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Further characterizations of property (V_{Π}) and some applications

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Abstract:

We carry out characterizations with techniques provided by the local spectral theory of bounded linear operators $T \in L(X)$, X infinite dimensional complex Banach space, which verify property (V_{Π}) introduced by Sanabria et al. (Open Math. 16(1) (2018), 289-297). We also carry out the study for polaroid operators and Drazin invertible operators that verify the property mentioned above.

Keywords: Semi-Fredholm operator; Semi-Weyl operator; Drazin invertible operator; Property (V_{Π}).

MSC (2020): Primary 47A10, 47A11; Secondary 47A53, 47A55.

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1. Introduction and preliminaries.

Let $L(X)$ be the algebra of all bounded linear operators acting on an infinite dimensional complex Banach space X. An operator $T \in L(X)$ is said to verify property (V_{Π}) if the upper semi-Weyl spectrum of T coincides with the Drazin spectrum of T. Property (V_{Π}) is a strong variant of classical Browder's theorem and their generalized versions, which was recently introduced by Sanabria et al. [20]. There are other strong versions of Browder's theorem that are equivalent to property (V_{Π}), such is the case of properties (Sb), (Sab), and (V_{Π_a}) introduced in [17], [18] and [20], respectively. In this paper we investigated new characterizations of property (V_{Π}) using the localized single-valued extension property and some topological conditions that satisfy the spectral subsets originated from Fredholm Theory and B-Fredholm Theory. Also, we study property (V_{Π}) for operators $T+K$ defined on a Banach space X, where $T+K$ is a (non necessarily commuting) compact operator on X. Moreover, we see how property (V_{Π}) is transmitted from an operator T to S^*, where S^* is the dual operator of the Drazin inverse S of T. In the last part of the paper we give examples of operator classes to which some of the results obtained can be applied. We start by explaining the relevant terminology. For $T \in L(X)$, we will denote by $\alpha(T)$ the dimension of the kernel $\ker T$ and by $\beta(T)$ the codimension of the range $T(X)$. Recall that an operator $T \in L(X)$ is said to be *upper semi-Fredholm*, denoted by $T \in \Phi^+(X)$, if $\alpha(T) < \infty$ and $T(X)$ is closed; while $T \in L(X)$ is said to be *lower semi-Fredholm*, denoted by $T \in \Phi^-(X)$, if $\beta(T) < \infty$. If T is either upper or lower semi-Fredholm then it is said to be *semi-Fredholm*; while if T is both upper and lower semi-Fredholm then it is said to be *Fredholm*. If T is semi-Fredholm, then the *index* of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. An operator $T \in L(X)$ is called a *Weyl operator*, denoted by $T \in W(X)$, if T is a Fredholm operator having index 0. The classes of *upper semi-Weyl* and *lower semi-Weyl* operators are defined, respectively, by

$$W^+(X) := \{T \in \Phi^+(X) : \operatorname{ind} T \leq 0\}$$

and

$$W^-(X) := \{T \in \Phi^-(X) : \operatorname{ind} T \geq 0\}.$$

Clearly, $W(X) = W^+(X) \cap W^-(X)$. The *Weyl spectrum*, *upper semi-Weyl spectrum* and *lower semi-Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin W(X)\},$$

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin W^+(X)\}$$

and

$$\sigma_{\mathrm{lw}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin W^-(X)\}.$$

Let $p(T)$ and $q(T)$ denote the *ascent* and the *descent* of $T \in L(X)$, respectively. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, if $\lambda \in \mathbf{C}$ the condition $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ is equivalent to saying that λ is a pole of the resolvent, see [13, Prop. 50.2]. An operator $T \in L(X)$ is said to be *Browder* (resp. *upper semi-Browder*, *lower semi-Browder*) if T is Fredholm and $p(T) = q(T) < \infty$ (resp. T is upper semi-Fredholm and $p(T) < \infty$, T is lower semi-Fredholm and $q(T) < \infty$). Denote by $B(X)$, $B^+(X)$ and $B^-(X)$ the classes of Browder operators, upper semi-Browder operators and lower semi-Browder operators, respectively. Clearly, $B(X) \subseteq W(X)$, $B^+(X) \subseteq W^+(X)$ and $B^-(X) \subseteq W^-(X)$. Let

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not Browder}\}$$

denote the *Browder spectrum* and $\sigma_{\mathrm{ub}}(T)$ denote the *upper semi-Browder spectrum* of T, defined as

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$

then $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ and $\sigma_{\mathrm{uw}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$.

The concept of semi-Fredholm operators has been generalized by Berkani ([11], [12]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$), then $T \in L(X)$ is said to be *semi B-Fredholm* (resp. *B-Fredholm, upper semi B-Fredholm*, *lower semi B-Fredholm*) if for some integer $n \geq 0$, the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm) operator. In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([12]). This enables one to define the index of a semi B-Fredholm operator T as $\operatorname{ind}(T) = \operatorname{ind}(T_{[n]})$. By [12, Proposition 2.5] every semi B-Fredholm operator on a Banach space is quasi-Fredholm. Note that T is B-Fredholm if and only if T^* is B-Fredholm. In this case $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$. An operator $T \in L(X)$ is said to be *B-Weyl* (resp. *upper semi B-Weyl, lower semi B-Weyl*) if for some integer $n \geq 0$, $T^n(X)$ is closed and $T_{[n]}$ is Weyl (resp. upper semi-Weyl, lower semi-Weyl). Analogously, an operator $T \in L(X)$ is said to be *B-Browder* (resp. *upper semi B-Browder, lower* *semi B-Browder*) if for some integer $n \geq 0$, $T^n(X)$ is closed and $T_{[n]}$ is Browder (resp. upper semi-Browder, lower semi-Browder). The classes of upper semi B-Weyl, B-Weyl, upper semi B-Browder and B-Browder operators are denoted, respectively, by $UBW(X)$, $BW(X)$, $UBB(X)$ and $BB(X)$, and their respective spectra are defined in the same order, as follows:

$$\sigma_{\text{ubw}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin UBW(X)\},$$

$$\sigma_{\text{bw}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin BW(X)\},$$

$$\sigma_{\text{ubb}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin UBB(X)\},$$

$$\sigma_{\text{bb}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin BB(X)\}.$$

According to [15, Chap. 3, Theorem 10], an operator $T \in L(X)$ is Drazin invertible (with a finite index) if there exists an operator $S \in L(X)$ and an integer $n \geq 0$ such that

$$TS = ST, \quad STS = S, \quad T^nST = T^n.$$

In this case the operator S is called Drazin inverse of T. In [15, Chap. 3, Theorem 10], it is shown that $T \in L(X)$ is Drazin invertible if and only if $p(T) = q(T) < \infty$. Also in [15, Chap. 3, Theorem 10] it is shown that $T \in L(X)$ is Drazin invertible if and only if there exist two closed invariant subspaces Y and Z such that $X = Y \oplus Z$ and, with respect to this decomposition,

$$T = T_1 \oplus T_2, \quad \text{with} \quad T_1 := T|Y \text{ nilpotent} \quad \text{and} \quad T_2 := T|Z \text{ invertible}.$$

According to this decomposition, the Drazin inverse S of T may be represented as the directed sum

$$S := 0 \oplus S_2 \quad \text{with} \quad S_2 := T_2^{-1}.$$

The concept of Drazin invertibility for bounded operators may be extended as follows.

