph obtained from $K_{3,r-2}$ (with partite sets $\{x, y, z\}$ and $\{y_1, y_2, \dots, y_{r-2}\}$) and $K_{1,k+1}$ (with partite sets $\{z\}$ and $\{x_1, x_2, \dots, x_{k+1}\}$) using z as a common vertex. This is the graph G as shown in Figure 3.9

Figure 3.9

Let $S = \{x_1, x_2, \dots, x_{k+1}\}$ be the set of all extreme vertices of G . By Theorem 2.1, every geodetic global dominating set contains S . It is clear that S is not geodetic global dominating set in G . But $S' = S \cup \{x, y\}$ is a minimum geodetic global dominating set of G and so $\bar{\gamma}_g(G) = |S \cup \{x, y\}| = a$. Let $S_1 = S \cup \{y_1, y_2, \dots, y_{r-2}\}$. It is clear that S_1 is not an geodetic global dominating set in G . Clearly, $S_2 = S_1 \cup \{z\}$ is a geodetic global dominating set of G . Now, we show that S_2 is a minimal geodetic global dominating set in G . Suppose on the contrary, there exists a proper subset W of S_2 such that W is a geodetic global dominating set in G . Chose a vertex $v \in S_2$ such that $v \notin W$. Clearly $v = x_i$ for $1 \leq i \leq k+1$. Suppose $v = y_i$ for

some $i \in \{1, 2, \dots, r-2\}$. Then the vertex v does not lie on any geodesic pair of vertices of W , which is a contradiction. If $v = z$, then W is not a global dominating set of G , a contradiction. Thus, S_2 is a minimal geodetic global dominating set of G and so $\bar{\gamma}_g^+(G) \geq |S_2| = b$. Moreover, any minimal geodetic global dominating set contains the set $\{x, y\}$ coincides S' . Therefore every minimal geodetic global dominating set other than S' contains either x or y or z . Hence, $\bar{\gamma}_g^+(G) = b$. Suppose that $p = b + l$, where $l \geq 3$. Let $a = 3 + k$ and $b = r + k$, $r \geq 3$ and $k \geq 0$. Consider the graph obtained as follows:

- (i) Take a copy of $K_{1,k+1}$ with $(k+1)$ -pendent vertices $w, z_1, z_2, \dots, z_k$ and the support vertex z .
- (ii) Take a copy of $K_{2,r-1}$ with partite sets $\{x, y\}$ and $\{y_1, y_2, \dots, y_{r-1}\}$.
- (iii) Add a new edge yw .

Let G be the graph in Figure 3.10 is obtained from H and a copy of complete graph K_{p-b-2} with $(p-b-2)$ vertices $x_1, x_2, \dots, x_{p-b-2}$ and joining each x_i ($1 \leq i \leq p-b-2$) with x, y and z .

Figure 3.10

Let $S = \{z_1, z_2, \dots, z_k\}$ be the set all extreme vertices of G . It is easily to verified that S is not a geodetic global dominating set in G . Clearly that $S \cup \{x, y\}$ is a $\bar{\gamma}_g$ -set of G and so $\bar{\gamma}_g(G) = |S| + 2 = k + 2 = a$. Also, it is easily seen that $S \cup \{y_1, y_2, \dots, y_{r-1}, z\}$ is a $\bar{\gamma}_g^+$ -set of G . In this case, $\bar{\gamma}_g(G) = a$ and $\bar{\gamma}_g^+(G) = r + k = b$. $\square$

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References

- [1] F. Buckley and F. Harary, *Distance in graphs*. Redwood City, CA: Addison-Wesley, 1990.
- [2] G. Chartrand, F. Harary, and P. Chang, "On the geodetic number of a graph", *Networks*, vol. 39, no.1, pp. 1-6, Jan. 2002, doi: 10.1002/net.10007
- [3] W. J. Desormeaux, P. E. Gibson, and T. W. Haynes, "Bounds on the global domination number", *Quaestiones mathematicae*, vol. 38, no. 4, pp. 563-572, Jun. 2015, doi: 10.2989/16073606.2014.981728
- [4] H. Escudro, R. Gera, A. Hansberg, N. Jafari Rad, and L. Volkmann, "Geodetic domination in graphs", *Journal of combinatorial mathematics and combinatorial computing*, vol. 77, pp. 88-101, May 2011. [On line]. Available: <https://bit.ly/3p6DuWZ>
- [5] T. W. Haynes, P. J. Slater, and S. T. Hedetniemi, *Fundamentals of domination in graphs*. Boca Raton, CA: CRC Press, 1998, doi: 10.1201/9781482246582
- [6] H. M. Nuenay and F. P. Jamil, "On minimal geodetic domination in graphs", *Discussiones mathematicae graph theory*, vol. 35, no. 3, pp. 403-418, Jul. 2015, doi: 10.7151/dmgt.1803
- [7] S. R. Chellathurai and L. Xaviour, "Geodetic global domination in graphs", *International journal of mathematical archive*, vol. 9, no. 4, pp. 29-36, 2018. [On line]. Available: <https://bit.ly/38mtIKv>
- [8] E. Sampath Kumar, "The global domination number of a graph", *Journal of mathematical and physical sciences*, vol. 23, no. 5, pp. 377-385, 1989. [On line]. Available: <https://bit.ly/3kbwpR6>
- [9] D. B. West, *Introduction to graph theory*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 2001.