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On $r-$ dynamic coloring of the gear graph families

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Abstract:

An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq \min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the $r-$dynamic chromatic number of the middle, central and line graphs of the gear graph.

Keywords: $r-$ dynamic coloring; gear graph; middle graph; central graph and line graph.

MSC (2020): 05C15.

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1. Introduction

Graphs in this paper are simple and finite. For undefined terminologies and notations see [5, 17]. Thus for a graph $G, \delta(G), \Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of G respectively. When the context is clear we write, δ, Δ and χ for brevity. For $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v in G and $d(v) = |N(v)|$. The r-dynamic chromatic number was first introduced by Montgomery [14].

An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$ and (ii) for each vertex $v \in V(G), |c(N(v))| \geq min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v and $d(v)$ its degree and r is a positive integer.

The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number denoted by $\chi_d(G)$ [1, 2, 3, 4, 8]. By simple observation, we can show that $\chi_r(G) \leq \chi_{r+1}(G)$, however $\chi_{r+1}(G) - \chi_r(G)$ can be arbitrarily large, for example $\chi(\text{Petersen}) = 2$, $\chi_d(\text{Petersen}) = 3$, but $\chi_3(\text{Petersen}) = 10$. Thus, finding an exact values of $\chi_r(G)$ is not trivially easy.

There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example, for a graph G with $\Delta(G) \geq 3$, Lai et al.[8] proved that $\chi_d(G) \leq \Delta(G)+1$. An upper bound for the dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in [7]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2log_2\alpha(G) + 3$. Taherkhani gave in [15] an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ. i.e., $\chi_2(G) - \chi(G) \leq \left\lceil (\Delta e)/\delta log\left(2e\left(\Delta^2+1\right)\right)\right\rceil$.

Li et al.proved in [10] that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Li and Zhou [9] showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with the maximum degree 3, is NP-complete.

N.Mohanapriya et al. [11, 12] studied the dynamic chromatic number for various graph families. Also, it was proven in [13] that the r-dynamic chromatic number of line graph of a helm graph H_n.

In this paper, we study $\chi_r(G)$, when $1 \leq r \leq \Delta$. We find the r- dynamic chromatic number of the middle, central and line graphs of the gear graph.

2. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [6] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The central graph [16] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The line graph [13] of G denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

The gear graph is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. The gear graph G_n has $2n + 1$ nodes and $3n$ edges.

Let $V(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{vv_i : 1 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n$ and the meaning of mod n is the obvious$\}$.

3. Main Theorem

Theorem 3.1. *Let $n \geq 5$, $M(G_n)$ be the middle graph of a gear graph G_n and let*
$\Delta = \Delta(M(G_n))$. *Then*

$$\chi_r(M(G_n)) = \begin{array}{c} n+1, \quad 1 \leq r \leq 4 \\ n+2, \quad 5 \leq r \leq \Delta - 2 \\ n+4, \quad r = \Delta - 1 \ \ and \ \ n \equiv 0 \ \ mod \ \ 3 \\ n+5, \quad r = \Delta - 1 \ \ and \ \ n \equiv 1 \ \ mod \ \ 3 \\ n+4, \quad r = \Delta - 1 \ \ and \ \ n \equiv 2 \ \ mod \ \ 3 \\ n+5, \quad r = \Delta \ \ and \ \ n \equiv 0 \ \ mod \ \ 3 \\ n+7, \quad r = \Delta \ \ and \ \ n \equiv 1 \ \ mod \ \ 3 \\ n+6, \quad r = \Delta \ \ and \ \ n \equiv 2 \ \ mod \ \ 3 \end{array}$$

Proof. By the definition of middle graph,
$V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq 2n\}$.

The vertices v and $\{e_i : 1 \leq i \leq n\}$ induces a clique of order K_{n+1} in $M(G_n)$.

Thus, $\chi_\delta(M(G_n)) \geq n+1$.
We divide the proof into some cases.

