A generalization of Fibonacci sequence

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Abstract:

We attempt to generalize Fibonacci sequence by generating certain number of sequences whose terms are obtained by adding the last two generated terms of the preceding sequence. When we consider the particular case of generating only one sequence we obtain the Fibonacci sequence. Moreover we prove some of the results which can be seen as generalized form of the results which hold for Fibonacci sequence. We also verify the results for the particular case of Fibonacci sequence.

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1. Introduction

Fibonacci sequence (F_k) is generated [1, 2] by the recursive formula $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$ with $F_1 = 1, F_2 = 1$. That is, each term in the sequence (third term onwards) is the sum of the two that immediately precede it. The Fibonacci sequence is the first known recursive sequence in mathematical work. First few terms of the sequence [3] are 1, 1, 2, 3, 5, 8, 13, 21, ...

The sequence has been generalized [4] by defining $F_1 = F_2 = F_3 = \dots = F_n = 1$ and thereafter each term is the sum of the n terms that immediately precede it. This sequence is called the n -nacci sequence and $n = 2$ gives the Fibonacci sequence. The limit of the ratio of successive terms of an n -nacci series tends to a root of the equation $x + x^{-n} = 2$. The limit of the ratio for any $n > 0$ is the positive root of the equation $x^n - \sum_{i=0}^{n-1} x^i = 0$.

Another generalization is (p, q) Fibonacci sequence, which is defined as $U(0) = 0, U(1) = 1, U(k) = pU(k-1) - qU(k-2)$. When $q = -1$, this sequence is called p -Fibonacci sequence, for e.g., Pell sequence is also called 2-Fibonacci sequence because $p = 2$. The normal Fibonacci sequence is the special case of $p = 1$ and $q = -1$.

The above generalizations available in the literature give us a single sequence. In this paper, we generate m number of sequences following certain recursive rules. When the number of sequence is one, i.e. $m = 1$, these rules coincide with those generating Fibonacci numbers and we get the Fibonacci Sequence.

We consider m interconnected sequences

$$(S_{1,k}), (S_{2,k}), (S_{3,k}), \dots, (S_{m,k})$$

which can be generated according to the following rule

$$\begin{aligned} S_{1,1} &= S_{2,1} = S_{3,1} = \dots = S_{m,1} = 1, \quad S_{1,2} = 1, \\ S_{i,k} &= S_{i-1,k-1} + S_{i-1,k}, & 1 < i \leq m, k \geq 2 \\ S_{1,k} &= S_{m,k-1} + S_{m,k-2}, & k \geq 3 \end{aligned}$$

For $m = 1$, we identify the only sequence $(S_{1,k})$ as the Fibonacci sequence (F_k) .

We generalize the following results along with others which hold for Fibonacci sequence and also verify that they are true for the particular case of Fibonacci sequence when $m = 1$.

- $F_k + F_{k+1} = F_{k+2}$
- $\sum_{k=1}^n F_k = F_{n+2} - 1$
- $\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$
- $\sum_{k=1}^n F_{2k} = F_{2n+1} - 1$

Notation: $S_{p,q}$ denotes the q^{th} term of the p^{th} sequence. $gcd_i(S_{i,(p+1)k})$ denotes the greatest common divisor of $(multiple\ of\ (p+1))^{th}$ terms of all the m sequences, i.e, i runs over all the sequences or $i = 1, 2, 3, \dots, m$. For e.g., $gcd_i(S_{i,(3+1)2}) = gcd(S_{1,8}, S_{2,8}, S_{3,8}, \dots, S_{m,8})$.

2. Generalization of few identities

Theorem 2.1. Sum of k^{th} terms in all the sequences and the following term, i.e. $(k+1)^{th}$ term in the first sequence equals the next i.e. $(k+2)^{th}$ term in the first sequence.

