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On graded 2-classical prime submodules of graded modules over graded commutative rings

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Abstract:

Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded 2-classical prime submodules. Various properties of graded 2-classical prime submodules are considered.

Keywords: Graded 2-classical prime submodule; Graded classical prime submodule; Graded prime submodule.

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1. Introduction and Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a G-graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called to be homogeneous of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Moreover, $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of R. Then I is called a graded ideal of (R, G) if $I = \bigoplus_{g\in G}(I \bigcap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$ (see [14].)

Let R be a G-graded ring and M an R-module. We say that M is a G-graded R-module (or graded R-module) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = g \in G \bigoplus M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = g \in G \bigcup M_g$ and the elements of $h(M)$ are called to be homogeneous. Let $M = g \in G \bigoplus M_g$ be a graded R-module and N a submodule of M. Then N is called a graded submodule of M if $N = g \in G \bigoplus N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$ (see [14].)

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. Then $(N :_R M)$ is defined as $(N :_R M) = \{r \in R | rM \subseteq N\}$. It is shown in [6, Lemma 2.1] that if N is a graded submodule of M, then $N :_R M) = \{r \in R : rM \subseteq N\}$ is a graded ideal of R.

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = g \in G \bigoplus (S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \ \ and\ g = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fractions $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = g \in G \bigoplus (S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S \ \ and\ g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = g \in G \cup (S^{-1}R)_g$ and $h(S^{-1}M) = g \in G \cup (S^{-1}M)_g$. Consider the graded homomorphism $\eta : M \to S^{-1}M$ defined by $\eta(m) = m/1$. For any graded submodule N of M, the submodule of $S^{-1}M$ generated by $\eta(N)$ is denoted by $S^{-1}N$. Similar to non graded case, one can prove that $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N :_R M) = \phi$. Indeed, $S^{-1}N = \{\beta \in S^{-1}M : \beta = m/s \text{ for } m \in N \text{ and } s \in S\}$. For more details, one can refer to [12, 13, 14, 15].

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. There is a wide variety of applications of graded algebras in geometry and physics. In differential geometry, a graded Lie algebra arising in the study of connections through using Frolicher-Nijenhuis or Nijenhuis-Richardson bracket (see [11].) They concern with many applications in noncommutative geometry. Furthermore, in physical sense and in studying supermanifolds, supersymmetries and quantizations of systems with symmetry, graded rings and modules play a key role (see [9, 17].)

The notion of graded prime ideals was introduced in [19] and studied in [7, 18, 20]. A proper graded ideal P of R is said to be a graded prime ideal if whenever $rs \in P$, we have $r \in P$ or $s \in P$, where $r, s \in h(R)$.

S.E. Atani in [6] extended graded prime ideals to graded prime submodules. A proper graded submodule P of M is said to be a graded prime submodule if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in P$, then either $r \in (P :_R M)$ or $m \in P$. Several authors investigated properties of graded prime submodules, for examples see [2, 5, 8, 16].

The notion of graded classical prime submodules as a generalization of graded prime submodules was introduced by A.Y. Darani and S. Motmaen in [10] and studied in [3, 4]. A proper graded submodule N of M is called a graded classical prime submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $sm \in N$.

Here, we generalize this concept to the concept of graded 2-classical prime submodules and give a number of its properties. For example, we give a characterization of a graded 2-classical prime submodules (see Theorem 2.2) and study the behavior of graded 2-classical prime submodules under graded homomorphisms (see Theorem 2.4) and under localization (see Theorem 2.6).

2. Results

Definition 2.1. *Let R be a G-graded ring and M a graded R-module. A proper graded submodule C of M is called a graded 2-classical prime submodule if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in C$,* *then either $r_g^2 m_\lambda \in C$ or $s_h^2 m_\lambda \in C$.*

The next theorem gives a characterization of being a graded 2-classical prime submodules.

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module and C a proper graded submodule of M. Then the following statements are equivalent.*

1. C is a graded 2-classical prime submodule of M.

2. For every graded submodule K of M and every pair of elements $r_g, s_h \in h(R)$ such that $r_g s_h K \subseteq C$, either $r_g^2 K \subseteq C$ or $s_h^2 K \subseteq C$.

