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New Hermite-Hadamard type inequalities for m and (α, m)-convex functions on the coordinates via generalized fractional integrals

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Abstract

In this paper, we obtained a new Hermite-Hadamard type inequality for functions of two independent variables that are m-convex on the coordinates via some generalized Katugampola type fractional integrals. We also established a new identity involving the second order mixed partial derivatives of functions of two independent variables via the generalized Katugampola fractional integrals. Using the identity, we established some new Hermite-Hadamard type inequalities for functions whose second order mixed partial derivatives in absolute value at some powers are (α, m)-convex on the coordinates. Our results are extensions of some earlier results in the literature for functions of two variables.

Keywords: *Hermite-Hadamard inequality, convex functions on the coordinates, m-convex functions on the coordinates, (α, m)-convex functions on the coordinates, Hölder's inequality, Katugampola fractional integrals.*

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1. Introduction

The following double inequality is the well-known Hermite-Hadamard inequality:

Let $f : [a, b] \to \mathbf{R}$ be a convex function. Then the following double inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The Hermite-Hadamard inequality has attracted the attention of many authors in the last decades due to its numerous applications in Mathematical Analysis. Several generalizations, refinements and extensions of the Hermite-Hadamard inequality has been introduced in the literature. For some recent results related to the Hermite-Hadamard inequality, we refer the interested reader to the papers [2, 1, 3, 10, 11, 14, 16, 17]. Recently, Chen and Katugampola [2] obtained the following generalizations of the Hermite-Hadamard inequality via the Katugampola fractional integrals.

Theorem 1.1. *Let $\beta, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbf{R}$ be a positive function with $0 \leq a < b$. If f is a convex function on $[a^\rho, b^\rho]$, then the following inequalities hold:*

$$f\left(\frac{a^\rho+b^\rho}{2}\right) \leq \frac{\rho^\beta \Gamma(\beta+1)}{2(b^\rho-a^\rho)^\beta}[{}^\rho I^\beta_{a+} f(b^\rho) + {}^\rho I^\beta_{b-} f(a^\rho)] \leq \frac{f(a^\rho)+f(b^\rho)}{2}$$

where ${}^\rho I^\beta_{a+} f$ and ${}^\rho I^\beta_{b-} f$ are the Katugampola fractional integrals defined in Definition 1.3.

Theorem 1.2. *Let $\beta, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbf{R}$ be a differentiable function on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f'|$ is a convex function on $[a^\rho, b^\rho]$, then the following inequality holds:*

$$\begin{aligned} \Bigg| \frac{f(a^\rho)+f(b^\rho)}{2} & - \frac{\rho^\beta \Gamma(\beta+1)}{2(b^\rho-a^\rho)^\beta}[{}^\rho I^\beta_{a+} f(b^\rho) + {}^\rho I^\beta_{b-} f(a^\rho)] \Bigg| \\ & \leq \frac{b^\rho - a^\rho}{2(b^\rho-a^\rho)^\beta}[|f'(a^\rho)| + |f'(b^\rho)|]. \end{aligned}$$

Motivated by the current research on the Hermite-Hadamard inequality, our goal in this paper is to extend Theorem 1.1 for functions of two variables that are m-convex on the coordinates and also extend Theorem 1.2 for functions of two variables whose second order mixed partial derivatives in absolute value at certain powers are (α, m)-convex on the coordinates. In what follows, we provide some preliminary definitions that will be useful to our work.

Definition 1.3 (See [5]). *Let $\beta, \rho > 0$ and f be a function a real-valued function of a single variable. The Katugampola fractional integrals of f are defined as follows:*

$$^{\rho}I_{a+}^{\beta}f(x) = \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_a^x \left(x^{\rho} - t^{\rho}\right)^{\beta-1} t^{\rho-1} f(t)dt$$

and

$$^{\rho}I_{b-}^{\beta}f(x) = \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_x^b \left(t^{\rho} - x^{\rho}\right)^{\beta-1} t^{\rho-1} f(t)dt$$

where $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

For some recent results related to the Katugampola fractional integral, we refer the interested reader to the papers [2, 5, 6, 8, 9, 10, 11].

The following fractional integrals for functions of two independent variables are natural extensions of the Katugampola fractional integrals in Definition 1.3.

Definition 1.4. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ and f be a function of two independent variables. We define, the Katugampola fractional integrals of f on the coordinates as follows:*

$$^{\rho_1}I_{a+}^{\beta_1}f(x,y) := \frac{\rho_1^{1-\beta_1}}{\Gamma(\beta_1)} \int_a^x \left(x^{\rho_1} - u^{\rho_1}\right)^{\beta_1-1} u^{\rho_1-1} f(u,y)du,$$

$$^{\rho_1}I_{b-}^{\beta_1}f(x,y) := \frac{\rho_1^{1-\beta_1}}{\Gamma(\beta_1)} \int_x^b \left(u^{\rho_1} - x^{\rho_1}\right)^{\beta_1-1} u^{\rho_1-1} f(u,y)du,$$

$$^{\rho_2}I_{c+}^{\beta_2}f(x,y) := \frac{\rho_2^{1-\beta_2}}{\Gamma(\beta_2)} \int_c^y \left(y^{\rho_2} - v^{\rho_2}\right)^{\beta_2-1} v^{\rho_2-1} f(x,v)dv$$

and

$$^{\rho_2}I_{d-}^{\beta_2}f(x,y) := \frac{\rho_2^{1-\beta_2}}{\Gamma(\beta_2)} \int_y^d \left(v^{\rho_2} - y^{\rho_2}\right)^{\beta_2-1} v^{\rho_2-1} f(x,v)dv.$$

We define the Katugampola fractional integrals of f in the two variables as follows:

$$^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_c^y \frac{u^{\rho_1-1}v^{\rho_2-1}}{(x^{\rho_1}-u^{\rho_1})^{1-\beta_1}(y^{\rho_2}-v^{\rho_2})^{1-\beta_2}} f(u,v)dvdu,$$

$$^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_y^d \frac{u^{\rho_1-1}v^{\rho_2-1}}{(x^{\rho_1}-u^{\rho_1})^{1-\beta_1}(v^{\rho_2}-y^{\rho_2})^{1-\beta_2}} f(u,v)dvdu,$$

$$^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_x^b \int_c^y \frac{u^{\rho_1-1}v^{\rho_2-1}}{(u^{\rho_1}-x^{\rho_1})^{1-\beta_1}(x^{\rho_2}-v^{\rho_2})^{1-\beta_2}} f(u,v)dvdu$$

and

$$^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2}f(x,y) := \frac{\rho_1^{1-\beta_1}\rho_2^{1-\beta_2}}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_x^b \int_y^d \frac{u^{\rho_1-1}v^{\rho_2-1}}{(u^{\rho_1}-x^{\rho_1})^{1-\beta_1}(v^{\rho_2}-y^{\rho_2})^{1-\beta_2}} f(u,v)dvdu.$$

Definition 1.5 (See [1]). *A function* $f : [0, d] \subset \mathbf{R} \to \mathbf{R}, d > 0$ *is said to be* (α, m)*-convex function where* $(\alpha, m) \in [0, 1]^2$ *if*

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y),$$

for all $x, y \in [0, d]$ *and* $t \in [0, 1]$.