Symbolically,

$$\sum_{i=1}^m S_{i,k} + S_{1,k+1} = S_{1,k+2}$$

Fibonacci equivalent: $F_k + F_{k+1} = F_{k+2}$

Proof.

$$\begin{aligned}
 \sum_{i=1}^m S_{i,k} + S_{1,k+1} &= (S_{1,k} + S_{2,k} + S_{3,k} + \dots + S_{m-1,k} + S_{m,k}) + S_{1,k+1} \\
 &= (S_{m,k} + S_{m-1,k} + \dots + S_{3,k} + S_{2,k} + S_{1,k}) + S_{1,k+1} \\
 &= (S_{m,k} + S_{m-1,k} + \dots + S_{3,k} + S_{2,k}) + (S_{1,k} + S_{1,k+1}) \\
 &= (S_{m,k} + S_{m-1,k} + \dots + S_{3,k} + S_{2,k}) + S_{2,k+1} \\
 &= (S_{m,k} + S_{m-1,k} + \dots + S_{3,k}) + (S_{2,k} + S_{2,k+1}) \\
 &= \dots \\
 &= S_{m,k} + S_{m,k+1} \\
 &= S_{1,k+2}
 \end{aligned}$$

Columns in the table below show the terms in six sequences (i.e. $m = 6$). The result is illustrated for $k = 2$ and $k = 6$, where the summands are bold and the sum is italicized.

1	1	1	1	1	1
1	2	3	4	5	6
7	8	10	13	17	22
<i>28</i>	35	43	53	66	83
105	133	168	211	264	330
413	518	651	819	1030	1294
1624	2037	2555	3206	4025	5055
<i>6349</i>	7973	10010	12565	15771	19796
24851	31200	39173	49183	61748	77519
...	...	...	...	...	...

When $m = 1$, we get,

$$\sum_{i=1}^m S_{i,k} + S_{1,k+1} = S_{1,k+2} \quad \text{or} \quad \sum_{i=1}^1 S_{i,k} + S_{1,k+1} = S_{1,k+2} \quad \text{or}$$

$$S_{1,k} + S_{1,k+1} = S_{1,k+2}$$

i.e. $F_k + F_{k+1} = F_{k+2}$ $\square$

Theorem 2.2. Sum of first n terms of all the sequences is one less than the $(n+2)^{th}$ term of the first sequence.

$$\text{Symbolically, } \sum_{i=1}^m \sum_{k=1}^n S_{i,k} = S_{1,n+2} - 1$$

$$\text{Fibonacci equivalent: } \sum_{k=1}^n F_k = F_{n+2} - 1$$

Proof.