Definition 1.1. *An operator $T \in L(X)$ is said to be* left Drazin invertible *if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be* right Drazin invertible *if $q := q(T) < \infty$ and $T^q(X)$ is closed.*

Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T) < \infty$ then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$, see [13, Prop. 50.2]. The left Drazin invertible spectrum is defined as

$$\sigma_{\mathrm{ld}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not left Drazin invertible}\}$$

and the right Drazin invertible spectrum is defined as

$$\sigma_{\mathrm{rd}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not right Drazin invertible}\}.$$

It is known that $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{ubb}}(T)$, $\sigma_{\mathrm{rd}}(T) = \sigma_{\mathrm{lbb}}(T)$ and $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{bb}}(T)$, see [4]. Let us denote by $\sigma_{\mathrm{a}}(T)$ the classical *approximate point spectrum* and by $\sigma_{\mathrm{s}}(T)$ the *surjectivity spectrum*. It is well known that $\sigma_{\mathrm{a}}(T^*) = \sigma_{\mathrm{s}}(T)$, and $\sigma_{\mathrm{s}}(T^*) = \sigma_{\mathrm{a}}(T)$. The concepts of Drazin's invertibility to the left or to the right lead to concepts of left or right pole, see [7].

Definition 1.2. *Let $T \in L(X)$. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{a}}(T)$ then λ is said to be a* left pole *of the resolvent of T. A left pole λ is said to have finite rank if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a* right pole *of the resolvent of T. A right pole λ is said to have finite rank if $\beta(\lambda I - T) < \infty$.*

Evidently, λ is a pole of T if and only if λ is both a left and a right pole of T. Moreover, λ is a pole of T if and only if λ is a pole of T^*.

Definition 1.3. *An operator $T \in L(X)$ is said to be* polaroid *if every isolated point of $\sigma(T)$ is a pole of the resolvent of T.*

The *quasi-nilpotent part* of $T \in L(X)$ is defined as the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in N$.

The following property has relevant role in local spectral theory, see the recent monographs by Laursen and Neumann [14] and Aiena [1].

Definition 1.4. *An operator $T \in L(X)$ is said to have* the single valued extension property *at $\lambda_0 \in \mathbf{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbf{D}$ centered at λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbf{D}$ is the function $f \equiv 0$.*

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbf{C}$. Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbf{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the points which belong to the boundary $\partial\sigma(T)$ of the spectrum. Also, both T and T^* have SVEP at every isolated point of the spectrum. We also have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1.1}$$

and dually,

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{1.2}$$

see [1, Theorem 3.8]. Furthermore, from definition of localized SVEP it easily seen that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{1.3}$$

and dually,

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{1.4}$$

Note that $H_0(T)$ generally is not closed and by [1, Theorem 2.31], we have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{1.5}$$

Remark 1.5. All the implications (1.1)-(1.5) are equivalences whenever $\lambda I - T$ is quasi-Fredholm, see [2]. In particular, theses equivalences hold whenever $\lambda I - T$ is a semi B-Fredholm operator.

2. Further characterizations of property (V_Π)

According to [20], an operator $T \in L(X)$ verifies property (V_Π) if $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \Pi(T)$, where $\Pi(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T)$, i.e. the set of all poles of the resolvent of T. In this section we establish new characterizations of property (V_Π) and investigate some results related to this property. The following lemma will be used in the sequel.

Lemma 2.1. *For every $T \in L(X)$ the following equivalences hold:*

1. *$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T)$ if and only if $\sigma_{\mathrm{a}}(T) = \sigma(T)$.*
2. *$\sigma_{\mathrm{a}}(T) = \sigma(T)$ if and only if $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$.*

3. $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{lb}}(T)$ *if and only if* $\sigma_{\mathrm{s}}(T) = \sigma(T)$.

4. $\sigma_{\mathrm{s}}(T) = \sigma(T)$ *if and only if* $\sigma_{\mathrm{rd}}(T) = \sigma_{\mathrm{d}}(T)$.

Proof. We will show only (1). The prof of (2) is similar to the proof of (1). The proofs of (3) and (4) follows by duality of (1) and (2), respectively. (1) Suppose that $\sigma_{\mathrm{a}}(T) = \sigma(T)$ and let $\lambda \notin \sigma_{\mathrm{ub}}(T)$. Then $\lambda I - T$ is upper semi-Fredholm and $p(\lambda I - T) < \infty$, so by Remark 1.5, $\sigma_{\mathrm{a}}(T)$ does not cluster at λ. If $\lambda \notin \sigma_{\mathrm{a}}(T) = \sigma(T)$, clearly $\lambda \notin \sigma_{\mathrm{b}}(T)$. If $\lambda \in \sigma_{\mathrm{a}}(T) = \sigma(T) = \sigma(T^*)$ then $\lambda \in \operatorname{iso}\sigma(T^*)$ and hence, T^* has SVEP at λ. Again by the Remark 1.5, $q(\lambda I - T) < \infty$. Being both $p(\lambda I - T)$ and $q(\lambda I - T)$ finites, we have $p(\lambda I - T) = q(\lambda I - T) < \infty$ and hence $\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty$ (see [1, Theorem 3.4]), it follows that $\lambda I - T \in B(X)$, which is $\lambda \notin \sigma_{\mathrm{b}}(T)$. This shows that $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T)$. The converse is clear, since if $\lambda \notin \sigma_{\mathrm{a}}(T)$ then $\lambda \notin \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{b}}(T)$ and consequently $\lambda \notin \sigma(T)$. $\square$

Theorem 2.2. *For* $T \in L(X)$*, the following statements are equivalent:*

1. T *verifies property* (V_Π).

2. *For every* $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)$*, there exists* $\nu := \nu(\lambda) \in \mathbf{N}$ *such that* $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$ *and* $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{w}}(T)$.

3. $H_0(\lambda I - T)$ *is closed for all* $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)$ *and* $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{w}}(T)$.