Remark 1.6. *If we choose* $(\alpha, m) = (1, m)$*, then we obtain* m*-convex functions and if* $(\alpha, m) = (1, 1)$*, then we have the ordinary convex functions.*

For some recent generalizations and results related to (α, m)-convex functions, we refer the interested reader to the papers [1, 14, 17, 15, 7].

Definition 1.7 (See [16]). *A function* $f : \Lambda := [0, b] \times [0, d] \to \mathbf{R}$ *is said to be* m*-convex on the coordinates on* Λ *for* $m \in [0, 1]$*, if*

$$\begin{aligned} f(tx + m(1-t)z, sy + m(1-s)w) &\leq stf(x, y) + mt(1-s)f(x, w) \\ &+ ms(1-t)f(z, y) + m^2(1-s)(1-t)f(z, w), \end{aligned}$$

for all $(x, y), (z, w) \in \Lambda$ *and* $(s, t) \in [0, 1] \times [0, 1]$*. Similarly,* $f : \Lambda := [0, b] \times [0, d] \to \mathbf{R}$ *is said to be* (α, m)*-convex on the coordinates on* Λ *for* $(\alpha, m) \in [0, 1] \times [0, 1]$*, if*

$$\begin{aligned} f(tx + m(1-t)z, sy + m(1-s)w) &\leq s^\alpha t^\alpha f(x, y) + mt^\alpha(1-s^\alpha)f(x, w) \\ &+ ms^\alpha(1-t^\alpha)f(z, y) + m^2(1-s^\alpha)(1-t^\alpha)f(z, w), \end{aligned}$$

for all $(x, y), (x, w), (z, y), (z, w) \in \Lambda$ *and* $(s, t) \in [0, 1] \times [0, 1]$.

Remark 1.8. *If we choose* $(\alpha, m) = (1, m)$*, then we obtain* m*-convex functions on the coordinates and if* $(\alpha, m) = (1, 1)$*, then we have the concept of convex functions on the coordinates.*

For some recent generalizations and results related to (α, m)-convex functions on the coordinates, we refer the interested reader to the papers [16, 3, 4, 13, 12].

2. Main results

The following identities will be very useful in our main results.

Lemma 2.1. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$, and $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbf{R}$ be a real-valued function of two independent variables. The following identities hold:*

$$\int_0^1 s^{\beta_2\rho_2-1} f(x^{\rho_1}, s^{\rho_2}d^{\rho_2} + (1-s^{\rho_2})c^{\rho_2})ds = \frac{\rho_2^{\beta_2-1}\Gamma(\beta_2)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_2}I_{d-}^{\beta_2} f(x^{\rho_1}, c^{\rho_2}),$$

$$\int_0^1 s^{\beta_2\rho_2-1} f(x^{\rho_1}, s^{\rho_2}c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2})ds = \frac{\rho_2^{\beta_2-1}\Gamma(\beta_2)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_2}I_{c+}^{\beta_2} f(x^{\rho_1}, d^{\rho_2}),$$

$$\int_0^1 t^{\beta_1\rho_1-1} f(t^{\rho_1}b^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, y^{\rho_2})dt = \frac{\rho_1^{\beta_1-1}\Gamma(\beta_1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\ {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, y^{\rho_2}),$$

$$\int_0^1 t^{\beta_1\rho_1-1} f(t^{\rho_1}a^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, y^{\rho_2})dt = \frac{\rho_1^{\beta_1-1}\Gamma(\beta_1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\ {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, y^{\rho_2}),$$

$$\begin{aligned}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} \quad & f(t^{\rho_1}x^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}y^{\rho_2} + (1-s^{\rho_2})d^{\rho_2})dsdt \\ &= \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I_{x+,y+}^{\beta_1,\beta_2} f(b^{\rho_1}, d^{\rho_2}),\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} \quad & f(t^{\rho_1}x^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}y^{\rho_2} + (1-s^{\rho_2})c^{\rho_2})dsdt \\ &= \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-x^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I_{x+,y-}^{\beta_1,\beta_2} f(b^{\rho_1}, c^{\rho_2}),\end{aligned}$$

$$\begin{aligned}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} \quad & f(t^{\rho_1}x^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}y^{\rho_2} + (1-s^{\rho_2})d^{\rho_2})dsdt \\ &= \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-y^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I_{x-,y+}^{\beta_1,\beta_2} f(a^{\rho_1}, d^{\rho_2})\end{aligned}$$

and

$$\begin{aligned}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} \quad & f(t^{\rho_1}x^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}y^{\rho_2} + (1-s^{\rho_2})c^{\rho_2})dsdt \\ &= \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(x^{\rho_1}-a^{\rho_1})^{\beta_1}(y^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I_{x-,y-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2}).\end{aligned}$$

Proof. The results follows directly by using change of variables and Definition 1.4. □

Theorem 2.2. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$, and $f : [0,\infty) \times [0,\infty) \to \mathbf{R}$ be an m-convex function on the coordinates for $m \in (0,1]$ with $0 \le a < b$ and $0 \le c < d$. Then the following inequalities hold:*

$$
\begin{aligned}
&\frac{1}{\beta_1\beta_2\rho_1\rho_2} f\left(\frac{a^{\rho_1}+b^{\rho_1}}{2}, \frac{c^{\rho_2}+d^{\rho_2}}{2}\right)\\
&\leq \frac{1}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+} f(b^{\rho_1}, d^{\rho_2})\\
&+\frac{m}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}\left(\frac{d^{\rho_2}}{m}-\frac{c^{\rho_2}}{m}\right)^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,(\frac{d}{\sqrt[\rho_2]{m}})-} f\left(b^{\rho_1}, \frac{c^{\rho_2}}{m}\right)\\
&+\frac{m}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{\left(\frac{b^{\rho_1}}{m}-\frac{a^{\rho_1}}{m}\right)^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{(\frac{b}{\sqrt[\rho_1]{m}})-,c+} f\left(\frac{a^{\rho_1}}{m}, d^{\rho_2}\right)\\
&+\frac{m^2}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{\left(\frac{b^{\rho_1}}{m}-\frac{a^{\rho_1}}{m}\right)^{\beta_1}\left(\frac{d^{\rho_2}}{m}-\frac{c^{\rho_2}}{m}\right)^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{(\frac{b}{\sqrt[\rho_1]{m}})-,(\frac{d}{\sqrt[\rho_2]{m}})-} f\left(\frac{a^{\rho_1}}{m}, \frac{c^{\rho_2}}{m}\right)\\
&\leq \frac{1}{(\beta_1\rho_1+\rho_1)(\beta_2\rho_2+\rho_2)}\left(\frac{f(a^{\rho_1},c^{\rho_2})}{4}+\frac{mf\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{mf\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)}{4}\right.\\
&\qquad\left.+\frac{m^2 f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}\right)\\
&+\frac{1}{\beta_1\rho_1+\rho_1}\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{mf\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^2 f\left(a^{\rho_1},\frac{c^{\rho_2}}{m^2}\right)}{4}\right.\\
&\qquad\left.+\frac{m^2 f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^3 f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m^2}\right)}{4}\right)\\
&+\frac{1}{\beta_2\rho_2+\rho_2}\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\left(\frac{mf\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)}{4}+\frac{m^2 f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}\right.\\
&\qquad\left.+\frac{m^2 f\left(\frac{a^{\rho_1}}{m^2},c^{\rho_2}\right)}{4}+\frac{m^3 f\left(\frac{a^{\rho_1}}{m^2},\frac{d^{\rho_2}}{m}\right)}{4}\right)\\
&+\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\left(\frac{m^2 f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^3 f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m^2}\right)}{4}\right.\\
&\qquad\left.+\frac{m^3 f\left(\frac{a^{\rho_1}}{m^2},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^4 f\left(\frac{a^{\rho_1}}{m^2},\frac{c^{\rho_2}}{m^2}\right)}{4}\right).
\end{aligned}
$$