$$S_{1,n} = S_{2,n+1} - S_{1,n+1}$$

$$S_{2,n} = S_{3,n+1} - S_{2,n+1}$$

$$S_{3,n} = S_{4,n+1} - S_{3,n+1}$$

$$\dots$$

$$\dots$$

$$S_{m-2,n} = S_{m-1,n+1} - S_{m-2,n+1}$$

$$S_{m-1,n} = S_{m,n+1} - S_{m-1,n+1}$$

$$S_{m,n} = S_{1,n+2} - S_{m,n+1}$$

$$S_{1,n-1} = S_{2,n} - S_{1,n}$$

$$S_{2,n-1} = S_{3,n} - S_{2,n}$$

$$S_{3,n-1} = S_{4,n} - S_{3,n}$$

$$\dots$$

$$\dots$$

$$S_{m-2,n-1} = S_{m-1,n} - S_{m-2,n}$$

$$S_{m-1,n-1} = S_{m,n} - S_{m-1,n}$$

$$S_{m,n-1} = S_{1,n+1} - S_{m,n}$$

$$S_{1,n-2} = S_{2,n-1} - S_{1,n-1}$$

$$S_{2,n-2} = S_{3,n-1} - S_{2,n-1}$$

$$S_{3,n-2} = S_{4,n-1} - S_{3,n-1}$$

$$\dots$$

$$\dots$$

$$S_{m-2,n-2} = S_{m-1,n-1} - S_{m-2,n-1}$$

$$S_{m-1,n-2} = S_{m,n-1} - S_{m-1,n-1}$$

$$S_{m,n-2} = S_{1,n} - S_{m,n-1}$$

$$\begin{array}{llll}
 \dots & \dots & \dots & S_{1,2} = S_{2,3} - S_{1,3} & S_{1,1} = S_{2,2} - S_{1,2} \\
 \dots & \dots & \dots & S_{2,2} = S_{3,3} - S_{2,3} & S_{2,1} = S_{3,2} - S_{2,2} \\
 \dots & \dots & \dots & S_{3,2} = S_{4,3} - S_{3,3} & S_{3,1} = S_{4,2} - S_{3,2} \\
 \dots & & & \dots & \dots \\
 \dots & & & \dots & \dots \\
 \dots & \dots & \dots & S_{m-2,2} = S_{m-1,3} - S_{m-2,3} & S_{m-2,1} = S_{m-1,2} - S_{m-2,2} \\
 \dots & \dots & \dots & S_{m-1,2} = S_{m,3} - S_{m-1,3} & S_{m-1,1} = S_{m,2} - S_{m-1,2} \\
 \dots & \dots & \dots & S_{m,2} = S_{1,4} - S_{m,3} & S_{m,1} = S_{1,3} - S_{m,2}
 \end{array}$$

Thus,

$$\begin{array}{l}
 \sum_{i=1}^m S_{i,n} = S_{1,n+2} - S_{1,n+1} \quad \sum_{i=1}^m S_{i,n-1} = S_{1,n+1} - S_{1,n} \quad \sum_{i=1}^m S_{i,n-2} = S_{1,n} - S_{1,n-1} \\
 \dots \dots \dots \sum_{i=1}^m S_{i,2} = S_{1,4} - S_{1,3} \quad \sum_{i=1}^m S_{i,1} = S_{1,3} - S_{1,2}
 \end{array}$$

Therefore

$$\sum_{i=1}^m \sum_{k=1}^n S_{i,k} = S_{1,n+2} - S_{1,2} = S_{1,n+2} - 1$$

The result is illustrated in the table for $m = 6, n = 6$, where the summands are bold and the sum is 6348, which is one less than the italicized term.

1	1	1	1	1	1
1	2	3	4	5	6
7	8	10	13	17	22
28	35	43	53	66	83
105	133	168	211	264	330
413	518	651	819	1030	1294
1624	2037	2555	3206	4025	5055
<i>6349</i>	7973	10010	12565	15771	19796
24851	31200	39173	49183	61748	77519
...	...	...	...	...	...

When $m = 1$, we get,

$$\sum_{i=1}^m \sum_{k=1}^n S_{i,k} = S_{1,n+2} - 1 \quad \text{or} \quad \sum_{i=1}^1 \sum_{k=1}^n S_{i,k} = S_{1,n+2} - 1 \quad \text{or}$$

$$\sum_{k=1}^n S_{1,k} = S_{1,n+2} - 1$$

$$\text{i.e. } \sum_{k=1}^n F_k = F_{n+2} - 1 \quad \square$$

Theorem 2.3. This result is a generalization of Theorem 2.2 above.

$$\sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \sum_{i=r}^m S_{i,1} + \sum_{i=1}^{r-1} S_{i,n} = S_{r,n+1} - r$$

Theorem 2.2 is a particular case for $r = 1$.

Proof.

$$\begin{aligned} & \sum_{i=1}^m \sum_{k=1}^n S_{i,k} &&= S_{1,n+2} - 1 \\ \Rightarrow & \sum_{i=1}^m S_{i,1} + \sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \sum_{i=1}^m S_{i,n} &&= S_{1,n+2} - 1 \\ \Rightarrow & \left(\sum_{i=1}^{r-1} S_{i,1} + \sum_{i=r}^m S_{i,1} \right) + \sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \left(\sum_{i=1}^{r-1} S_{i,n} + \sum_{i=r}^m S_{i,n} \right) &&= S_{1,n+2} - 1 \\ \Rightarrow & \sum_{i=1}^{r-1} 1 + \sum_{i=r}^m S_{i,1} + \sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \sum_{i=1}^{r-1} S_{i,n} + \sum_{i=r}^m S_{i,n} &&= S_{1,n+2} - 1 \\ \Rightarrow & \sum_{i=r}^m S_{i,1} + \sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \sum_{i=1}^{r-1} S_{i,n} &&= S_{1,n+2} - 1 - \sum_{i=1}^{r-1} 1 - \sum_{i=r}^m S_{i,n} \\ \Rightarrow & \sum_{i=r}^m S_{i,1} + \sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \sum_{i=1}^{r-1} S_{i,n} &&= S_{1,n+2} - 1 - (r-1) - \sum_{i=r}^m S_{i,n} \\ \Rightarrow & \sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \sum_{i=r}^m S_{i,1} + \sum_{i=1}^{r-1} S_{i,n} &&= S_{1,n+2} - \sum_{i=r}^m S_{i,n} - r \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=1}^m \sum_{k=2}^{n-1} S_{i,k} + \sum_{i=r}^m S_{i,1} + \sum_{i=1}^{r-1} S_{i,n} &= (S_{m,n} + S_{m,n+1}) - \left(\sum_{i=r}^{m-1} S_{i,n} + S_{m,n} \right) - r \\ &= S_{m,n+1} - \sum_{i=r}^{m-1} S_{i,n} - r \\ &= (S_{m-1,n} + S_{m-1,n+1}) - \left(\sum_{i=r}^{m-2} S_{i,n} + S_{m-1,n} \right) - r \\ &= S_{m-1,n+1} - \sum_{i=r}^{m-2} S_{i,n} - r \\ &\dots \\ &= S_{r,n+1} - r \end{aligned}$$