Proof. (1) $\Rightarrow$ (2). If T verifies property (V_Π), $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{d}}(T)$ by [20, Theorem 2.27]. Now, if $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)$ then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T)$, so λ is a pole of the resolvent of T, it follows that there exists $\nu(\lambda)$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^{\nu(\lambda)}$, see [3, Corollary 2.47].
(2) $\Rightarrow$ (3). It is clear.
(3) $\Rightarrow$ (1). Let $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)$ and since $H_0(\lambda I - T)$ is closed, by implication (5) it follows that T has SVEP in λ. Now, since $\lambda I - T$ is Weyl, then by Remark 1.5, we have $p(\lambda I - T) < \infty$ and consequently, $q(\lambda I - T) < \infty$. Hence, $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T)$ and so, $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \Pi(T)$.

On the other hand, let $\lambda \in \Pi(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$ and so, $\Pi(T) \subseteq \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$. Therefore, $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \Pi(T)$ and T verifies property (V_Π). $\square$

In the following theorem we see that property (V_Π) is one of the strongest variants of a-Browder's theorem.

Theorem 2.3. *For $T \in L(X)$, the following statements are equivalent:*

1. *T verifies property (V_Π).*
2. *T verifies a-Browder's theorem and $\sigma_{\mathrm{w}}(T) \cap (\sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)) = \emptyset$.*
3. *T verifies generalized a-Browder's theorem and $\sigma_{\mathrm{w}}(T) \cap (\sigma(T) \setminus \sigma_{\mathrm{ubw}}(T)) = \emptyset$.*

Proof. The proofs of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from [20, Theorem 2.27] and the fact that generalized a-Browder's theorem and a-Browder's theorem are equivalent.
(3) $\Rightarrow$ (1). By hypothesis, $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$ and $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{w}}(T)$, which implies that $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{ld}}(T)$. We will show that $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$ and then T will verify property (V_Π). Indeed, if $\lambda \notin \sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{w}}(T)$ then $p(\lambda I - T) < \infty$ and $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, it follows that $p(\lambda I - T) = q(\lambda I - T) < \infty$ and hence $\lambda I - T$ is Drazin invertible, so $\lambda \notin \sigma_{\mathrm{d}}(T)$. Consequently, $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$ and $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T)$. $\square$

For $T \in L(X)$, define $p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$ (the set of all poles of the resolvent of T having finite rank), $p_{00}^a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T)$ (the set of all left poles of T having finite rank), $\Pi_a(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ld}}(T)$ (the set of all left poles of T). The following theorem gives additional characterizations for property (V_Π).

Theorem 2.4. *For $T \in L(X)$, the following statements are equivalent:*

1. *T verifies property (V_Π).*
2. *T^* has SVEP at each $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and $p_{00}(T) = \Pi(T)$.*
3. *T^* has SVEP at each $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and $p_{00}^a(T) = \Pi(T)$.*

Proof. The proofs of (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follows from [20, Theorem 2.27] and Remark 1.5.
(2) $\Rightarrow$ (1). Let $\lambda \in \Pi(T) = p_{00}(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{b}}(T) \subseteq \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$ and hence, $\Pi(T) \subseteq \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$. To show the opposite inclusion $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \Pi(T)$, let $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$. Then $\lambda I - T \in \Phi_+(X)$ and $\alpha(\lambda I - T) \leq \beta(\lambda I - T)$. Since T^* has SVEP in λ, by Remark 1.5 it follows that $q(\lambda I - T) < \infty$ and hence $\beta(\lambda I - T) \leq \alpha(\lambda I - T) < \infty$. Consequently,

$p(\lambda I - T) = q(\lambda I - T) < \infty$ and so $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \Pi(T)$. Therefore, $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \Pi(T)$ and the equality $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \Pi(T)$ holds.
(3) $\Rightarrow$ (1). Clearly the inclusion $\Pi(T) \subseteq \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$ follows from equality $\Pi(T) = p_{00}^a(T)$. The opposite inclusion $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \Pi(T)$ is obtained similarly to that of (2) $\Rightarrow$ (1). $\square$

Corollary 2.5. *Let $T \in L(X)$. If T^* has SVEP at each $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and* $\mathrm{iso}\,\sigma_a(T) = \emptyset$*, then T verifies property (V_Π).*

Proof. It is clear since in this case $p_{00}^a(T) = \Pi(T) = \emptyset$. $\square$

Theorem 2.6. *Let $T \in L(X)$. Then the following assertions hold:*

1. *If T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ and T verifies property (V_Π), then T^* verifies property (V_Π).*

2. *If T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and T^* verifies property (V_Π), then T verifies property (V_Π).*

Proof. (1). Suppose that T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$ and T verifies property (V_Π), then we have $\sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{w}}(T)$ and $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{d}}(T)$, it follows that

$$\begin{aligned}\sigma(T^*) \setminus \sigma_{\mathrm{uw}}(T^*) &= \sigma(T) \setminus \sigma_{\mathrm{lw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) \\ &= \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \sigma(T^*) \setminus \sigma_{\mathrm{d}}(T^*).\end{aligned}$$

Hence, T^* verifies property (V_Π).

(2). Suppose that T^* has SVEP at every $\lambda \notin \sigma_{\mathrm{uw}}(T)$ and T^* verifies property (V_Π), then $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{w}}(T)$ and $\sigma_{\mathrm{w}}(T^*) = \sigma_{\mathrm{d}}(T^*)$, which implies that

$$\begin{aligned}\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) &= \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \sigma(T^*) \setminus \sigma_{\mathrm{w}}(T^*) \\ &= \sigma(T^*) \setminus \sigma_{\mathrm{d}}(T^*) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T).\end{aligned}$$

Thus, T verifies property (V_Π). $\square$

3. Topological conditions

Throughout this section we will write $\Delta_+(T) = \sigma(T)\setminus\sigma_{uw}(T)$ and $\Delta_+^g(T) = \sigma(T) \setminus \sigma_{ubw}(T)$. It is well known that the following equalities are satisfied in any metric space: $Cl(A) = A \cup \operatorname{acc} A$, $Cl(A) = Int(A) \cup \partial(A)$, $Cl(A) = \operatorname{iso} A \cup \operatorname{acc} A$ and $\partial(A) = Cl(A) \setminus Int(A)$, for any $A \subseteq X$. In the case that $A \subseteq \mathbf{C}$, we have $Int(\Pi(T)) = \emptyset$, which implies that $\Pi(T) \subseteq \Pi(T) \cup \operatorname{acc}\Pi(T) = Cl(\Pi(T)) = \partial(\Pi(T))$. Also, $\Pi(T) \cap \operatorname{acc}\Delta_+(T) \subseteq \operatorname{iso}\sigma(T) \cap \operatorname{acc}\sigma(T) = \emptyset$, so $\Pi(T) \cap \operatorname{acc}\Delta_+(T) = \emptyset$.