Proof. $(i) \Rightarrow (ii)$ Suppose that C is a graded 2-classical prime submodule of M. Let $r_g s_h K \subseteq C$ for some graded submodule K of M and $r_g, s_h \in h(R)$. Assume that $r_g^2 KC$ and $s_h^2 KC$. Since K is generated by $K \cap h(M)$, there exist $k_{g_1}, k_{g_2} \in K \cap h(M)$ such that $r_g^2 k_{g_1}, s_h^2 k_{g_2} \notin C$. Since C is a graded 2-classical prime submodule, $r_g s_h k_{g_1} \in C$ and $r_g^2 k_{g_1} \notin C$, we conclude that $s_h^2 k_{g_1} \in C$. By a similar argument, since $r_g s_h k_{g_2} \in C$ and $s_h^2 k_{g_2} \notin C$, then we deduce that $r_g^2 k_{g_2} \in C$. Since $r_g s_h (k_{g_1} + k_{g_2}) \in C$, we have either $r_g^2 (k_{g_1} + k_{g_2}) \in C$ or $s_h^2 (k_{g_1} + k_{g_2}) \in C$. If $r_g^2 (k_{g_1} + k_{g_2}) = r_g^2 k_{g_1} + r_g^2 k_{g_2} \in C$, then since $r_g^2 k_{g_2} \in C$, we have $r_g^2 k_{g_1} \in C$, a contradiction. If $s_h^2 (k_{g_1} + k_{g_2}) = s_h^2 k_{g_1} + s_h^2 k_{g_2} \in C$, then since $s_h^2 k_{g_1} \in C$, we have $s_h^2 k_{g_2} \in C$, which again is a contradiction. Therefore $r_g^2 K \subseteq C$ or $s_h^2 K \subseteq C$.
$(ii) \Rightarrow (i)$ Let $r_g s_h m_\lambda \in C$ for some $m_\lambda \in h(M)$ and $r_g, s_h \in h(R)$. So we have $r_g s_h (m_\lambda) \subseteq C$ where (m_λ) is a graded submodule of M generated by m_λ. By our assumption we obtain either $r_g^2 (m_\lambda) \subseteq C$ or $s_h^2 (m_\lambda) \subseteq C$ and hence either $r_g^2 m_\lambda \in C$ or $s_h^2 m_\lambda \in C$. Thus C is a graded 2-classical prime submodule of M.

Let R be a G-graded ring. The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g \in G} x_g \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in \mathrm{Gr}(I)$ if and only if $r^n \in I$ for some $n \in N$ (see [18].)

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module and C a graded 2-classical prime submodule of M. Then $\mathrm{Gr}((C :_R M))$ is a graded prime ideal of R.*

Proof. Assume that C is a graded 2-classical prime submodule of M. Let $r_g s_h \in$Gr$((C :_R M))$ for some $r_g, s_h \in h(R)$. Then $r_g^n s_h^n M \subseteq C$ for some $n \in Z^+$. Since C is a graded 2-classical prime submodule, by Theorem 2.2, we get either $r_g^{2n} M \subseteq C$ or $s_h^{2n} M \subseteq C$. Which means $r_g \in$Gr$((C :_R M))$ or $s_h \in$Gr$((C :_R M)$. Therefore Gr$((C :_R M))$ is a graded prime ideal.

Let R be a G-graded ring and M, M' graded R-modules. Let $\varphi : M \to M'$ be an R-module homomorphism. Then φ is said to be a graded homomorphism if $\varphi(M_g) \subseteq M'_g$ for all $g \in G$ (see [13].)

The following results study the behavior of graded 2-classical prime submodules under graded homomorphisms.

Theorem 2.4. *Let R be a G-graded ring and M, M' be two graded R-modules and let $\varphi : M \to M'$ be a graded homomorphism.*

1. If C is a graded 2-classical prime submodule of M', then $\varphi^{-1}(C)$ is a graded 2-classical prime submodule of M.
2. If $\varphi : M \to M'$ is a graded epimorphism and C is a graded 2-classical prime submodule of M containing Kerφ, then $\varphi(C)$ is a graded 2-classical prime submodule of M'.