Case 1 : For $1 \leq r \leq 4$
The $r-$ dynamic $(n+1)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i and v_i.
$|N(u_i)| = d(u_i) = 2 = \delta$,
$|N(v_i)| = d(v_i) = 3$
$|N(v)| = d(v) = n$,
$|N(e_i)| = d(e_i) = n+3$
and $|N(s_i)| = d(s_i) = 5$
For $1 \leq i \leq 2n$, assign the allowed colors to the vertex s_i and also it must satisfies the $r-$ adjacency condition.

- color the vertices $s_1, s_3, s_5, s_7, \cdots s_{2n-5}, s_{2n-3}, s_{2n-1}$ with colors $c_3, c_4, c_5, \cdots c_n, c_1, c_2$(the order of assigned color is important).
- color the vertices $s_2, s_4, s_6, s_8, \cdots s_{2n-4}, s_{2n-2}, s_{2n}$ with colors $c_n, c_1, c_2, c_3, \cdots c_{n-3}, c_{n-2}, c_{n-1}$(the order of assigned color is important).

We know that the $|N(v)| = d(v) = n$, so we need the color $n+1$.
It is easy to verify that adjacency and r-adjacency conditions are fulfilled.
Hence, $\chi_r(M(G_n)) = n+1$, for $n \geq 5$ and $1 \leq r \leq 4$.

Case 2 : For $5 \leq r \leq \Delta - 2$
The $r-$ dynamic $(n+2)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i.
For $1 \leq i \leq 2n$, if any, assign the vertex s_i to one of the allowed colors - such color exists, because $|N(s_i)| = d(s_i) = 5$

- color the vertices $s_1, s_3, s_5, s_7, \cdots s_{2n-5}, s_{2n-3}, s_{2n-1}$ with colors $c_3, c_4, c_5, \cdots c_n, c_1, c_2$(the order of assigned color is important).
- color the vertices $s_2, s_4, s_6, s_8, \cdots s_{2n-4}, s_{2n-2}, s_{2n}$ with colors $c_n, c_1, c_2, c_3, \cdots c_{n-3}, c_{n-2}, c_{n-1}$(the order of assigned color is important).

- color the vertex v_i with the color c_{n+2}.

Now $|N(s_i)|$ satisfies the $r-$adjacency condition.
But $d(e_i) = n + 3$, so $N(e_i)$ having $n + 2$ colors.
It is easy to verify that the r-adjacency condition is fulfilled.
Hence, $\chi_r(M(G_n)) = n + 2$, for $n \geq 5$ and $5 \leq r \leq \Delta - 2$.

Case 3 : For $r = \Delta - 1$ and $n \equiv 0 \mod 3$
The $r-$ dynamic $(n + 4)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i.
For $1 \leq i \leq n$, assign the color c_{n+2} to v_i.
$|N(e_i)|$ having $n + 1$ colors only. So we assign one new color to s_i.

- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-5}, s_{2n-2}$ with color c_{n+3}.
- color the vertices $s_2, s_5, s_8, s_{11}, \cdots s_{2n-4}, s_{2n-1}$ with colors c_{n+4}.

Now $s_3, s_6, s_9, \cdots s_{2n-3}, s_{2n}$ are uncolored. So assign these vertices to any one of the allowed colors-such color exists.

- color the vertices $s_3, s_6, s_9, \cdots s_{2n-3}, s_{2n}$ with colors $c_5, c_7, c_9, \cdots c_{n-1}, c_1, c_3$ (the order of assigned color is important).

Now neighbours of e_i having $n + 4$ colors and an easy check shows that the $r-$adjacency condition is fulfilled.

Hence, $\chi_r(M(G_n)) = n + 4$, for $n \geq 5$, $r = \Delta - 1$ and $n \equiv 0 \mod 3$.