Proof. By using the m-convexity of f on the coordinates, we have that for any $x^{\rho_1}, y^{\rho_1} \in [a^{\rho_1}, b^{\rho_1}]$ and $u^{\rho_2}, v^{\rho_2} \in [c^{\rho_2}, d^{\rho_2}]$

$$
\begin{aligned}
f\left(\frac{x^{\rho_1}+y^{\rho_2}}{2}, \frac{u_2^{\rho}+v^{\rho_2}}{2}\right) \quad &\leq \frac{1}{4}f(x^{\rho_1}, u^{\rho_2}) + \frac{m}{4}f\left(x^{\rho_1}, \frac{v^{\rho_2}}{m}\right)\\
&\quad + \frac{m}{4}f\left(\frac{y^{\rho_1}}{m}, u^{\rho_2}\right) + \frac{m^2}{4}f\left(\frac{y^{\rho_1}}{m}, \frac{v^{\rho_2}}{m}\right).
\end{aligned}
$$

(2.1)

In particular, if we choose $x^{\rho_1} = t^{\rho_1}a^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, y^{\rho_1} = t^{\rho_1}b^{\rho_1} + (1-t^{\rho_1})a^{\rho_1}, u^{\rho_1} = s^{\rho_2}c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2}$ and $v^{\rho_2} = s^{\rho_2}d^{\rho_2} + (1-s^{\rho_2})c^{\rho_2}$ in (2.1), then we have

$$\begin{aligned} f\left(\frac{a^{\rho_1}+b^{\rho_2}}{2}, \frac{c^{\rho_2}+d^{\rho_2}}{2}\right) \quad & \leq \frac{1}{4} f(t^{\rho_1}a^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2}) \\ & + \frac{m}{4} f\left(t^{\rho_1}a^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}\frac{d^{\rho_2}}{m} + (1-s^{\rho_2})\frac{c^{\rho_2}}{m}\right) \\ & + \frac{m}{4} f\left(t^{\rho_1}\frac{b^{\rho_1}}{m} + (1-t^{\rho_1})\frac{a^{\rho_1}}{m}, s^{\rho_2}c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2}\right) \\ & + \frac{m^2}{4} f\left(t^{\rho_1}\frac{b^{\rho_1}}{m} + (1-t^{\rho_1})\frac{a^{\rho_1}}{m}, s^{\rho_2}\frac{d^{\rho_2}}{m} + (1-s^{\rho_2})\frac{c^{\rho_2}}{m}\right). \end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by $t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}$ and integrating with respect to (s,t) over $[0,1]\times[0,1]$, we have

$$\begin{aligned} & \frac{1}{\beta_1\beta_2\rho_1\rho_2} f\left(\frac{a^{\rho_1}+b^{\rho_1}}{2}, \frac{c^{\rho_2}+d^{\rho_2}}{2}\right) \\ & \leq \frac{1}{4}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f(t^{\rho_1}a^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2})dsdt \\ & + \frac{m}{4}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f\left(t^{\rho_1}a^{\rho_1} + (1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}\frac{d^{\rho_2}}{m} + (1-s^{\rho_2})\frac{c^{\rho_2}}{m}\right)dsdt \\ & + \frac{m}{4}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f\left(t^{\rho_1}\frac{b^{\rho_1}}{m} + (1-t^{\rho_1})\frac{a^{\rho_1}}{m}, s^{\rho_2}c^{\rho_2} + (1-s^{\rho_2})d^{\rho_2}\right)dsdt \\ & + \frac{m^2}{4}\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1} f\left(t^{\rho_1}\frac{b^{\rho_1}}{m} + (1-t^{\rho_1})\frac{a^{\rho_1}}{m}, s^{\rho_2}\frac{d^{\rho_2}}{m} + (1-s^{\rho_2})\frac{c^{\rho_2}}{m}\right)dsdt. \end{aligned} \tag{2.3}$$

Now, by using Lemma 2.1 with some slight modifications and (2.3), we have

$$\begin{aligned} & \frac{1}{\beta_1\beta_2\rho_1\rho_2} f\left(\frac{a^{\rho_1}+b^{\rho_2}}{2}, \frac{c^{\rho_2}+d^{\rho_2}}{2}\right) \\ & \leq \frac{1}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+} f(b^{\rho_1}, d^{\rho_2}) \\ & + \frac{m}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}\left(\frac{d^{\rho_2}}{m}-\frac{c^{\rho_2}}{m}\right)^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,(\frac{d}{\sqrt[\rho_2]{m}})-} f\left(b^{\rho_1}, \frac{c^{\rho_2}}{m}\right) \\ & + \frac{m}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{\left(\frac{b^{\rho_1}}{m}-\frac{a^{\rho_1}}{m}\right)^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{(\frac{b}{\sqrt[\rho_1]{m}})-,c+} f\left(\frac{a^{\rho_1}}{m}, d^{\rho_2}\right) \\ & + \frac{m^2}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{\left(\frac{b^{\rho_1}}{m}-\frac{a^{\rho_1}}{m}\right)^{\beta_1}\left(\frac{d^{\rho_2}}{m}-\frac{c^{\rho_2}}{m}\right)^{\beta_2}}\ {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{(\frac{b}{\sqrt[\rho_1]{m}})-,(\frac{d}{\sqrt[\rho_2]{m}})-} f\left(\frac{a^{\rho_1}}{m}, \frac{c^{\rho_2}}{m}\right) \end{aligned}$$

This proves the first inequality. On the other hand, by using the fact that f is m-convex on the coordinates, we have