Lemma 3.1. *Let $T \in L(X)$. If $Int(\Delta_+^g(T)) = \emptyset$, then $\sigma(T) = \sigma_{\mathrm{a}}(T)$.*

Proof. Suppose that $Int(\Delta_+^g(T)) = \emptyset$ and let $\lambda_0 \notin \sigma_{\mathrm{a}}(T)$ such that $\lambda_0 \in \sigma(T)$. Then, $\lambda_0 \in \rho_{\mathrm{a}}(T)$ and as $\rho_{\mathrm{a}}(T)$ is open, there exists an open disc $\mathbf{D}_{\lambda_0}$ centered at λ_0 such that $\mathbf{D}_{\lambda_0} \subseteq \rho_{\mathrm{a}}(T)$ and $\mathbf{D}_{\lambda_0} \cap \sigma(T) = \emptyset$. We assert that $\mathbf{D}_{\lambda_0} \cap \rho(T) = \emptyset$. Otherwise, if $\mathbf{D}_{\lambda_0} \cap \rho(T) = \emptyset$, then $\mathbf{D}_{\lambda_0} \cap \partial\sigma(T) = \emptyset$ and as $\partial\sigma(T) \subseteq \sigma_{\mathrm{a}}(T)$, it follows that $\mathbf{D}_{\lambda_0} \cap \sigma_{\mathrm{a}}(T) = \emptyset$, which is a contradiction. Hence, $\mathbf{D}_{\lambda_0} \cap \rho(T) = \emptyset$ and so $\mathbf{D}_{\lambda_0} \subseteq \sigma(T)$. Since $\mathbf{D}_{\lambda_0} \subseteq \rho_{\mathrm{a}}(T) \subseteq \mathbf{C} \setminus \sigma_{\mathrm{ubw}}(T)$, it follows that $\mathbf{D}_{\lambda_0} \subseteq \sigma(T) \cap (\mathbf{C} \setminus \sigma_{\mathrm{ubw}}(T)) = \Delta_+^g(T)$ and so $\lambda_0 \in Int(\Delta_+^g(T))$, which contradicts the hypothesis $Int(\Delta_+^g(T)) = \emptyset$. Thus, we conclude that $\sigma(T) = \sigma_{\mathrm{a}}(T)$ whenever $Int(\Delta_+^g(T)) = \emptyset$. □

Theorem 3.2. *For $T \in L(X)$, the following statements are equivalent:*

1. *T verifies property (V_Π).*
2. *$Cl(\Delta_+(T)) = \partial(\Pi(T))$ and $\Delta_+(T) \cap \operatorname{acc}\Pi(T) = \emptyset$.*
3. *$Int(\Delta_+^g(T)) = \emptyset$ and $\partial(\Pi(T)) \subseteq \partial(\Delta_+(T))$.*
4. *$Int(\Delta_+^g(T)) = \emptyset$ and $\Pi(T) = p_{00}(T)$.*
5. *$\Delta_+(T) \subseteq \operatorname{iso}\sigma(T)$ and $\partial(\Pi(T)) \subseteq Cl(\Delta_+(T))$.*
6. *$\Delta_+(T) \subseteq \partial\sigma(T)$ and $\partial(\Pi(T)) \subseteq Cl(\Delta_+(T))$.*

Proof. The implications (1) ⇒ (2), (1) ⇒ (3), (1) ⇒ (4), (1) ⇒ (5) and (1) ⇒ (6) are clear, because $\Delta_+^g(T) = \Delta_+(T) = \Pi(T) = p_{00}(T)$ by [20, Theorem 2.27].
(2) ⇒ (1). Suppose that $Cl(\Delta_+(T)) = \partial(\Pi(T))$ and $\Delta_+(T) \cap \operatorname{acc}\Pi(T) = \emptyset$. Then, $\Delta_+(T) \subseteq \partial(\Pi(T)) = \Pi(T) \cup \operatorname{acc}\Pi(T)$ and so $\Delta_+(T) \subseteq \Pi(T)$. On the other hand, $\Pi(T) \subseteq \partial(\Pi(T)) = Cl(\Delta_+(T)) = \Delta_+(T) \cup \operatorname{acc}\Delta_+(T)$ and

hence, $\Pi(T) \subseteq \Delta_+(T)$.
(3) $\Rightarrow$ (1). Suppose that $Int(\Delta_+^g(T)) = \emptyset$ and $\partial(\Pi(T)) \subseteq \partial(\Delta_+(T))$. By Lemma 3.1, we have $\sigma(T) = \sigma_{\mathrm{a}}(T)$ and then by Lemma 2.1, $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{d}}(T)$. Since $Int(\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T)) = \emptyset$, by [3, Theorem 5.40] it follows that $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{ubw}}(T)$. Thus, $\Delta_+(T) \subseteq \sigma(T) \setminus \sigma_{\mathrm{ubw}}(T) = \sigma(T) \setminus \sigma_{\mathrm{ld}}(T) = \Pi(T)$. On the other hand, we have $\Pi(T) \subseteq \partial(\Pi(T)) \subseteq \partial(\Delta_+(T)) \subseteq Cl(\Delta_+(T))$, which implies that $\Pi(T) \subseteq \Delta_+(T)$.
(4) $\Rightarrow$ (1). The proof is similar to the proof of (3) $\Rightarrow$ (1). Just use the fact that if $\Pi(T) = p_{00}(T) \subseteq \Delta_+(T)$.
(5) $\Rightarrow$ (1). Suppose that $\partial(\Pi(T)) \subseteq Cl(\Delta_+(T))$ and $\Delta_+(T) \subseteq \operatorname{iso} \sigma(T)$. Then, $\Pi(T) \subseteq \partial(\Pi(T)) \subseteq Cl(\Delta_+(T)) = \Delta_+(T) \cup \operatorname{acc} \Delta_+(T)$ and hence, $\Pi(T) \subseteq \Delta_+(T)$. On the other hand, if $\lambda \in \Delta_+(T)$ then by hypothesis, $\lambda \in \operatorname{iso} \sigma(T)$ and so both T and T^* have SVEP at λ. Since $\lambda \in \Delta_+(T)$, by Remark 1.5 it follows that $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Therefore, $\lambda \in \Pi(T)$ and so $\Delta_+(T) \subseteq \Pi(T)$.
(6) $\Rightarrow$ (1). The proof is similar to the proof of (5) $\Rightarrow$ (1). Just use the fact that if $\lambda \in \partial\sigma(T)$ then both T and T^* have SVEP at λ. $\square$