Case 4 : For $r = \Delta - 1$ and $n \equiv 1 \mod 3$
The $r-$ dynamic $(n + 5)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i.
For $1 \leq i \leq n$, assign the color c_{n+2} to v_i.

$N(e_i)$ having $n + 1$ colors. So we have to assign one new color to s_i.

- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-4}$ with color c_{n+3}.
- color the vertices $s_2, s_5, s_8, s_11, \cdots s_{2n-3}$ with color c_{n+4}

But neighbours of e_n having $n+1$ colors only. So we have to assign a new color c_{n+5} to s_{2n-2}.

Now neighbours of e_i having $n+2$ colors. But the vertices s_{2n-1} and s_{2n} are uncolored.

So we have to assign any one of the allowed colors to s_{2n-1} and s_{2n}.

- color the vertex s_{2n-1} with the color c_2 and color the vertex s_{2n} with the color c_3.

Now an easy check shows that the r-adjacency condition is fulfilled.
Hence, $\chi_r(M(G_n)) = n+5$, for $n \geq 5$ and $r = \Delta - 1$ and $n \equiv 1 \mod 3$.

Case 5 : For $r = \Delta - 1$ and $n \equiv 2 \mod 3$
The $r-$ dynamic $(n+4)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i.
For $1 \leq i \leq n$, assign the color c_{n+2} to v_i.

Now $N(e_i)$ having $n+1$ colors.so we have to assign one new color to s_i.

- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-3}$ with color c_{n+3}.
- color the vertices $s_2, s_5, s_8, s_1 1, \cdots s_{2n-2}$ with color c_{n+4}

But the vertices $s_3, s_6, s_9, \cdots s_{2n-4}, s_{2n-1}$ and s_{2n} are uncolored. So we have to assign any one of the allowed colors to these vertices.

- color the vertices $s_3, s_6, s_9, \cdots s_{2n-1}, s_{2n}$ with colors $c_4, c_1 1, c_7, c_3, \cdots, c_2, c_8$ respectively.(the order of assigned color is important).

Now an easy check shows that the $r-$ adjacency condition is fulfilled.
Hence, $\chi_r(M(G_n)) = n+4$, for $n \geq 5$, $r = \Delta - 1$ and $n \equiv 2 \mod 3$.

Case 6 : For $r = \Delta$ and $n \equiv 0 \mod 3$
The $r-$ dynamic $(n+5)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i.
For $1 \leq i \leq n$, assign the color c_{n+2} to v_i.
For $r = \Delta$, we have to assign two new colors to neighbours of e_i.

- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-5}, s_{2n-2}$ with color c_{n+3}
- color the vertices $s_{2n}, s_3, s_6, s_9, \cdots s_{2n-3}$ with color c_{n+4}
- color the vertices $s_2, s_5, s_8, s_{11}, \cdots s_{2n-1}$ with color c_{n+5}

Now an easy check shows that the $r-$adjacency condition is fulfilled for all the vertices.

Hence, $\chi_r(M(G_n)) = n+5$, for $n \geq 5$, $r = \Delta$ and $n \equiv 0 \mod 3$.

Case 7 : For $r = \Delta$ and $n \equiv 1 \mod 3$

The $r-$ dynamic $(n+7)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i.
For $1 \leq i \leq n$, assign the color c_{n+2} to v_i.
For $r = \Delta$, we have to assign two new colors to neighbours of e_i.

- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-4}$ with color c_{n+3}.
- color the vertices $s_{2n}, s_3, s_6, s_9, \cdots s_{2n-5}$ with color c_{n+4}.
- color the vertices $s_2, s_5, s_8, s_{11}, \cdots s_{2n-3}$ with color c_{n+5}.

But neighbours of e_n does not satisfies the r-adjacency condition.

So we have to assign two new colors to the vertices s_{2n-2} and s_{2n-1} respectively.

- color the vertex s_{2n-2} with the color c_{n+6} and color the vertex s_{2n-1} with the color c_{n+7}.