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dsdt\\
&\leq f(a^{\rho_1},c^{\rho_2})\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}s^{\rho_2}t^{\rho_1}dsdt\\
&+mf\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}(1-s^{\rho_2})t^{\rho_1}dsdt\\
&+mf\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}s^{\rho_2}(1-t^{\rho_1})dsdt\\
&+m^2f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}(1-s^{\rho_2})(1-t^{\rho_1})dsdt.
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dsdt\\
&\leq f(a^{\rho_1},c^{\rho_2})\frac{1}{(\beta_1\rho_1+\rho_1)(\beta_2\rho_2+\rho_2)}\\
&+mf\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)\frac{1}{\beta_1\rho_1+\rho_1}\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\\
&+mf\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)\frac{1}{\beta_2\rho_2+\rho_2}\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\\
&+m^2f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right).
\end{aligned} \tag{2.4}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f\left(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}\frac{d^{\rho_2}}{m}+(1-s^{\rho_2})\frac{c^{\rho_2}}{m}\right)dsdt\\
&\leq f\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)\frac{1}{(\beta_1\rho_1+\rho_1)(\beta_2\rho_2+\rho_2)}\\
&+mf\left(a^{\rho_1},\frac{c^{\rho_2}}{m^2}\right)\frac{1}{\beta_1\rho_1+\rho_1}\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\\
&+mf\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\frac{1}{\beta_2\rho_2+\rho_2}\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\\
&+m^2f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m^2}\right)\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f\left(t^{\rho_1}\frac{b^{\rho_1}}{m}+(1-t^{\rho_1})\frac{a_1^{\rho}}{m}, s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2}\right)dsdt\\
&\leq f\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)\frac{1}{(\beta_1\rho_1+\rho_1)(\beta_2\rho_2+\rho_2)}\\
&+mf\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\frac{1}{\beta_1\rho_1+\rho_1}\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\\
&+mf\left(\frac{a^{\rho_1}}{m^2},c^{\rho_2}\right)\frac{1}{\beta_2\rho_2+\rho_2}\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\\
&+m^2f\left(\frac{a^{\rho_1}}{m^2},\frac{d^{\rho_2}}{m}\right)\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f\left(t^{\rho_1}\frac{b^{\rho_1}}{m}+(1-t^{\rho_1})\frac{a^{\rho_1}}{m}, s^{\rho_2}\frac{d^{\rho_2}}{m}+(1-s^{\rho_2})\frac{c^{\rho_2}}{m}\right)dsdt\\
&\leq f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\frac{1}{(\beta_1\rho_1+\rho_1)(\beta_2\rho_2+\rho_2)}\\
&+mf\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m^2}\right)\frac{1}{\beta_1\rho_1+\rho_1}\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\\
&+mf\left(\frac{a^{\rho_1}}{m^2},\frac{d^{\rho_2}}{m}\right)\frac{1}{\beta_2\rho_2+\rho_2}\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\\
&+m^2f\left(\frac{a^{\rho_1}}{m^2},\frac{c^{\rho_2}}{m^2}\right)\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)
\end{aligned} \tag{2.7}$$

Now, we multiply (2.4) by $\frac{1}{4}$, multiply (2.5) and (2.6) by $\frac{m}{4}$, and multiply (2.7) by $\frac{m^2}{4}$, then add the results and use Lemma 2.1 to get

$$\begin{aligned}
&\frac{1}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\;{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+}f(b^{\rho_1},d^{\rho_2})\\
&+\frac{m}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}\left(\frac{d^{\rho_2}}{m}-\frac{c^{\rho_2}}{m}\right)^{\beta_2}}\;{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,(\frac{d}{\sqrt[\rho_2]{m}})-}f\left(b^{\rho_1},\frac{c^{\rho_2}}{m}\right)\\
&+\frac{m}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{\left(\frac{b^{\rho_1}}{m}-\frac{a^{\rho_1}}{m}\right)^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\;{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{(\frac{b}{\sqrt[\rho_1]{m}})-,c+}f\left(\frac{a^{\rho_1}}{m},d^{\rho_2}\right)\\
&+\frac{m^2}{4}\frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1)\Gamma(\beta_2)}{\left(\frac{b^{\rho_1}}{m}-\frac{a^{\rho_1}}{m}\right)^{\beta_1}\left(\frac{d^{\rho_2}}{m}-\frac{c^{\rho_2}}{m}\right)^{\beta_2}}\;{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{(\frac{b}{\sqrt[\rho_1]{m}})-,(\frac{d}{\sqrt[\rho_2]{m}})-}f\left(\frac{a^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\\
&\leq\frac{1}{(\beta_1\rho_1+\rho_1)(\beta_2\rho_2+\rho_2)}\left(\frac{f(a^{\rho_1},c^{\rho_2})}{4}+\frac{mf\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{mf\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)}{4}\right.\\
&\qquad\left.+\frac{m^2f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}\right)\\
&+\frac{1}{\beta_1\rho_1+\rho_1}\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{mf\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^2f\left(a^{\rho_1},\frac{c^{\rho_2}}{m^2}\right)}{4}\right.\\
&\qquad\left.+\frac{m^2f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^3f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m^2}\right)}{4}\right)\\
&+\frac{1}{\beta_2\rho_2+\rho_2}\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\left(\frac{mf\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)}{4}+\frac{m^2f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}\right.\\
&\qquad\left.+\frac{m^2f\left(\frac{a^{\rho_1}}{m^2},c^{\rho_2}\right)}{4}+\frac{m^3f\left(\frac{a^{\rho_1}}{m^2},\frac{d^{\rho_2}}{m}\right)}{4}\right)\\
&+\left(\frac{1}{\beta_2\rho_2}-\frac{1}{\beta_2\rho_2+\rho_2}\right)\left(\frac{1}{\beta_1\rho_1}-\frac{1}{\beta_1\rho_1+\rho_1}\right)\left(\frac{m^2f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^3f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m^2}\right)}{4}\right.\\
&\qquad\left.+\frac{m^3f\left(\frac{a^{\rho_1}}{m^2},\frac{d^{\rho_2}}{m}\right)}{4}+\frac{m^4f\left(\frac{a^{\rho_1}}{m^2},\frac{c^{\rho_2}}{m^2}\right)}{4}\right).
\end{aligned}$$

This proves the second inequality. Hence the proof is complete. □

Corollary 2.3. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$, and $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbf{R}$ be a convex function on the coordinates with $0 \le a < b$ and $0 \le c < d$. Then the following inequalities hold:*

$$
\begin{aligned}
& f\left(\tfrac{a^{\rho_1}+b^{\rho_2}}{2}, \tfrac{c^{\rho_2}+d^{\rho_2}}{2}\right) \\
& \le \tfrac{1}{4} \tfrac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \Bigg({}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,c+} f(a^{\rho_1}, d^{\rho_2}) \\
& \qquad + {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,d-} f(b^{\rho_1}, c^{\rho_2}) + {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+} f(b^{\rho_1}, d^{\rho_2}) \Bigg) \\
& \le \tfrac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}.
\end{aligned}
$$

Proof. The result follows directly from Theorem 2.2 if we choose $m = 1$. □

Lemma 2.4. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ and $f : [a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}] \to \mathbf{R}$ be a twice partial differentiable mapping on $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ with $0 \le a < b$ and $0 \le c < d$, and $\dfrac{\partial^2 f}{\partial t \partial s} \in L_1([a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}])$. Then the following equality holds:*

$$
\begin{aligned}
& \tfrac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4} \\
& \quad - \tfrac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \Big[{}^{\rho_2}I^{\beta_2}_{c+} f(a^{\rho_1}, d^{\rho_2}) + {}^{\rho_2}I^{\beta_2}_{c+} f(b^{\rho_1}, d^{\rho_2}) \\
& \qquad + {}^{\rho_2}I^{\beta_2}_{d-} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_2}I^{\beta_2}_{d-} f(b^{\rho_1}, c^{\rho_2}) \Big] \\
& \quad - \tfrac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \Big[{}^{\rho_1}I^{\beta_1}_{a+} f(b^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I^{\beta_1}_{a+} f(b^{\rho_1}, d^{\rho_2}) \\
& \qquad + {}^{\rho_1}I^{\beta_1}_{b-} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I^{\beta_1}_{b-} f(a^{\rho_1}, d^{\rho_2}) \Big] \\
& \quad + \tfrac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \Big[{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+} f(b^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,d-} f(b^{\rho_1}, c^{\rho_2}) \\
& \qquad + {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,c+} f(c^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-} f(a^{\rho_1}, c^{\rho_2}) \Big] \\
& = \tfrac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4} \Big(I_1 - I_2 - I_3 + I_4 \Big),
\end{aligned}
$$

where

$$I_1 = \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dtds,$$
$$I_2 = \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dtds,$$
$$I_3 = \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dtds$$

and

$$I_4 = \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}a^{\rho_1}+(1-t^{\rho_1})b^{\rho_1}, s^{\rho_2}c^{\rho_2}+(1-s^{\rho_2})d^{\rho_2})dtds.$$