The result is illustrated in the table for $m = 6, n = 5$ and $r = 4$, where the summands are bold and the sum is 815, which is 4 (i.e. r) less than the italicized term.

1	1	1	1	1	1
1	2	3	4	5	6
7	8	10	13	17	22
28	35	43	53	66	83
105	133	168	211	264	330
413	518	651	<i>819</i>	1030	1294
1624	2037	2555	3206	4025	5055
6349	7973	10010	12565	15771	19796
24851	31200	39173	49183	61748	77519
...	...	...	...	...	...

□

Theorem 2.4. Sum of all the terms in odd positions (upto $(2n + 1)^{th}$ position) in all the sequences equals the $(2n)^{th}$ term of the last sequence.

$$\text{Symbolically, } \sum_{i=1}^m \sum_{k=0}^{n-1} S_{i,2k+1} = S_{m,2n}$$

$$\text{Fibonacci equivalent: } \sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$$

Proof.

$$\begin{array}{lll}
 S_{1,1} = S_{2,2} - S_{1,2} & S_{1,3} = S_{2,4} - S_{1,4} & S_{1,5} = S_{2,6} - S_{1,6} \\
 \dots S_{1,2n-1} = S_{2,2n} - S_{1,2n} & & \\
 S_{2,1} = S_{3,2} - S_{2,2} & S_{2,3} = S_{3,4} - S_{2,4} & S_{2,5} = S_{3,6} - S_{2,6} \\
 \dots S_{2,2n-1} = S_{3,2n} - S_{2,2n} & & \\
 S_{3,1} = S_{4,2} - S_{3,2} & S_{3,3} = S_{4,4} - S_{3,4} & S_{3,5} = S_{4,6} - S_{3,6} \\
 \dots S_{3,2n-1} = S_{4,2n} - S_{3,2n} & & \\
 \dots & \dots & \\
 \dots & \dots & \\
 S_{m-1,1} = S_{m,2} - S_{m-1,2} & S_{m-1,3} = S_{m,4} - S_{m-1,4} & S_{m-1,5} = S_{m,6} - S_{m-1,6} \\
 \dots S_{m-1,2n-1} = S_{m,2n} - S_{m-1,2n} & & \\
 S_{m,1} = S_{1,3} - S_{m,2} & S_{m,3} = S_{1,5} - S_{m,4} & S_{m,5} = S_{1,7} - S_{m,6} \\
 \dots S_{m,2n-1} = S_{1,2n+1} - S_{m,2n} & &
 \end{array}$$

Thus,

$$\begin{aligned}
 \sum_{i=1}^m S_{i,1} &= S_{1,3} - S_{1,2} = S_{m,1} + S_{m,2} - S_{1,2} &= 1 + S_{m,2} - 1 \\
 \sum_{i=1}^m S_{i,3} &= S_{1,5} - S_{1,4} = S_{m,3} + S_{m,4} - S_{m,2} - S_{m,3} &= S_{m,4} - S_{m,2} \\
 \sum_{i=1}^m S_{i,5} &= S_{1,7} - S_{1,6} = S_{m,5} + S_{m,6} - S_{m,4} - S_{m,5} &= S_{m,6} - S_{m,4} \\
 \dots &\dots \dots \dots \dots \\
 \sum_{i=1}^m S_{i,2n-1} &= S_{1,2n+1} - S_{1,2n} = S_{m,2n-1} + S_{m,2n} - S_{m,2n-2} - S_{m,2n-1} = S_{m,2n} - S_{m,2n-2}
 \end{aligned}$$