So we have to assign any one of the allowed colors to s_{2n-1} and s_{2n}.
Now an easy check shows that the r-adjacency condition is fulfilled.
Hence, $\chi_r(M(G_n)) = n+7$, for $n \geq 5$, $r = \Delta$ and $n \equiv 1 \mod 3$.

Case 8 : For $r = \Delta$ and $n \equiv 2 \mod 3$

The $r-$ dynamic $(n+6)$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n$, assign the color c_{n+1} to u_i.
For $1 \leq i \leq n$, assign the color c_{n+2} to v_i.
For $r = \Delta$, we have to assign two new colors to neighbours of e_i.

- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-3}$ with color c_{n+3}.
- color the vertices $s_{2n}, s_3, s_6, s_9, \cdots s_{2n-4}$ with color c_{n+4}.
- color the vertices $s_2, s_5, s_8, s_{11}, \cdots s_{2n-2}$ with color c_{n+5}.

Now neighbours of e_n does not satisfies the r- adjacency condition.

- color the vertex s_{2n-1} with the new color c_{n+6}

Now an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(M(G_n)) = n + 6$, for $n \geq 5$, $r = \Delta$ and $n \equiv 2 \mod 3$. □

Theorem 3.2. *Let $n \geq 5$, $C(G_n)$ be the central graph of a Gear graph G_n and let*
$\Delta = \Delta(C(G_n))$. Then

$$\chi_r(C(G_n)) = \begin{cases} n+1, & r = 1 \\ 2n+1, & \delta \leq r \leq \Delta - 2 \\ 2n+2, & r = \Delta - 1 \\ 3n+3, & r = \Delta \end{cases}$$

Proof. By the definition of central graph, subdividing each edge of G_n exactly once and then joining each pair of vertices of G_n which were non-adjacent.

Let $V(C(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq 2n\}$

We divide the proof into some cases.

Case 1 : For $r = 1$

The r- dynamic $(n+1)-$ coloring is as follows:

For $1 \leq i \leq n$, assign the color c_i to v_i and u_i.

For $1 \leq i \leq n-1$, assign the color c_i to e_{i+1} and assign the color c_n to e_1.

$|N(u_i)| = d(u_i) = 2n$
$|N(v_i)| = d(v_i) = 2n$
$|N(v)| = d(v) = 2n$,
$|N(e_i)| = d(e_i) = 2$
and $|N(s_i)| = d(s_i) = 2$

For $1 \leq i \leq 2n$, assign the color c_{n+1} to the vertex s_i and assign the color c_{n+1} to v.

Now an easy check shows that the $r-$ adjacency condition is fulfilled.
Hence, $\chi_r(C(G_n)) = n + 1$, for $r = 1$.

Case 2 : For $\delta \leq r \leq \Delta - 2$

The r- dynamic $(2n + 1)-$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to v_i.
For $1 \leq i \leq 2n$, assign the color c_{n+1} to s_i.
For $1 \leq i \leq n - 1$, assign the color c_i to e_{i+1} and assign the color c_n to e_1 and also assign the color c_{n+1} to v.

- Color the vertices $u_1, u_2, u_3, \cdots u_{n-1}, u_n$ with colors $c_{n+2}, c_{n+3}, \cdots c_{2n}, c_{2n+1}$ (the order of assigned color is important).

Now an easy check shows that the $r-$adjacency condition is fulfilled.
Hence, $\chi_r(C(G_n)) = 2n + 1$, for $\delta \leq r \leq \Delta - 2$

Case 3 : For $r = \Delta - 1$

The r- dynamic $(2n + 2)-$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to v_i and assign the color c_{n+1} to v.
For $1 \leq i \leq n - 1$, assign the color c_i to e_{i+1} and assign the color c_n to e_1.