Let $\rho_{\mathrm{a}}(T) = \mathbf{C} \setminus \sigma_{\mathrm{a}}(T)$, $\rho_{\mathrm{uw}}(T) = \mathbf{C} \setminus \sigma_{\mathrm{uw}}(T)$ and $\rho_{\mathrm{sf}}(T) = \mathbf{C} \setminus \sigma_{\mathrm{sf}}(T)$, where $\sigma_{\mathrm{sf}}(T)$ is the semi-Fredholm spectrum of T. It is proved in [9, Theorem 3.4] that $\rho_{uw}(T)$ is connected if and only if $\rho_{\mathrm{a}}(T)$ is connected and T verifies a-Browder's theorem, i.e. $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$ (or equivalently generalized a-Browder's theorem, i.e. $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$). Now, the fact that $\rho_{\mathrm{a}}(T)$ is connected implies that $\sigma_{\mathrm{a}}(T) = \sigma(T)$. Indeed, if $\rho_{\mathrm{a}}(T)$ is connected then it has an unique component, says Ω; since $\rho(T) \subseteq \rho_{\mathrm{a}}(T)$, we have $\rho(T) \subseteq \Omega$ and as $\rho_{\mathrm{a}}(T) \subseteq \rho_{\mathrm{sf}}(T)$, there exists a component Ω' of $\rho_{\mathrm{sf}}(T)$ which contains Ω and hence, $\rho_{\mathrm{a}}(T) \subseteq \Omega'$. It is clear that Ω' contains $\rho(T)$. Since both T and T^* have SVEP at every point of $\rho(T)$, by [9, Theorem 2.4] it follows that both T and T^* have SVEP at every point of Ω'; in particular, they have SVEP at every point of Ω. If $\lambda \notin \sigma_{\mathrm{a}}(T)$ then $\lambda \in \Omega$, so both T and T^* have SVEP at λ and as $\lambda I - T$ is a semi-Fredholm operator, by Remark 1.5 it follows that $p(\lambda I - T) = q(\lambda I - T) = 0$ and hence, $\lambda \notin \sigma(T)$. This shows what we had said before.

Theorem 3.3. *Let* $T \in L(X)$*. If* $\rho_{\mathrm{uw}}(T)$ *is connected and* $p_{00}(T) = \Pi(T)$*, then* T *verifies property* (V_{Π})*.*

Proof. Suppose that $\rho_{\mathrm{uw}}(T)$ is connected. From the previous discussion we have $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{a}}(T) = \sigma(T)$. By Lemma 2.1, we conclude that $\sigma_{\rm uw}(T) = \sigma_{\rm b}(T)$ and hence, $\sigma(T) \setminus \sigma_{\rm uw}(T) \subseteq \sigma(T) \setminus \sigma_{\rm d}(T) = \Pi(T)$. On the other hand, let $\lambda \in \Pi(T) = p_{00}(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{\rm b}(T) = \sigma(T) \setminus \sigma_{\rm uw}(T)$ and so, $\Pi(T) \subseteq \sigma(T) \setminus \sigma_{\rm uw}(T)$. Therefore, T verifies property (V_Π). $\square$

Theorem 3.4. *Let $T \in L(X)$. If $\rho_{\rm uw}(T)$ is connected and $\sigma_{\rm ubw}(T) = \sigma_{\rm w}(T)$ then, T verifies property (V_Π).*

Proof. The hypotheses $\rho_{\rm uw}(T)$ is connected and $\sigma_{\rm ubw}(T) = \sigma_{\rm w}(T)$ implies that $\sigma_{\rm a}(T) = \sigma(T)$ and $\sigma_{\rm uw}(T) = \sigma_{\rm ubw}(T) = \sigma_{\rm ld}(T)$, respectively. Then, by Lemma 2.1, the conclusion follows easily. $\square$

Let $\mathcal{H}(\sigma(T))$ be the set of all analytic functions defined over an open neighborhood of $\sigma(T)$.

Theorem 3.5. *Let $T \in L(X)$ and $f \in \mathcal{H}(\sigma(T))$. If $f(T)$ verifies property (V_Π) then $\sigma_{\rm uw}(f(T)) = f(\sigma_{\rm uw}(T))$.*

Proof. Suppose that $f(T)$ verifies property (V_Π). Then by [20, Theorem 2.27], $\sigma_{\rm uw}(f(T)) = \sigma_{\rm b}(f(T))$, and as $\sigma_{\rm b}(T)$ is a regularity (see [3, Theorem 3.109]), we have $f(\sigma_{\rm uw}(T)) \subseteq f(\sigma_{\rm b}(T)) = \sigma_{\rm uw}(f(T))$. By [3, Theorem 3.115], we have $\sigma_{\rm uw}(f(T)) \subseteq f(\sigma_{\rm uw}(T))$ and hence, $\sigma_{\rm uw}(f(T)) = f(\sigma_{\rm uw}(T))$. $\square$

Let $\mathcal{H}_{nc}(\sigma(T))$ be the set of all analytic functions defined on an open neighborhood of $\sigma(T)$, such that f is non-constant on each of the components of its domain of definition. Applying the same techniques in the proof of [3, Theorem 5.8], we obtain the following result.

Theorem 3.6. *Suppose that $T \in L(X)$ verifies property (V_Π) and $f \in \mathcal{H}_{nc}(\sigma(T))$. Then $f(T)$ verifies property (V_Π).*

Proof. Suppose that $f \in \mathcal{H}_{nc}(\sigma(T))$ and $f(\lambda_0) \in \sigma(f(T)) \setminus \sigma_{\rm uw}(f(T))$. There is a $v \in \mathbf{N}$ and two polynomials h and g in $\mathcal{H}_{nc}(\sigma(T))$ with no zero in $\sigma(T)$, such that

$$f(\lambda) - f(\lambda_0) = (\lambda_0 - \lambda)^v h(\lambda) g(\lambda),$$

with $h(\lambda_0) = 0$ and $h(\lambda_0) \notin g(\sigma(T))$, which implies that