Therefore

$$\sum_{i=1}^m \sum_{k=0}^{n-1} S_{i,2k+1} = S_{m,2n}$$

The result is illustrated in the table for $m = 6, n = 4$, where the summands are bold and the sum is the italicized term.

1	1	1	1	1	1
1	2	3	4	5	6
7	8	10	13	17	22
28	35	43	53	66	83
105	133	168	211	264	330
413	518	651	819	1030	1294
1624	2037	2555	3206	4025	5055
6349	7973	10010	12565	15771	<i>19796</i>
24851	31200	39173	49183	61748	77519
...	...	...	...	...	...

When $m = 1$, we get,

$$\begin{aligned}
 \sum_{i=1}^m \sum_{k=0}^{n-1} S_{i,2k+1} &= S_{m,2n} \quad \text{or} \quad \sum_{i=1}^1 \sum_{k=0}^{n-1} S_{i,2k+1} = S_{1,2n} \quad \text{or} \\
 \sum_{k=0}^{n-1} S_{1,2k+1} &= S_{1,2n} \text{ i.e. } \sum_{k=0}^{n-1} F_{2k+1} = F_{2n} \quad \square
 \end{aligned}$$

Corollary 2.4.1. *Sum of all the terms in even positions (upto $(2n)^{th}$ position) in all the sequences is one less than the $(2n + 1)^{th}$ term of the last*

sequence.

$$\text{Symbolically, } \sum_{i=1}^m \sum_{k=1}^n S_{i,2k} = S_{m,2n+1} - 1$$

$$\text{Fibonacci equivalent: } \sum_{k=1}^n F_{2k} = F_{2n+1} - 1$$

$$\begin{aligned} \text{Proof. } \sum_{i=1}^m \sum_{k=1}^n S_{i,2k} &= \sum_{i=1}^m \sum_{k=1}^{2n} S_{i,k} - \sum_{i=1}^m \sum_{k=0}^{n-1} S_{i,2k+1} = S_{1,2n+2} - 1 - \\ S_{m,2n} &= S_{m,2n+1} - 1 \end{aligned}$$

The result is illustrated in the table for $m = 6, n = 3$, where the summands are bold and the sum is 5054, which is one less than the italicized term.

1	1	1	1	1	1
1	2	3	4	5	6
7	8	10	13	17	22
28	35	43	53	66	83
105	133	168	211	264	330
413	518	651	819	1030	1294
1624	2037	2555	3206	4025	<i>5055</i>
6349	7973	10010	12565	15771	19796
24851	31200	39173	49183	61748	77519
...	...	...	...	...	...

When $m = 1$, we get,

$$\sum_{i=1}^m \sum_{k=1}^n S_{i,2k} = S_{m,2n+1} - 1 \quad \text{or} \quad \sum_{i=1}^1 \sum_{k=1}^n S_{i,2k} = S_{1,2n+1} - 1 \quad \text{or}$$

$$\sum_{k=1}^n S_{1,2k} = S_{1,2n+1} - 1$$

$$\text{i.e. } \sum_{k=1}^n F_{2k} = F_{2n+1} - 1 \quad \square$$

Theorem 2.5. $\lim_{k \rightarrow \infty} \frac{S_{i,k}}{S_{i,k+1}}$ is a root of the polynomial $x(x+1)^m - 1$ for $i = 1, 2, \dots, m$.

For Fibonacci sequence, $m = 1$, $i = 1$ and therefore $\lim_{k \rightarrow \infty} \frac{S_{1,k}}{S_{1,k+1}} =$
 $\lim_{k \rightarrow \infty} \frac{F_k}{F_{k+1}} = (\text{Golden ratio})^{-1}$, which is a root of the polynomial $x(x+1)-1$.

Proof. Suppose $\lim_{k \rightarrow \infty} \frac{S_{i,k}}{S_{i,k+1}} = l$.