- Color the vertices $u_1, u_2, u_3, \cdots u_{n-1}, u_n$ with colors $c_{n+2}, c_{n+3}, \cdots c_{2n}, c_{2n+1}$ (the order of assigned color is important).
- Color the vertices $s_2, s_4, s_6, \cdots s_{2n-2}, s_{2n}$ with color c_{n+1}.
- Color the vertices $s_1, s_3, s_5, \cdots s_{2n-3}, s_{2n-1}$ with colors c_{2n+2}.

Now an easy check shows that the $r-$adjacency condition is fulfilled.
Hence, $\chi_r(C(G_n)) = 2n + 2$, for $r = \Delta - 1$

Case 4 : For $r = \Delta$

The r- dynamic $(3n + 3)-$ coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to v_i and assign the color c_{n+1} to v.

- Color the vertices $u_1, u_2, u_3, \cdots u_{n-1}, u_n$ with colors $c_{n+2}, c_{n+3}, \cdots c_{2n}, c_{2n+1}$ (the order of assigned color is important).

- Color the vertices $s_1, s_3, s_5, \cdots s_{2n-3}, s_{2n-1}$ with colors c_{2n+2} and color the vertices $s_2, s_4, s_6, \cdots s_{2n-2}, s_{2n}$ with color c_{2n+3}.
- Color the vertices $e_1, e_2, e_3, \cdots e_{n-1}, e_n$ with colors $c_{2n+4}, c_{2n+5}, c_{2n+6} \cdots c_{3n+2}, c_{3n+3}$ respectively.(the order of assigned color is important).

Now an easy check shows that the $r-$adjacency condition is fulfilled.
Hence, $\chi_r(C(G_n)) = 3n + 3$, for $r = \Delta$ □

Result:

Let us consider the line graphs built on the base of Gear graph.
By the definition of line graph
$V(L(G_n)) = E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq 2n\}$.
Note that $d(e_i) = n + 1, d(s_i) = 3$. Hence $\delta(L(G_n)) = 3$.

Next, observe that the vertices $\{e_1, e_2, e_3, .., e_n\}$ induces a clique K_n in $L(G_n)$. Thus,

$$\chi_\delta(L(G_n)) \geq n \tag{3.1}$$

for any r. Let us start with $r = \delta$.

Proposition 3.3. *Let $n \geq 5$.Let $L(G_n)$ be the line graph of a Gear graph G_n.Then $\chi_\delta(L(G_n)) = n$.*

Proof. Due to (1), we have $\chi_\delta(L(G_n)) \geq n$.
So, we need to fix only appropriate coloring.

For $1 \leq i \leq n$, assign the color i to e_i. Next,assign the colors to s_i such that partial coloring is proper and the r- adjacency condition for $r = \delta$ is also fulfilled.

That is we should assign one of the allowed colors from $\{1, 2, \cdots n\}$ to vertex s_i of degree 3, $1 \leq i \leq n$.

The coloring we obtained is $\delta-$ dynamic coloring of $L(G_n)$.

The result from proposition can be extended to r- dynamic coloring for line graph of Gear graph for all r, where $1 \leq r \leq \Delta$. □

Theorem 3.4. *Let $n \geq 6$, $L(G_n)$ be the line graph of a Gear graph G_n and*
let $\Delta = \Delta(L(G_n))$. Then

$$\chi_r(L(G_n)) = \begin{cases} n, & 1 \le r \le n-1 \\ n+2, & r = n \;\; and \;\; n \not\equiv 1 \;\; mod \;\; 3 \\ n+3, & r = n \;\; and \;\; n \equiv 1 \;\; mod \;\; 3 \\ n+3, & r = n+1 = \Delta, \;\; n \ge 5 \;\; and \;\; 2n \equiv 0 \;\; mod \;\; 3 \\ n+4, & r = n+1 = \Delta, \;\; n \ge 5 \;\; and \;\; 2n \equiv 1 \;\; mod \;\; 3 \\ n+5, & r = n+1 = \Delta, \;\; n \ge 5 \;\; and \;\; 2n \equiv 2 \;\; mod \;\; 3 \end{cases}$$

Proof. We divide the proof into some cases.