$$f(T) - f(\lambda_0 I) = (\lambda_0 I - T)^v h(T) g(T) \in W^+(X),$$

with $0 \notin \sigma(h(T)g(T))$ and so, $\lambda_0 \notin \sigma_{\mathrm{uw}}(T)$. By Theorem 2.4, we have T^* has SVEP at λ_0 and $p_{00}(T) = \Pi(T)$. Now, by [3, Theorem 2.88], $f(T^*) = f(T)^*$ has SVEP at $f(\lambda_0)$ and, on the other hand, as $\sigma_{\mathrm{b}}(T)$ and $\sigma_{\mathrm{d}}(T)$ are regularities, then by [3, Theorem 3.109] it follows that $\Pi(f(T)) \cup \sigma_{\mathrm{d}}(f(T)) = \sigma(f(T)) = f(\sigma(T)) = f(\Pi(T) \cup \sigma_{\mathrm{d}}(T)) = f(\Pi(T)) \cup f(\sigma_{\mathrm{d}}(T)) = f(\Pi(T)) \cup \sigma_{\mathrm{d}}(f(T))$, which implies that $\Pi(f(T)) = f(\Pi((T))$ and $p_{00}(f(T)) \cup \sigma_{\mathrm{b}}(f(T)) = \sigma(f(T)) = f(\sigma(T)) = f(p_{00}(T) \cup \sigma_{\mathrm{b}}(T)) = f(p_{00}(T)) \cup f(\sigma_{\mathrm{b}}(T)) = f(p_{00}(T)) \cup \sigma_{\mathrm{b}}(f(T))$, which implies that $p_{00}(f(T)) = f(p_{00}((T))$. By [20, Theorem 2.27] it follows that $\Pi(f(T)) = p_{00}(f(T))$ and again, by Theorem 2.4, we conclude that $f(T)$ verifies property (V_Π). $\square$

Lemma 3.7. *Let $T \in L(X)$ and suppose that $\lambda I - T$ is injective for all $\lambda \in \sigma(T)$. Then, $\sigma_{\mathrm{uw}}(T^*) = \sigma_{\mathrm{d}}(T^*)$ or equivalently, $\sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{d}}(T)$.*

Proof. Let $\lambda \notin \sigma_{\mathrm{d}}(T^*) = \sigma_{\mathrm{d}}(T)$. Then, $\lambda I - T$ is a Drazin invertible operator and as $\lambda I - T$ is injective, then $0 = p(\lambda I - T) = q(\lambda I - T) < \infty$, which implies that $\mathrm{ind}(\lambda I - T) = 0$ and $\lambda I - T \in W(X)$. Thus, $\lambda I - T \in W_-(X)$ and $\lambda \notin \sigma_{\mathrm{lw}}(T) = \sigma_{\mathrm{uw}}(T^*)$. This shows the inclusion $\sigma_{\mathrm{uw}}(T^*) \subseteq \sigma_{\mathrm{d}}(T^*)$.

To show the opposite inclusion $\sigma_{\mathrm{d}}(T^*) \subseteq \sigma_{\mathrm{uw}}(T^*)$, let $\lambda \notin \sigma_{\mathrm{uw}}(T^*) = \sigma_{\mathrm{lw}}(T)$. Then, $\lambda I - T$ is a lower semi-Weyl operator and so $\mathrm{ind}(\lambda I - T) \geq 0$. By the injectivity of $\lambda I - T$ it follows that $0 = p(\lambda I - T) < \infty$ and hence $\beta(\lambda I - T) \leq \alpha(\lambda I - T) = 0$. Thus, $0 = \alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$ and consequently, $0 = p(\lambda I - T) = q(\lambda I - T) < \infty$. Therefore, $\lambda I - T$ is a Drazin invertible operator and so $\lambda \notin \sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{d}}(T^*)$. $\square$

For an operator $T \in L(X)$, the analytic core of T is the set $K(T)$ of all $x \in X$ such that there exists a constant $c > 0$ and a sequence $(u_n) \subset X$, such that $x = u_0$, $Tu_{n+1} = u_n$, and $\|u_n\| < c^n \|x\|$ for all $n \in \mathbf{N}$. Note that inclusion $\ker(\lambda I - T) \subseteq K(\lambda I - T)$ is obvious for every $\lambda \in \mathbf{C}$. We denote by $\sigma_{\mathrm{p}}(T)$ the point spectrum of T.

Theorem 3.8. *Let $T \in L(X)$. If there exists $\lambda_0 \in \mathbf{C}$ such that $K(\lambda_0 I - T) = \{0\}$ and $\ker(\lambda_0 I - T) = \{0\}$, then $f(T^*) = f(T)^*$ verifies property (V_Π) for each $f \in \mathcal{H}(\sigma(T))$.*

Proof. Since $\ker(\lambda I - T) \subseteq K(\lambda_0 I - T)$ for each $\lambda = \lambda_0$, it follows that $\ker(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbf{C}$, which implies that $\sigma_{\mathrm{p}}(T) = \emptyset$. First we will show that $\sigma_{\mathrm{p}}(f(T)) = \emptyset$. Suppose that there exists a $\mu \in \sigma_{\mathrm{p}}(f(T))$, then

$$\mu - f(\lambda) = p(\lambda)g(\lambda),$$

where g is an analytical function on an open set $\mathcal{U}$ containing $\sigma(T)$ and without zeros in $\sigma(T)$, and p is a polynomial of the form $p(\lambda) = \prod_{k=1}^{n}(\lambda_k - \lambda)^{v_k}$ with different roots $\lambda_1, \cdots, \lambda_k$ in $\sigma(T)$. Clearly, $\mu - f(T) = p(T)g(T)$ and $g(T)$ is invertible. As $\sigma_{\mathrm{p}}(T) = \emptyset$, it then follows that $\ker(\lambda I - f(T)) = \{0\}$ for all $\lambda \in \mathbf{C}$, and hence $\sigma_{\mathrm{p}}(f(T)) = \emptyset$. Consequently, $\lambda I - f(T)$ is injective for all $\lambda \in \mathbf{C}$ and by Lemma 4.14, we conclude that $\sigma_{\mathrm{uw}}(f(T)^*) = \sigma_{\mathrm{d}}(f(T)^*)$. Since the spectral mapping theorem holds for T, the proof follows. $\square$

4. Property (V_Π) for Drazin invertible and polaroid operators

It is shown in [8] that the nonzero points of some other spectra of a Drazin invertible operator $T \in L(X)$ and its Drazin inverse S, that originated from Fredholm theory, satisfy a relationship of reciprocity. The details for the proof of the following lemma are obtained from [8].