Proof. By using integration by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 s^{(\beta_2+1)\rho_2-1}\left[\int_0^1 t^{\beta_1\rho_1}t^{\rho_1-1}\frac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dt\right]ds\\
&= \int_0^1 s^{(\beta_2+1)\rho_2-1}\left[\frac{1}{(b^{\rho_1}-a^{\rho_1})\rho_1}t^{\beta_1\rho_1}\frac{\partial}{\partial s}f(t^{\rho}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\Big|_{t=0}^{t=1}\right.\\
&\qquad\left.-\frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\frac{\partial}{\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dt\right]ds\\
&= \int_0^1 s^{(\beta_2+1)\rho_2-1}\left[\frac{1}{(b^{\rho_1}-a^{\rho_1})\rho_1}\frac{\partial}{\partial s}f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\right.\\
&\qquad\left.-\frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\int_0^1 t^{\beta_1\rho_1-1}\frac{\partial}{\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dt\right]ds\\
&= \frac{1}{(b^{\rho_1}-a^{\rho_1})\rho_1}\int_0^1 s^{\beta_2\rho_2}s^{\rho_2-1}\frac{\partial}{\partial s}f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds-\frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\\
&\quad\times\int_0^1 t^{\beta_1\rho_1-1}\left[\int_0^1 s^{\beta_2\rho_2}s^{\rho_2-1}\frac{\partial}{\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds\right]dt\\
&= \frac{1}{(b^{\rho_1}-a^{\rho_1})(y^{\rho_2}-c^{\rho_2})\rho_1\rho_2}s^{\beta_2\rho_2}f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\Big|_{s=0}^{s=1}\\
&\quad-\frac{\beta_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1}\int_0^1 s^{\beta_2\rho_2-1}f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds\\
&\quad\frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})}\\
&\quad\times\int_0^1 t^{\beta_1\rho_1-1}\left[\frac{1}{(d^{\rho_2}-c^{\rho_2})\rho_2}s^{\beta_2\rho_2}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\Big|_{s=0}^{s=1}\right.\\
&\qquad\left.-\frac{\beta_2}{(d^{\rho_2}-c^{\rho_2})}\int_0^1 s^{\beta_2\rho_2-1}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds\right]dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1 &= \frac{1}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1\rho_2}f(b^{\rho_1}, d^{\rho_2})\\
&\quad-\frac{\beta_2}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1}\int_0^1 s^{\beta_2\rho_2-1}f(b^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})ds\\
&\quad-\frac{\beta_1}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_2}\int_0^1 t^{\beta_1\rho_1-1}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, d^{\rho_2})dt\\
&\quad+\frac{\beta_1\beta_2}{(b^{\rho_1}-a^{\rho_1})(b^{\rho_2}-c^{\rho_2})}\\
&\quad\times\int_0^1\int_0^1 t^{\beta_1\rho_1-1}s^{\beta_2\rho_2-1}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})dsdt
\end{aligned}$$

By using Lemma 2.1, we have

$$\begin{aligned} I_1 = &\frac{1}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})\rho_1\rho_2} f(b^{\rho_1}, d^{\rho_2}) \\ &- \frac{\rho_2^{\beta_2-1}\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})^{\beta_2+1}\rho_1} \ {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1}, c^{\rho_2}) \\ &- \frac{\rho_1^{\beta_1-1}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1+1}(d^{\rho_2}-c^{\rho_2})\rho_2} \ {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, d^{\rho_2}) \\ &+ \frac{\rho_1^{\beta_1-1}\rho_2^{\beta_2-1}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1+1}(d^{\rho_2}-c^{\rho_2})^{\beta_2+1}} \ {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2}). \end{aligned}$$

So, it follows that

$$\begin{aligned} &(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2 I_1 \\ &= f(b^{\rho_1}, d^{\rho_2}) - \frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1}, c^{\rho_2}) - \frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \ {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, d^{\rho_2}) \\ &\quad + \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2}). \end{aligned} \tag{2.8}$$

By using similar arguments as in the above, we obtained the following.

$$\begin{aligned} &(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2 I_2 \\ &= -f(b^{\rho_1}, c^{\rho_2}) + \frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_2}I_{c+}^{\beta_2} f(b^{\rho_1}, d^{\rho_2}) + \frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \ {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, c^{\rho_2}) \\ &\quad - \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(a^{\rho_1}, d^{\rho_2}), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2 I_3 \\ &= -f(a^{\rho_1}, d^{\rho_2}) + \frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_2}I_{d-}^{\beta_2} f(a^{\rho_1}, c^{\rho_2}) + \frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \ {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, d^{\rho_2}) \\ &\quad - \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(b^{\rho_1}, c^{\rho_2}), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &(b^{\rho_1} - a^{\rho_1})(d^{\rho_2} - c^{\rho_2})\rho_1\rho_2 I_4 \\ &= f(a^{\rho_1}, c^{\rho_2}) - \frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_2}I_{c+}^{\beta_2} f(a^{\rho_1}, d^{\rho_2}) - \frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \ {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, c^{\rho_2}) \\ &\quad + \frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \ {}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(b^{\rho_1}, d^{\rho_2}). \end{aligned} \tag{2.11}$$