Then

$$\lim_{k \rightarrow \infty} \frac{S_{i+1,k}}{S_{i+1,k+1}} = \lim_{k \rightarrow \infty} \frac{S_{i,k-1} + S_{i,k}}{S_{i,k} + S_{i,k+1}} = \lim_{k \rightarrow \infty} \frac{\frac{S_{i,k-1}}{S_{i,k}} + 1}{1 + \frac{S_{i,k+1}}{S_{i,k}}} = \frac{l+1}{1+\frac{1}{l}} = l$$

Thus, $\lim_{k \rightarrow \infty} \frac{S_{1,k}}{S_{1,k+1}} = \lim_{k \rightarrow \infty} \frac{S_{2,k}}{S_{2,k+1}} = \dots = \lim_{k \rightarrow \infty} \frac{S_{m,k}}{S_{m,k+1}} = l$.

Also

$$\lim_{k \rightarrow \infty} \frac{S_{i,k}}{S_{i+1,k}} = \lim_{k \rightarrow \infty} \frac{S_{i,k}}{S_{i,k-1} + S_{i,k}} = \lim_{k \rightarrow \infty} \frac{1}{\frac{S_{i,k-1}}{S_{i,k}} + \frac{S_{i,k}}{S_{i,k}}} = \frac{1}{l+1}$$

Now combining the rules for the interconnected sequences, we get,

$$\begin{aligned} S_{1,k+1} &= S_{m,k-1} + S_{m,k} \\ &= \dots \\ &= a_m S_{1,k} + a_{m-1} S_{1,k-1} + a_{m-2} S_{1,k-2} + \dots + a_0 S_{1,k-m} \end{aligned}$$

where $a_i = \binom{m}{i}$.

Thus,

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{S_{1,k-1}}{S_{1,k}} &= \lim_{k \rightarrow \infty} \frac{S_{1,k}}{S_{1,k+1}} \\ \Rightarrow \lim_{k \rightarrow \infty} \frac{S_{1,k-1}}{S_{1,k}} &= \lim_{k \rightarrow \infty} \frac{S_{1,k}}{a_m S_{1,k} + a_{m-1} S_{1,k-1} + a_{m-2} S_{1,k-2} + \dots + a_0 S_{1,k-m}} \\ \Rightarrow \lim_{k \rightarrow \infty} \frac{S_{1,k-1}}{S_{1,k}} &= \lim_{k \rightarrow \infty} \frac{1}{a_m \frac{S_{1,k}}{S_{1,k}} + a_{m-1} \frac{S_{1,k-1}}{S_{1,k}} + a_{m-2} \frac{S_{1,k-2}}{S_{1,k}} + \dots + a_0 \frac{S_{1,k-m}}{S_{1,k}}} \end{aligned}$$

or,

$$l = \frac{1}{a_m + a_{m-1}l + a_{m-2}l^2 + \dots + a_0l^m} \quad \Rightarrow l = \frac{1}{(1+l)^m} \quad \Rightarrow l(1+l)^m = 1$$

Therefore l is a root of the polynomial $x(x+1)^m - 1$. $\square$

Corollary 2.5.1. $\lim_{k \rightarrow \infty} \frac{S_{i,k+1}}{S_{i,k}}$ is a root of the polynomial $x^{m+1} - (x+1)^m$

for $i = 1, 2, \dots, m$.

For Fibonacci sequence, $m = 1$, $i = 1$ and therefore $\lim_{k \rightarrow \infty} \frac{S_{1,k+1}}{S_{1,k}} = \lim_{k \rightarrow \infty} \frac{F_{k+1}}{F_k} =$

Golden ratio, which is a root of the polynomial $x^2 - (x+1)$.

Proof. The proof is obvious. $\square$

Theorem 2.6. If $m = p - 1$, p prime, then $p | \gcd_i (S_{i, (p+1)k}), k \geq 1$.

Suppose $p = 2$. Then $m = 1$ i.e. we have the Fibonacci sequence. In this case, $2 | \gcd(S_{1, 3k})$ i.e. $2 | S_{1, 3k}$ i.e. $2 | F_{3k}$. In other words, every third term in Fibonacci sequence is even.