Lemma 4.1. *Suppose that $T \in L(X)$ is Drazin invertible with Drazin inverse S. Then we have:*

1. $0 \in \sigma(T)$ *if and only if* $0 \in \sigma(S)$.
2. $0 \in \operatorname{iso}\sigma(T)$ *if and only if* $0 \in \operatorname{iso}\sigma(S)$.
3. $\alpha(T) < \infty$ *if and only if* $\alpha(S) < \infty$.
4. $\sigma_{\mathrm{uw}}(S) \setminus \{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{uw}}(T) \setminus \{0\}\right\}$.
5. $\sigma_{\mathrm{ub}}(S) \setminus \{0\} = \left\{\frac{1}{\lambda} : \lambda \in \sigma_{\mathrm{ub}}(T) \setminus \{0\}\right\}$.
6. $\ker(\lambda I - S)^k = \ker\left(\frac{1}{\lambda}I - T\right)^k$ *for all* $k \in \mathbf{N}$.
7. $(\lambda I - S)^k(X) = \left(\frac{1}{\lambda}I - T\right)^k(X)$ *for all* $k \in \mathbf{N}$.
8. $p(\lambda I - T) = p\left(\frac{1}{\lambda}I - S\right)$ *and* $q(\lambda I - T) = q\left(\frac{1}{\lambda}I - S\right)$.
9. $\lambda \in \sigma(T)$ *if and only if* $\frac{1}{\lambda} \in \sigma(S)$.

Now, we make some considerations for the special case where $\lambda = 0$.

Theorem 4.2. *Suppose that $T \in L(X)$ is Drazin invertible with Drazin inverse S. Then we have:*

1. $0 \in \sigma(S) \setminus \sigma_{\mathrm{uw}}(S)$ *if and only if* $0 \in \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$.
2. $0 \in \sigma(S) \setminus \sigma_{\mathrm{ub}}(S)$ *if and only if* $0 \in \sigma(T) \setminus \sigma_{\mathrm{ub}}(T)$.

Proof. (1). If $0 \in \sigma(T) \setminus \sigma_{uw}(T)$ then $0 \in \sigma(T)$ and $0 \notin \sigma_{uw}(T)$. It follows that $\alpha(T) < \infty$ and hence $\alpha(S) < \infty$ by Lemma 4.1. Now, as $p(S) = q(S) < \infty$ then $\alpha(S) = \beta(S) < \infty$ and $S \in W(X)$; in particular, $S \in W^{+}(X)$ and so $0 \notin \sigma_{\mathrm{uw}}(S)$. Also by Lemma 4.1, we have $0 \in \sigma(S)$ and hence $0 \in \sigma(S) \setminus \sigma_{uw}(S)$. The converse may be proved in a similar way. (2) The proof is similar to part (1). □

The next result show that property (V_{Π}) is transmitted from a Drazin invertible operator to its Drazin inverse.

Theorem 4.3. *Suppose that $T \in L(X)$ is Drazin invertible with Drazin inverse S. Then, T verifies property (V_{Π}) if and only if S verifies property (V_{Π}).*

Proof. Suppose that T verifies property (V_{Π}). Let $\lambda \in \sigma(S) \setminus \sigma_{\mathrm{uw}}(S)$. If $\lambda = 0$ then from Theorem 4.2 it follows that $0 \in \sigma(S) \setminus \sigma_{\mathrm{uw}}(S)$ if and only if $0 \in \sigma(S) \setminus \sigma_{\mathrm{d}}(S)$. Suppose that $\lambda = 0$. By Lemma 4.1, we have the following equivalences:

$$\begin{aligned} \lambda \in \sigma(S) \setminus \sigma_{\mathrm{uw}}(S) \quad &\Leftrightarrow \quad \frac{1}{\lambda} \in \sigma(T) \setminus \sigma_{\mathrm{uw}}(T) \\ &\Leftrightarrow \quad \frac{1}{\lambda} \in \sigma(T) \setminus \sigma_{\mathrm{d}}(T) \\ &\Leftrightarrow \quad \lambda \in \sigma(S) \setminus \sigma_{\mathrm{d}}(S). \end{aligned}$$

Therefore, S verifies property (V_{Π}). The converse may be proved by using similar arguments. □

Observe that if $T \in L(X)$ is Drazin invertible with Drazin inverse S, then T^* is Drazin invertible and its Drazin inverse is S^*, since $(T^*)^n = T^*S^*(T^n)^* = T^*S^*T^*(T^{n-1})^* = T^*T^*S^*(T^{n-1})^* = (T^*)^2S^*(T^{n-1})^* = \cdots = (T^*)^nS^*T^*$. Property (V_{Π}) is also transmitted from a Drazin invertible operator T to the dual of its Drazin inverse under assumption that T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$.

Theorem 4.4. *Suppose that $T \in L(X)$ is Drazin invertible with Drazin inverse S and T has SVEP at every $\lambda \notin \sigma_{\mathrm{lw}}(T)$. If T verifies property (V_{Π}), then S^* verifies property (V_{Π}).*

Proof. Follows from Theorems 2.6 and 4.3. □

For $T \in L(X)$, define $\pi_{00}(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}$ (the set if all eigenvalues of T which are isolated points of the spectrum and have finite multiplicity) and $E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(\lambda I - T)\}$ (the set of all eigenvalues of T which are isolated points of the spectrum).

According to [16] (resp. [10]), we say that an operator $T \in L(X)$ verifies property (w) (resp. property (gw)), if $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ (resp. $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ubw}}(T) = E(T)$). It is shown in [10, Theorem 2.3] that (gw) implies (w) but not conversely. According to [5], we say that $T \in L(X)$ verifies *property* (R) if $p_{00}^{a}(T) = \pi_{00}(T)$. It is shown in [5, Theorem 2.4] that property (w) implies property (R) but the converse in general does not hold. Following [6], we say that $T \in L(X)$ verifies property (gR), if $\Pi_a(T) = E(T)$. In [6, Theorem 2.2], it is shown that property (gR) implies property (R), but the converse is not true in general. The following results shows that under the condition that T is polaroid, property (V_{Π}) implies properties (gR) and (gw). According to [19], we say that $T \in L(X)$ verifies *property* (V_E) if $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = E(T)$. It is shown in [20, Theorem 2.3] that property (V_E) implies property (V_{Π}) but the converse in general does not hold.

Theorem 4.5. *Let $T \in L(X)$ polaroid. Then, T verifies property (V_E) if and only if T verifies property (V_{Π}).*

Proof. It is clear, because if T is polaroid then $E(T) = \Pi(T)$. □

Theorem 4.6. *If $T \in L(X)$ verifies property (V_E), then T verifies property (gR).*

Proof. Property (V_E) implies by [19, Theorem 2.27] that generalized a-Browder's theorem and property (gw) are equivalent, and T verifies these properties. Consequently, $\Pi_a(T) = E(T)$ and T verifies property (gR). □

The following example shows that, in general, property (gR) does not imply property (V_E).