Proof. (i) Assume that C is a graded 2-classical prime submodule of M'. Let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in \varphi^{-1}(C)$. Then $\varphi(r_g s_h m_\lambda) \in C$. Since φ is a graded homomorphism, we have $r_g s_h \varphi(m_\lambda) \in C$. Since C is a graded 2-classical prime submodule, we get either $r_g^2 \varphi(m_\lambda) \in C$ or $s_h^2 \varphi(m_\lambda) \in C$. This implies that $r_g^2 m_\lambda \in \varphi^{-1}(C)$ or $s_h^2 m_\lambda \in \varphi^{-1}(C)$. Hence $\varphi^{-1}(C)$ is a graded 2-classical prime submodule. (ii) Assume that C is a graded 2-classical prime submodule of M containing Kerφ. Let $r_g, s_h \in h(R)$ and $m'_\lambda \in h(M')$ such that $r_g s_h m'_\lambda \in \varphi(C)$. Then there exists $c \in C \cap h(M)$ such that $\varphi(c) = r_g s_h m'_\lambda$. Since $m'_\lambda \in h(M')$ and φ is a graded epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'_\lambda$. Then, we have $\varphi(r_g s_h m_\lambda) = \varphi(c)$ and so $\varphi(c - r_g s_h m_\lambda) = 0$. Hence $c - r_g s_h m_\lambda \in$ Ker$\varphi \subseteq C$. This yields that $r_g s_h m_\lambda \in C$. Since C is a graded 2-classical prime submodule, either $r_g^2 m_\lambda \in C$ or $s_h^2 m_\lambda \in C$. So $r_g^2 m'_\lambda = r_g^2 \varphi(m) = \varphi(r_g^2 m) \in \varphi(C)$ or $s_h^2 m'_\lambda = s_h^2 \varphi(m) = \varphi(s_h^2 m) \in \varphi(C)$. Therefore $\varphi(C)$ is a graded 2-classical prime submodule of M'.

As an immediate consequence of Theorem 2.4 we have the following corollary.

Corollary 2.5. *Let R be a G-graded ring, M a graded R-module and $N \subseteq C$ a graded submodules of M. Then C is a graded 2-classical prime submodule of M if and only if C/N is a graded 2-classical prime submodule of M/N.*

Recall that a graded zero-divisor on a graded R-module M is an element $r_g \in h(R)$ for which there exists $m_\lambda \in h(M)$ such that $m_\lambda \neq 0$ but $r_g m_\lambda = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$ (see [1].)

The following results study the behavior of graded 2-classical prime submodules under localization.

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ a multiplication closed subset of R. Then the following hold:*

1. *If C is a graded 2-classical prime submodule of M such that $(C :_R M) \cap S = \phi$, then $S^{-1}C$ is a graded 2-classical prime submodule of $S^{-1}M$.*
2. *If $S^{-1}C$ is a graded 2-classical prime submodule of $S^{-1}M$ and $S\cap$ $G\text{-}Zdv_R(M/C) = \phi$, then C is a graded 2-classical prime submodule of M.*

Proof. (i) Suppose that C is a graded 2-classical prime submodule of M and S a multiplicatively closed subset of $h(R)$. Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in h(S^{-1}R)$ and $\frac{m}{s_3} \in h(S^{-1}M)$ satisfying $\frac{r_1}{s_1}\frac{r_2}{s_2}\frac{m}{s_3} \in S^{-1}C$. Hence there exists $u \in S$ such that $r_1r_2(um) \in C$. Since C is a graded 2-classical prime submodule of M, we conclude that either $r_1^2 um \in C$ or $r_2^2 um \in C$. Thus $\frac{r_1^2(um)}{s_1^2 s_3 u} = (\frac{r_1}{s_1})^2 \frac{m}{s_3} \in S^{-1}C$ or $\frac{r_2^2(um)}{s_2^2 s_3 u} = (\frac{r_2}{s_2})^2 \frac{m}{s_3} \in S^{-1}C$. Therefore $S^{-1}C$ is a graded 2-classical prime submodule of $S^{-1}M$. (ii) Assume that $S^{-1}C$ is a graded 2-classical prime submodule of $S^{-1}M$ and $S\cap$ $G\text{-}Zdv_R(M/C) = \phi$. Let $r_1r_2m \in C$ for some $r_1, r_2 \in h(R)$ and for some $m \in h(M)$. Then $\frac{r_1r_2m}{1} = \frac{r_1}{1}\frac{r_2}{1}\frac{m}{1} \in S^{-1}C$. Since $S^{-1}C$ is a graded 2-classical prime submodule of $S^{-1}M$, we conclude that either $(\frac{r_1}{1})^2\frac{m}{1} = \frac{r_1^2 m}{1} \in S^{-1}C$ or $(\frac{r_2}{1})^2\frac{m}{1} = \frac{r_2^2 m}{1} \in S^{-1}C$. If $\frac{r_1^2 m}{1} \in S^{-1}C$, then there exists $s \in S$ such that ${}_1^2 m \in C$ and since $S\cap$ $G\text{-}Zdv_R(M/C) = \phi$, we have $r_1^2 m \in C$. With a same argument, we can show that if $\frac{r_2^2 m}{1} \in S^{-1}C$, then $r_2^2 m \in C$. Therefore C is a graded 2-classical prime submodule of M.