Proof. For prime p , $p | \binom{p-1}{i} + \binom{p-1}{i+1}$ because $\binom{p-1}{i} + \binom{p-1}{i+1} = \binom{p}{i+1} = \frac{p!}{(i+1)!(p-i-1)!}$. Thus $p | \binom{m}{i} + \binom{m}{i+1}$.

Now,

$$S_{1, k+1} = \binom{m}{m} S_{1, k} + \binom{m}{m-1} S_{1, k-1} + \binom{m}{m-2} S_{1, k-2} + \dots + \binom{m}{0} S_{1, k-m}$$

with $S_{1, 1} = 1, S_{1, q} = 0, q \leq 0$.

Hence, we get

$$\begin{aligned} S_{1, 1} &= 1(\text{mod } p), & S_{1, 2} &= 1(\text{mod } p), & S_{1, 3} &= \binom{m}{1} + \binom{m}{0}(\text{mod } p), \\ S_{1, 4} &= \binom{m}{2} + \binom{m}{1}(\text{mod } p), & \dots & \dots & S_{1, m+2} &= \binom{m}{m} + \binom{m}{m-1}(\text{mod } p) \end{aligned}$$

That is,

$$\begin{aligned} S_{1, 1} &= 1(\text{mod } p), & S_{1, 2} &= 1(\text{mod } p), & S_{1, 3} &= 0(\text{mod } p), & S_{1, 4} &= 0(\text{mod } p), \\ \dots, & S_{1, m+2} &= 0(\text{mod } p) \end{aligned}$$

Therefore,

$$\begin{aligned} S_{1, m+3} &= 1(\text{mod } p), & S_{1, m+4} &= 1(\text{mod } p), & S_{1, m+5} &= 0(\text{mod } p), & S_{1, m+6} &= \\ 0(\text{mod } p), & \dots, & S_{1, 2(m+2)} &= 0(\text{mod } p) \end{aligned}$$

By the sequence generating rule, we get,

$$\begin{aligned} S_{1, 2} &= 1(\text{mod } p), \\ S_{1, 3} &= 0(\text{mod } p), & S_{2, 3} &= 1(\text{mod } p), \\ S_{1, 4} &= 0(\text{mod } p), & S_{2, 4} &= 0(\text{mod } p), & S_{3, 4} &= 1(\text{mod } p), & \dots \end{aligned}$$

$$\begin{array}{cccc}
S_{1,5} = 0(\bmod p), & S_{2,5} = 0(\bmod p), & S_{3,5} = 0(\bmod p), & \dots \\
\dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots \\
S_{1,m+2} = 0(\bmod p), & S_{2,m+2} = 0(\bmod p), & S_{3,m+2} = 0(\bmod p), & \\
& \dots & S_{m,m+2} = 0(\bmod p) &
\end{array}$$

and so on. Thus $S_{i,m+2} = 0(\bmod p) \forall i$.

That is $p|S_{i,m+2} \forall i$, or $p|S_{i,p+1} \forall i$. Hence $p|gcd_i(S_{i,(p+1)})$.

Finally, using the expression for $S_{1,k+1}$ provided in the beginning of this proof, we can easily prove that $p|gcd_i(S_{i,(p+1)k}), k \geq 1$. $\square$

Theorem 2.7. *If $m = p^\alpha - 1$, p prime, then $p|gcd_i(S_{i,(p^\alpha+1)k}), k \geq 1$.*

Proof. The proof is in the same line as in case of Theorem 2.6 using the fact that for prime p , $p| \binom{p^\alpha - 1}{i} + \binom{p^\alpha - 1}{i+1}$. $\square$

Conclusion

The generalization of Fibonacci sequence as discussed in this paper, has opened a new wider scope to explore all the results involving Fibonacci sequence. Binet's Formula, Cassini's identity etc. can be explored to be generalized accordingly. The last two results dealt with the gcd of the terms in the m sequences generated for the particular cases when $m+1$ is a positive power of a prime p . We conclude this paper with an open problem which deals with the general case when $m+1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$.

Problem: If $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t} - 1$, where p_i 's are primes and $t \geq 2$, then **is it true** that $gcd_i(S_{i,k}) = 1, \forall k$?

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