The desired identity follows from adding (2.8), (2.9), (2.10) and (2.11). □

Theorem 2.5. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ and $f : [0,\infty) \times [0,\infty) \to \mathbf{R}$ be a twice partial differentiable mapping on $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ with $0 \le a < b$ and $0 \le c < d$, and $\frac{\partial^2 f}{\partial t \partial s} \in L_1([a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}])$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is (α, m)-convex on the coordinates on $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ for $(\alpha, m) \in (0,1] \times (0,1]$ and $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| \tfrac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4} \\
&\quad - \tfrac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \Big[{}^{\rho_2}I_{c+}^{\beta_2} f(a^{\rho_1}, d^{\rho_2}) + {}^{\rho_2}I_{c+}^{\beta_2} f(b^{\rho_1}, d^{\rho_2}) \\
&\qquad\qquad + {}^{\rho_2}I_{d-}^{\beta_2} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1}, c^{\rho_2}) \Big] \\
&\quad - \tfrac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}} \Big[{}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1}, d^{\rho_2}) \\
&\qquad\qquad + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, c^{\rho_2}) + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1}, d^{\rho_2}) \Big] \\
&\quad + \tfrac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}} \Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(b^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(b^{\rho_1}, c^{\rho_2}) \\
&\qquad\qquad + {}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(c^{\rho_1}, d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1}, c^{\rho_2}) \Big] \Bigg| \\
&\le \tfrac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4} \left(\tfrac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2} \right)^{1-\frac{1}{q}} \\
&\quad \times \Bigg\{ \Bigg(\tfrac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2} \left| \tfrac{\partial^2}{\partial t \partial s} f(b^{\rho_1}, d^{\rho_2}) \right|^q \\
&\quad + \tfrac{1}{(\beta_1+\alpha+1)\rho_1} \left(\tfrac{1}{(\beta_2+1)\rho_2} - \tfrac{1}{(\beta_2+\alpha+1)\rho_2} \right) m \left| \tfrac{\partial^2}{\partial t \partial s} f\left(b^{\rho_1}, \tfrac{c^{\rho_2}}{m}\right) \right|^q \\
&\quad + \tfrac{1}{(\beta_2+\alpha+1)\rho_2} \left(\tfrac{1}{(\beta_1+1)\rho_1} - \tfrac{1}{(\beta_1+\alpha+1)\rho_1} \right) m \left| \tfrac{\partial^2}{\partial t \partial s} f\left(\tfrac{a^{\rho_1}}{m}, d^{\rho_2}\right) \right|^q \\
&\quad + \left(\tfrac{1}{(\beta_2+1)\rho_2} - \tfrac{1}{(\beta_2+\alpha+1)\rho_2} \right) \left(\tfrac{1}{(\beta_1+1)\rho_1} - \tfrac{1}{(\beta_1+\alpha+1)\rho_1} \right) \\
&\quad \times m^2 \left| \tfrac{\partial^2}{\partial t \partial s} f\left(\tfrac{a^{\rho_1}}{m}, \tfrac{c^{\rho_2}}{m}\right) \right|^q \Bigg)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg(\frac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q \\
&+\frac{1}{(\beta_1+\alpha+1)\rho_1}\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q \\
&+\frac{1}{(\beta_2+\alpha+1)\rho_2}\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},c^{\rho_2}\right)\right|^q \\
&+\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right) \\
&\times m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\frac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q \\
&+\frac{1}{(\beta_1+\alpha+1)\rho_1}\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q \\
&+\frac{1}{(\beta_2+\alpha+1)\rho_2}\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},d^{\rho_2}\right)\right|^q \\
&+\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right) \\
&\times m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\frac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q \\
&+\frac{1}{(\beta_1+\alpha+1)\rho_1}\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q \\
&+\frac{1}{(\beta_2+\alpha+1)\rho_2}\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)\right|^q \\
&+\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right) \\
&\times m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Proof. By using Lemma 2.4 and the properties of the absolute value, we obtain

$$\begin{aligned}
&\Bigg| \tfrac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4} \\
&\quad -\tfrac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Bigg[{}^{\rho_2}I_{c+}^{\beta_2} f(a^{\rho_1},d^{\rho_2}) + {}^{\rho_2}I_{c+}^{\beta_2} f(b^{\rho_1},d^{\rho_2}) \\
&\qquad\qquad + {}^{\rho_2}I_{d-}^{\beta_2} f(a^{\rho_1},c^{\rho_2}) + {}^{\rho_2}I_{d-}^{\beta_2} f(b^{\rho_1},c^{\rho_2})\Bigg] \\
&\quad -\tfrac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Bigg[{}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1},c^{\rho_2}) + {}^{\rho_1}I_{a+}^{\beta_1} f(b^{\rho_1},d^{\rho_2}) \\
&\qquad\qquad + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1},c^{\rho_2}) + {}^{\rho_1}I_{b-}^{\beta_1} f(a^{\rho_1},d^{\rho_2})\Bigg] \\
&\quad +\tfrac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Bigg[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2} f(b^{\rho_1},d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2} f(b^{\rho_1},c^{\rho_2}) \\
&\qquad\qquad + {}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2} f(c^{\rho_1},d^{\rho_2}) + {}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2} f(a^{\rho_1},c^{\rho_2})\Bigg]\Bigg| \\
&\le \tfrac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4}\Big(|I_1|+|I_2|+|I_3|+I_4|\Big).
\end{aligned} \tag{2.12}$$

By using the power mean inequality and the (α,m)-convexity of $\left|\dfrac{\partial^2 f}{\partial t\partial s}\right|^q$ on the coordinates, we have

$$\begin{aligned}
|I_1| &\le \left(\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}dsdt\right)^{1-\frac{1}{q}}\Bigg(\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1} \\
&\qquad \times\left|\tfrac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\right|^q dtds\Bigg)^{\frac{1}{q}} \\
&\le \left(\tfrac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \\
&\quad \times\Bigg(\left|\tfrac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}s^{\alpha\rho_2}t^{\alpha\rho_1}dtds \\
&\quad +m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\tfrac{c^{\rho_2}}{m}\right)\right|^q \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}(1-s^{\alpha\rho_2})t^{\alpha\rho_1}dtds \\
&\quad +m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{a^{\rho_1}}{m},d^{\rho_2}\right)\right|^q \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}s^{\alpha\rho_2}(1-t^{\alpha\rho_1})dtds \\
&\quad +m^2\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{a^{\rho_1}}{m},\tfrac{c^{\rho_2}}{m}\right)\right|^q \int_0^1\int_0^1 s^{(\beta_2+1)\rho_2-1}t^{(\beta_1+1)\rho_1-1}(1-s^{\alpha\rho_2})(1-t^{\alpha\rho_1})dtds\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

Thus,

$$\begin{aligned}|I_1| &\leq \left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \left(\frac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q\right.\\ &+\frac{1}{(\beta_1+\alpha+1)\rho_1}\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q\\ &+\frac{1}{(\beta_2+\alpha+1)\rho_2}\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},d^{\rho_2}\right)\right|^q\\ &+\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)\\ &\left.\times m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{2.13}$$

Using similar argument, we have

$$\begin{aligned}|I_2| &\leq \left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \left(\frac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q\right.\\ &+\frac{1}{(\beta_1+\alpha+1)\rho_1}\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q\\ &+\frac{1}{(\beta_2+\alpha+1)\rho_2}\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},c^{\rho_2}\right)\right|^q\\ &+\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)\\ &\left.\times m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\right)^{\frac{1}{q}},\end{aligned} \tag{2.14}$$

$$\begin{aligned}|I_3| &\leq \left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \left(\frac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q\right.\\ &+\frac{1}{(\beta_1+\alpha+1)\rho_1}\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q\\ &+\frac{1}{(\beta_2+\alpha+1)\rho_2}\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},d^{\rho_2}\right)\right|^q\\ &+\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)\\ &\left.\times m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\right)^{\frac{1}{q}}\end{aligned} \tag{2.15}$$

and

$$
\begin{aligned}
|I_4| \leq & \left(\frac{1}{(\beta_1+1)(\beta_2+1)\rho_1\rho_2}\right)^{1-\frac{1}{q}} \left(\frac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q\right. \\
& +\frac{1}{(\beta_1+\alpha+1)\rho_1}\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q \\
& +\frac{1}{(\beta_2+\alpha+1)\rho_2}\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)\right|^q \\
& +\left(\frac{1}{(\beta_2+1)\rho_2}-\frac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\frac{1}{(\beta_1+1)\rho_1}-\frac{1}{(\beta_1+\alpha+1)\rho_1}\right) \\
& \left.\times m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.16}
$$