Let R_i be a graded commutative ring with identity and M_i be a graded R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is a graded R-module and each graded submodule of M is of the form $C = C_1 \times C_2$ for some graded submodules C_1 of M_1 and C_2 of M_2.

Theorem 2.7. *Let $R = R_1 \times R_2$ be a graded ring and $M = M_1 \times M_2$ be a graded R-module where M_1 is a graded R_1-module and M_2 is a graded R_2-module. Let C_1 and C_2 be a proper graded submodules of M_1 and M_2, respectively.*

1. *C_1 is a graded 2-classical prime submodule of M_1if and only if $C = C_1 \times M_2$ is a graded 2-classical prime of M.*
2. *C_2 is a graded 2-classical prime submodule of M_2 if and only if $C = M_1 \times C_2$ is a graded 2-classical prime submodule of M.*
3. *If $C = C_1 \times C_2$ is a graded 2-classical prime submodule of M, then C_1 and C_2 are graded 2-classical prime submodules of M_1 and M_2, respectively.*

Proof. (i) Assume that C_1 is a graded 2-classical prime submodule of M_1. Let $(r_1, r_2), (s_1, s_2) \in h(R)$ and $(n, m) \in h(M)$ such that $(r_1, r_2)(s_1, s_2)(n, m) \in C$. Then $r_1 s_1 n \in C_1$. Since C_1 is a graded 2-classical prime submodule of M_1, we have either $r_1^2 n \in C_1$ or $s_1^2 n \in N_1$. This implies $(r_1, r_2)^2(n, m) \in C$ or $(s_1, s_2)^2(n, m) \in C$. Thus C is a graded 2-classical prime submodule of M. Conversely, assume C is a graded 2-classical prime submodule of M. Let r, $s \in h(R_1)$ and $m \in h(M_1)$ such that $rsm \in C_1$. Then $(r, 1)(s, 1)(m, 0) = (rsm, 0) \in C$ and since C is a graded 2-classical prime submodule of M, we have either $(r, 1)^2(m, 0) = (r^2 m, 0) \in C$ or $(s, 1)^2(m, 0) = (s^2 m, 0) \in C$. This implies $r^2 m \in C_1$ or $s^2 m \in C_1$. Therefore C_1 is a graded 2-classical prime submodule of M_1. (ii) It can be easily verified similar to (i). (iii) Assume $C = C_1 \times C_2$ is a graded 2-classical prime submodule of M. Let r, $s \in h(R_1)$ and $m \in h(M_1)$ such that $rsm \in C_1$. Then $(rsm, 0) = (r, 1)(s, 1)(m, 0) \in C$. Since C is a graded 2-classical prime submodule, we get either $(r^2 m, 0) = (r, 1)^2(m, 0) \in C$ or $(s^2 m, 0) = (s, 1)^2(m, 0) \in C$. Therefore $r^2 m \in C_1$ or $s^2 m \in C_1$. Consequently C_1 is a graded 2-classical prime submodule of M_1. Similarly, one can show that C_2 is a graded 2-classical prime submodule of M_1.

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