Case 1 : For $1 \le r \le n-1$
The r- dynamic $(n)-$ coloring is as follows:
$|N(e_i)| = d(e_i) = n-1$,
$|N(s_i)| = d(s_i) = 3 = \delta$.
Now an easy check shows that the $r-$adjacency condition is fulfilled.
Hence, $\chi_r(L(G_n)) = n$, for $1 \le r \le n-1$

Case 2 : For $r = n$ and $n \not\equiv 1 \;\; mod \;\; 3$
The r- dynamic $(n+2)-$ coloring is as follows:

- Color vertex e_i with color i, $1 \le i \le n$.

Let us notice that vertices adjacent to each vertex e_i mustbe colored with $r = n$ different colors. After this step each vertex e_i has $n-1$ neighbours in different colors and exactly its two neighbours are uncolored: s_{i-1}, s_i.

We have to color them with atleast one new color to vertex s_i to fulfill r adjacenct condition for vertex s_i. so $\chi_r\,(L(G_n)) \ge n+2$.

To color vertices $s_i, 1 \le i \le n$.

Now the number of vertices s_i, forming a cycle C_{2n}, is not divisibly by 3, so color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-2}$ with color $n+1$.

Now another neighbour of e_1 has uncolored. So we have to assign one of the allowed colors $c_1, c_2, c_3, \cdots, c_n$ to vertex s_{2n}.

Next, the two neighbours of e_2 are uncolored. We have to color them with atleast one new color to vertex s_2 to fulfill r- adjacent condition for vertex e_i.

- color the vertices $s_2, s_5, s_8, \cdots s_{2n-1}$ with color $n+2$.

Now the neighbours of e_i has atleast n colors.

Now $s_3, s_6, s_9, s_{12}, \cdots s_{2n}$ vertices get any one of the allowed colors $c_1, c_2, c_3, \cdots c_n$.

Now an easy check shows that the $r-$adjacency condition is fulfilled.

Hence, $\chi_r(L(G_n)) = n+2$, for $r = n$ and $n \not\equiv 1 \mod 3$

Case 3 : For $r = n$ and $n \equiv 1 \mod 3$

The r- dynamic $(n+3)-$ coloring is as follows:

- Color vertex e_i with color i, $1 \leq i \leq n$.

Let us notice that vertices adjacent to each vertex e_i mustbe colored with $r = n$ different colors. After this step each vertex e_i has $n-1$ neighbours in different colors and exactly its two neighbours are uncolored: s_{i-1}, s_i.

We have to color them with atleast one new color to vertex s_i to fulfill r adjacenct condition for vertex s_i. so $\chi_r\ (L(G_n)) \geq n+2$.

To color vertices $s_i, 1 \leq i \leq n$.

Now the number of vertices s_i, forming a cycle C_{2n}, is not divisibly by 3, so color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-4}$ with color $n+1$.

Now another neighbour of e_1 has uncolored. So we have to assign one of the allowed colors $1, 2, 3, \cdots n$ to vertex s_{2n}.

Next, the two neighbours of e_2 are uncolored. We have to color them with atleast one new color to vertex s_2 to fulfill r- adjacent condition for vertex e_i.

- color the vertices $s_2, s_5, s_8, \cdots s_{2n-3}$ with color $n+2$.

But the neighbours of e_n having only $n-1$ colors. So we have to assign any one of the new color to the vertices s_{2n-1}, s_{2n-2}.

Suppose to assign color $n+3$ to s_{2n-2}, next assign the uncolored vertices to the any one of the allowed colors $1, 2 \cdots n$ to fulfill r- adjacent condition for vertex e_i.