The desired inequality follows from (2.12) and using (2.13)-(2.16). □

Corollary 2.6. *Let $\beta_1,\beta_2,\rho_1,\rho_2>0$ and $f:[0,\infty)\times[0,\infty)\to\mathbf{R}$ be a twice partial differentiable mapping on $(a^{\rho_1},b^{\rho_1})\times(c^{\rho_2},d^{\rho_2})$ with $0\leq a<b$ and $0\leq c<d$, and $\dfrac{\partial^2 f}{\partial t\partial s}\in L_1([a^{\rho_1},b^{\rho_1}]\times[c^{\rho_2},d^{\rho_2}])$. If $\left|\dfrac{\partial^2 f}{\partial t\partial s}\right|$ is (α,m)-convex on the coordinates on $(a^{\rho_1},b^{\rho_1})\times(c^{\rho_2},d^{\rho_2})$ for $(\alpha,m)\in(0,1]\times(0,1]$, then the following inequality holds:*

$$
\begin{aligned}
& \left|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}\right. \\
& -\frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\left[{}^{\rho_2}I_{c+}^{\beta_2}f(a^{\rho_1},d^{\rho_2})+{}^{\rho_2}I_{c+}^{\beta_2}f(b^{\rho_1},d^{\rho_2})\right. \\
& \qquad\left.+{}^{\rho_2}I_{d-}^{\beta_2}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_2}I_{d-}^{\beta_2}f(b^{\rho_1},c^{\rho_2})\right] \\
& -\frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\left[{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},d^{\rho_2})\right. \\
& \qquad\left.+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},d^{\rho_2})\right] \\
& +\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\left[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2}f(b^{\rho_1},c^{\rho_2})\right. \\
& \qquad\left.\left.+{}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2}f(c^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2}f(a^{\rho_1},c^{\rho_2})\right]\right| \\
\leq & \frac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg[\tfrac{1}{(\beta_1+\alpha+1)(\beta_2+\alpha+1)\rho_1\rho_2}\left(\left|\tfrac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|+\left|\tfrac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|\right.\\
&\qquad\left.+\left|\tfrac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|+\left|\tfrac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|\right)\\
&+\tfrac{1}{(\beta_1+\alpha+1)\rho_1}\left(\tfrac{1}{(\beta_2+1)\rho_2}-\tfrac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\tfrac{c^{\rho_2}}{m}\right)\right|\right.\\
&\qquad\left.+m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\tfrac{d^{\rho_2}}{m}\right)\right|+m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\tfrac{c^{\rho_2}}{m}\right)\right|+m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\tfrac{d^{\rho_2}}{m}\right)\right|\right)\\
&+\tfrac{1}{(\beta_2+\alpha+1)\rho_2}\left(\tfrac{1}{(\beta_1+1)\rho_1}-\tfrac{1}{(\beta_1+\alpha+1)\rho_1}\right)\left(m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{a^{\rho_1}}{m},d^{\rho_2}\right)\right|\right.\\
&\qquad\left.+m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{a^{\rho_1}}{m},c^{\rho_2}\right)\right|+m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{b^{\rho_1}}{m},d^{\rho_2}\right)\right|+m\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{b^{\rho_1}}{m},c^{\rho_2}\right)\right|\right)\\
&+\left(\tfrac{1}{(\beta_2+1)\rho_2}-\tfrac{1}{(\beta_2+\alpha+1)\rho_2}\right)\left(\tfrac{1}{(\beta_1+1)\rho_1}-\tfrac{1}{(\beta_1+\alpha+1)\rho_1}\right)\\
&\times\left(m^2\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{a^{\rho_1}}{m},\tfrac{c^{\rho_2}}{m}\right)\right|+m^2\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{a^{\rho_1}}{m},\tfrac{d^{\rho_2}}{m}\right)\right|\right.\\
&\qquad\left.+m^2\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{b^{\rho_1}}{m},\tfrac{c^{\rho_2}}{m}\right)\right|+m^2\left|\tfrac{\partial^2}{\partial t\partial s}f\left(\tfrac{b^{\rho_1}}{m},\tfrac{d^{\rho_2}}{m}\right)\right|\right)\Bigg].
\end{aligned}
$$

Proof. The result follows directly from Theorem 2.5, if we take $q = 1$. $\square$

Theorem 2.7. *Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ and $f : [0,\infty) \times [0,\infty) \to \mathbf{R}$ be a twice partial differentiable mapping on $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ with $0 \le a < b$ and $0 \le c < d$, and $\dfrac{\partial^2 f}{\partial t \partial s} \in L_1([a^{\rho_1}, b^{\rho_1}] \times [c^{\rho_2}, d^{\rho_2}])$. If $\left|\dfrac{\partial^2 f}{\partial t\partial s}\right|^q$ is (α, m)-convex on the coordinates on $(a^{\rho_1}, b^{\rho_1}) \times (c^{\rho_2}, d^{\rho_2})$ for $(\alpha, m) \in (0,1] \times (0,1]$ and $q > 1$, then the following inequality holds:*

$$\Bigg|\frac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4}$$

$$-\frac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Bigg[{}^{\rho_2}I^{\beta_2}_{c+}f(a^{\rho_1},d^{\rho_2})+{}^{\rho_2}I^{\beta_2}_{c+}f(b^{\rho_1},d^{\rho_2})$$

$$+{}^{\rho_2}I^{\beta_2}_{d-}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_2}I^{\beta_2}_{d-}f(b^{\rho_1},c^{\rho_2})\Bigg]$$

$$-\frac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Bigg[{}^{\rho_1}I^{\beta_1}_{a+}f(b^{\rho_1},c^{\rho_2})+{}^{\rho_1}I^{\beta_1}_{a+}f(b^{\rho_1},d^{\rho_2})$$

$$+{}^{\rho_1}I^{\beta_1}_{b-}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_1}I^{\beta_1}_{b-}f(a^{\rho_1},d^{\rho_2})\Bigg]$$

$$+\frac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Bigg[{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,c+}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{a+,d-}f(b^{\rho_1},c^{\rho_2})$$

$$+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,c+}f(c^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I^{\beta_1,\beta_2}_{b-,d-}f(a^{\rho_1},c^{\rho_2})\Bigg]\Bigg|$$

$$\leq\frac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4}\left(\frac{1}{((\beta_1+1)\rho_1r-r+1)((\beta_2+1)\rho_2r-r+1)}\right)^{\frac{1}{r}}$$

$$\times\left(\frac{1}{(\alpha\rho_1+1)(\alpha\rho_2+1)}\right)^{\frac{1}{q}}\Bigg\{\Bigg(\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q+\alpha\rho_2m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q$$

$$+\alpha\rho_1m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},d^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q+\alpha\rho_2m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q$$

$$+\alpha\rho_1m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},c^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q+\alpha\rho_2m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q$$

$$+\alpha\rho_1m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},d^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q+\alpha\rho_2m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q$$

$$+\alpha\rho_1m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}}\Bigg\},$$

where $\dfrac{1}{r}+\dfrac{1}{q}=1$.

Proof. *By using Lemma 2.4 and the properties of the absolute value, we obtain*

$$\begin{aligned}
&\Bigg|\tfrac{f(a^{\rho_1},c^{\rho_2})+f(a^{\rho_1},d^{\rho_2})+f(b^{\rho_1},c^{\rho_2})+f(b^{\rho_1},d^{\rho_2})}{4} \\
&\quad -\tfrac{\rho_2^{\beta_2}\Gamma(\beta_2+1)}{4(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_2}I_{c+}^{\beta_2}f(a^{\rho_1},d^{\rho_2})+{}^{\rho_2}I_{c+}^{\beta_2}f(b^{\rho_1},d^{\rho_2}) \\
&\qquad\qquad +{}^{\rho_2}I_{d-}^{\beta_2}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_2}I_{d-}^{\beta_2}f(b^{\rho_1},c^{\rho_2})\Big] \\
&\quad -\tfrac{\rho_1^{\beta_1}\Gamma(\beta_1+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}}\Big[{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{a+}^{\beta_1}f(b^{\rho_1},d^{\rho_2}) \\
&\qquad\qquad +{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},c^{\rho_2})+{}^{\rho_1}I_{b-}^{\beta_1}f(a^{\rho_1},d^{\rho_2})\Big] \\
&\quad +\tfrac{\rho_1^{\beta_1}\rho_2^{\beta_2}\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{4(b^{\rho_1}-a^{\rho_1})^{\beta_1}(d^{\rho_2}-c^{\rho_2})^{\beta_2}}\Big[{}^{\rho_1,\rho_2}I_{a+,c+}^{\beta_1,\beta_2}f(b^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{a+,d-}^{\beta_1,\beta_2}f(b^{\rho_1},c^{\rho_2}) \\
&\qquad\qquad +{}^{\rho_1,\rho_2}I_{b-,c+}^{\beta_1,\beta_2}f(c^{\rho_1},d^{\rho_2})+{}^{\rho_1,\rho_2}I_{b-,d-}^{\beta_1,\beta_2}f(a^{\rho_1},c^{\rho_2})\Big]\Bigg| \\
&\leq \tfrac{\rho_1\rho_2(b^{\rho_1}-a^{\rho_1})(d^{\rho_2}-c^{\rho_2})}{4}\Big(|I_1|+|I_2|+|I_3|+I_4|\Big).
\end{aligned}\tag{2.17}$$

By using the Hölder's inequality and the (α, m)*-convexity of* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *on the coordinates, we have*

$$\begin{aligned}
|I_1| &\leq \left(\int_0^1\int_0^1 s^{(\beta_2+1)\rho_2 r-r}t^{(\beta_1+1)\rho_1 r-r}dsdt\right)^{\frac{1}{r}} \\
&\quad\times\left(\int_0^1\int_0^1\left|\tfrac{\partial^2}{\partial t\partial s}f(t^{\rho_1}b^{\rho_1}+(1-t^{\rho_1})a^{\rho_1}, s^{\rho_2}d^{\rho_2}+(1-s^{\rho_2})c^{\rho_2})\right|^q dtds\right)^{\frac{1}{q}} \\
&\leq \left(\tfrac{1}{((\beta_1+1)\rho_1 r-r+1)((\beta_2+1)\rho_2 r-r+1)}\right)^{\frac{1}{r}} \\
&\quad\times\Bigg(\left|\tfrac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q\int_0^1\int_0^1 s^{\alpha\rho_2}t^{\alpha\rho_1}dtds
\end{aligned}$$

$$
\begin{aligned}
&+m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q\int_0^1\int_0^1(1-s^{\alpha\rho_2})t^{\alpha\rho_1}dtds\\
&+m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},d^{\rho_2}\right)\right|^q\int_0^1\int_0^1 s^{\alpha\rho_2}(1-t^{\alpha\rho_1})dtds\\
&+m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\int_0^1\int_0^1(1-s^{\alpha\rho_2})(1-t^{\alpha\rho_1})dtds\Bigg)^{\frac{1}{q}}\\
=&\left(\frac{1}{((\beta_1+1)\rho_1r-r+1)((\beta_2+1)\rho_2r-r+1)}\right)^{\frac{1}{r}}\Bigg(\frac{1}{(\alpha\rho_1+1)(\alpha\rho_2+1)}\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q\\
&+\frac{1}{(\alpha\rho_1+1)}\left(\frac{\alpha\rho_2}{(\alpha\rho_2+1)}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q\\
&+\frac{1}{(\alpha\rho_2+1)}\left(\frac{\alpha\rho_1}{(\alpha\rho_1+1)}\right)m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},d^{\rho_2}\right)\right|^q\\
&+\left(\frac{\alpha\rho_2}{(\alpha\rho_2+1)}\right)\left(\frac{\alpha\rho_1}{(\alpha\rho_1+1)}\right)m^2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|I_1| \le&\left(\frac{1}{((\beta_1+1)\rho_1r-r+1)((\beta_2+1)\rho_2r-r+1)}\right)^{\frac{1}{r}}\left(\frac{1}{(\alpha\rho_1+1)(\alpha\rho_2+1)}\right)^{\frac{1}{q}}\\
&\times\Bigg(\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},d^{\rho_2})\right|^q+\alpha\rho_2m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q\\
&+\alpha\rho_1m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},d^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned}
\tag{2.18}
$$

Using similar arguments, we have

$$
\begin{aligned}
|I_2| \le&\left(\frac{1}{((\beta_1+1)\rho_1r-r+1)((\beta_2+1)\rho_2r-r+1)}\right)^{\frac{1}{r}}\left(\frac{1}{(\alpha\rho_1+1)(\alpha\rho_2+1)}\right)^{\frac{1}{q}}\\
&\times\Bigg(\left|\frac{\partial^2}{\partial t\partial s}f(b^{\rho_1},c^{\rho_2})\right|^q+\alpha\rho_2m\left|\frac{\partial^2}{\partial t\partial s}f\left(b^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q\\
&+\alpha\rho_1m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},c^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{a^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\Bigg)^{\frac{1}{q}},
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
|I_3| \leq & \left(\frac{1}{((\beta_1+1)\rho_1 r-r+1)((\beta_2+1)\rho_2 r-r+1)}\right)^{\frac{1}{r}}\left(\frac{1}{(\alpha\rho_1+1)(\alpha\rho_2+1)}\right)^{\frac{1}{q}} \\
& \times\left(\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},d^{\rho_2})\right|^q+\alpha\rho_2 m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{c^{\rho_2}}{m}\right)\right|^q\right. \\
& \left.+\alpha\rho_1 m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},d^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{c^{\rho_2}}{m}\right)\right|^q\right)^{\frac{1}{q}}
\end{aligned}
\tag{2.20}
$$

and

$$
\begin{aligned}
|I_4| \leq & \left(\frac{1}{((\beta_1+1)\rho_1 r-r+1)((\beta_2+1)\rho_2 r-r+1)}\right)^{\frac{1}{r}}\left(\frac{1}{(\alpha\rho_1+1)(\alpha\rho_2+1)}\right)^{\frac{1}{q}} \\
& \times\left(\left|\frac{\partial^2}{\partial t\partial s}f(a^{\rho_1},c^{\rho_2})\right|^q+\alpha\rho_2 m\left|\frac{\partial^2}{\partial t\partial s}f\left(a^{\rho_1},\frac{d^{\rho_2}}{m}\right)\right|^q\right. \\
& \left.+\alpha\rho_1 m\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},c^{\rho_2}\right)\right|^q+\alpha^2m^2\rho_1\rho_2\left|\frac{\partial^2}{\partial t\partial s}f\left(\frac{b^{\rho_1}}{m},\frac{d^{\rho_2}}{m}\right)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.21}
$$

The desired inequality follows from (2.17) and using (2.18)-(2.21). □

3. Conclusion

We established a new integral inequality of Hermite-Hadamard type for functions of two variables that are m-convex on the coordinates via generalized Katugampola type fractional integrals of functions of two variables. We also established three new integral inequalities of Hermite-Hadamard type for functions whose second order mixed partial derivatives in absolute value at certain powers are (α, m)-convex on the coordinates. If we choose $(\alpha, m) = (1, 1)$, then our results hold for functions whose second order mixed partial derivatives are convex on the coordinates. We believe that, results in this paper we inspire further research on the inequalities involving functions of two independent variables via generalized fractional integrals.

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