Now an easy check shows that the $r-$adjacency condition is fulfilled.

Hence, $\chi_r(L(G_n)) = n+3$, for $r = n$ and $n \equiv 1 \mod 3$

Case 4 : $r = n+1 = \Delta$ and $2n \equiv 0 \mod 3$

The r dynamic $(n+3)$-coloring is as follows:

- color the vertex e_i with color i, $1 \leq i \leq n$.

It is clear that to color $2n$ remaining vertices: s_i we have to use colors $n, \cdots \chi_r$.

we have to still take care of the r- adjacency condition for all vertices. The r- adjacency condition for vertices $s_i, 1 \leq i \leq n$, we must use atleast two new colors to vertex s_i. So $\chi_r(L(G_n)) \geq n+3$.

- Color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-2}$ with color $n+1$.
- color the vertices $s_3, s_6, s_9, s_{12}, \cdots s_{2n}$ with new color $n+2$.

Now the vertex s_2 is uncolored. So we have to assign the new color $n+3$ to the vertices $s_2, s_5, s_8, \cdots s_{2n-1}$.

Now an easy check shows that the $r-$adjacency condition is fulfilled.

Hence, $\chi_r(L(G_n)) = n+3$, for $r = n+1 = \Delta$ and $2n \equiv 0 \mod 3$.

Case 5 : $r = n+1 = \Delta$ and $2n \equiv 1 \mod 3$

The r dynamic $(n+4)$-coloring is as follows:

- color the vertex e_i with color i, $1 \leq i \leq n$.

It is clear that to color $2n$ remaining vertices: s_i we have to use colors $n, \cdots \chi_r$.

we have to still take care of the r- adjacency condition for all vertices. The r- adjacency condition for vertices $s_i, 1 \leq i \leq n$, we must use atleast two new colors to vertex s_i.

- Color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-3}$ with color $n+1$.
- color the vertices $s_{2n}, s_3, s_6, s_9, s_{12}, \cdots s_{2n-4}$ with new color $n+2$.

Now the vertices $s_2, s_5, s_8, \cdots s_{2n-2}, s_{2n-1}$ are uncolored.

- color the vertices $s_2, s_5, s_8, \cdots s_{2n-1}$ with the color $n+3$.

Now s_{2n-2} is uncolored. So we have to assign the new color $n+4$ to s_{2n-2}.

Now an easy check shows that the $r-$adjacency condition is fulfilled.

Hence, $\chi_r(L(G_n)) = n+4$, for $r = n+1 = \Delta$ and $2n \equiv 1 \mod 3$.

Case 6 : $r = n+1 = \Delta$ and $2n \equiv 2 \mod 3$

The r dynamic $(n+5)$-coloring is as follows:

- color the vertex e_i with color i, $1 \leq i \leq n$.

It is clear that to color $2n$ remaining vertices: s_i we have to use colors $n, \cdots \chi_r$.

we have to still take care of the r- adjacency condition for all vertices. The r- adjacency condition for vertices $s_i, 1 \leq i \leq n$, we must use atleast two new colors to vertex s_i.

- Color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-4}$ with color $n + 1$.
- color the vertices $s_{2n}, s_3, s_6, s_9, s_{12}, \cdots s_{2n-5}$ with new color $n + 2$.
- color the vertices $s_2, s_5, s_8, \cdots s_{2n-3}$ with the color $n + 3$.

Now s_{2n-2}, s_{2n-1} are uncolored.

So we have to assign the new color $n+4$ to s_{2n-2} and to assign the new color $n + 5$ to s_{2n-1}.

Now an easy check shows that the $r-$adjacency condition is fulfilled.

Hence, $\chi_r(L(G_n)) = n + 5$, for $r = n + 1 = \Delta$ and $2n \equiv 2 \mod 3$.

In all cases the order of the assigned colors is important. One can verify that the adjacency and r-adjacency conditions are fulfilled.

□

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