Example 4.7. *Consider the operator* $T = I \oplus S$ *that is defined on the Banach space* $X = \ell^2(\mathbf{N}) \oplus \ell^2(\mathbf{N})$ *where* I *is the identify operator on* $\ell^2(\mathbf{N})$ *and* S *is an injective quasinilpotent operator on* $\ell^2(\mathbf{N})$ *which is not nilpotent. Then* $\sigma(T) = \{0, 1\}, \sigma_{uw}(T) = \{0, 1\}, \prod_a(T) = \{1\}$ *and* $E(T) = \{1\}$*. Thus,* $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = E(T)$ *and* $\prod_a(T) = E(T)$*, which implies that* T *verifies property* (gR) *but not verifies property* (V_E)*.*

Corollary 4.8. *Let* $T \in L(X)$ *polaroid. If* T *verifies property* (V_Π)*, then* T *verifies property* (gR)*.*

Proof. Follows from Theorems 4.5 and 4.6. □

Corollary 4.9. *Let* $T \in L(X)$ *polaroid. If* T *verifies property* (V_Π)*, then* T *verifies property* (gw)*.*

Proof. Follows from Theorem 4.5 and [19, Theorem 2.7]. □

Corollary 4.10. *Let* $T \in L(X)$ *polaroid. If* T *verifies property* (V_Π)*, then* T *verifies property* (R)*.*

Proof. It is clear, since property (gR) implies property (R). □

Corollary 4.11. *Let* $T \in L(X)$ *polaroid. If* T *verifies property* (V_Π)*, then* T *verifies property* (w)*.*

Proof. It is clear, since property (gw) implies property (w). □

Theorem 4.12. *Let* $T \in L(X)$ *polaroid. Then,* T *verifies property* (V_Π) *if and only if* T *verifies property* (w) *and* $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{b}}(T)$*.*

Proof. Sufficiency: Follows from Corollary 4.11 and [20, Theorem 2.27]. Necessity: The equality $\sigma_{\mathrm{ld}}(T) = \sigma_{\mathrm{b}}(T)$ implies that $\sigma_{\mathrm{d}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T)$ and the last equality implies by Lemma 2.1 that $\sigma(T) = \sigma_{\mathrm{a}}(T)$. Then, $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ and as T is polaroid, $\pi_{00}(T) \subseteq \Pi(T)$. Hence, $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) \subseteq \Pi(T)$. On the other hand, $\Pi(T) = \sigma(T) \setminus \sigma_{\mathrm{d}}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) \subseteq \sigma(T) \setminus \sigma_{\mathrm{uw}}(T)$. Thus, T verifies property (V_Π). □

In the classical Fredholm theory, it is well known that given an operator $T \in L(X)$ and a compact operator $K \in L(X)$, then $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T + K)$

and $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T+K)$, where commutativity between T and K is not required. The following theorem was motivated by the results of Aiena and Triolo [9].

Theorem 4.13. *Let $T \in L(X)$ and suppose that $K \in L(X)$ is a compact operator such that $p_{00}(T+K) = \Pi(T+K)$. If $\rho_{\mathrm{uw}}(T)$ is connected, then $T+K$ verifies property (V_{Π}). In addition, if $\operatorname{iso}\sigma_{\mathrm{uw}}(T) = \emptyset$, then $T+K$ verifies properties (gR) and (gw).*

Proof. Let $S := T+K$ where K is a compact operator on $L(X)$ such that $p_{00}(T+K) = \Pi(T+K)$ and suppose that $\rho_{\mathrm{uw}}(T)$ is connected. Note that

$$\rho_{\mathrm{uw}}(S) \;=\; \rho_{\mathrm{uw}}(T+K) = \mathbf{C} \setminus \sigma_{\mathrm{uw}}(T+K) = \mathbf{C} \setminus \sigma_{\mathrm{uw}}(T) = \rho_{\mathrm{uw}}(T),$$

which implies that $\rho_{\mathrm{uw}}(S)$ is connected. Since $p_{00}(S) = \Pi(S)$, by Theorem 2.6 it follows that $S : T+K$ verifies property (V_{Π}).

In addition, assuming that $\operatorname{iso}\sigma_{\mathrm{uw}}(T) = \emptyset$, we have $\operatorname{iso}\sigma_{\mathrm{uw}}(S) = \emptyset$ and hence $S := T+K$ is polaroid, see [9, Lemma 3.15]. Now by Corollaries 4.8 and 4.9, we obtain that $S := T+K$ verifies properties (gR) and (gw), respectively. $\square$

Recall that an operator $T \in L(X)$ is called finite-polaroid if every isolated point of the spectrum $\sigma(T)$ is a pole a finite rank, i.e. $\operatorname{iso}\sigma(T) \subseteq p_{00}(T)$. Clearly, if T is finite-polaroid, then T is polaroid and $p_{00}(T) = \Pi(T) = E(T)$.

Lemma 4.14. *If $T \in L(X)$ is such that $\operatorname{iso}\sigma_{\mathrm{b}}(T) = \emptyset$, then T is finite-polaroid and hence polaroid.*

Proof. Let $\lambda \in \operatorname{iso}\sigma(T)$. Then, either $\lambda \in \sigma_{\mathrm{b}}(T)$ or $\lambda \notin \sigma_{\mathrm{b}}(T)$. If $\lambda \in \sigma_{\mathrm{b}}(T)$, we have $\lambda \in \operatorname{iso}\sigma_{\mathrm{b}}(T)$, which is impossible. Hence, $\lambda \notin \sigma_{\mathrm{b}}(T)$ and so $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T)$, that is, λ is a pole of finite rank. $\square$

Theorem 4.15. *Let $T \in L(X)$ be such that $\rho_{\mathrm{w}}(T)$ is connected and $\operatorname{iso}\sigma_{\mathrm{w}}(T) = \emptyset$. If $K \in L(X)$ is a compact operator, then $T+K$ is finite-polaroid.*

Proof. Let $S := T + K$ where K is a compact operator on $L(X)$. Since $\rho_{\mathrm{w}}(T)$ is connected, by [9, Theorem 3.6] it follows that $S := T + K$ verifies a-Browder's theorem and so $\sigma_{\mathrm{b}}(S) = \sigma_{\mathrm{w}}(S) = \sigma_{\mathrm{w}}(T)$, which implies that $\operatorname{iso}\sigma_{\mathrm{b}}(S) = \operatorname{iso}\sigma_{\mathrm{w}}(T) = \emptyset$, and hence, by Lemma 4.14, $S := T + K$ is finite-polaroid. $\square$

Corollary 4.16. *Let* $T \in L(X)$ *be such that* $\rho_{\mathrm{w}}(T)$ *is connected and* $\operatorname{iso}\sigma_{\mathrm{w}}(T) = \emptyset$. *Then* $T + K$ *verifies property* (V_{Π}) *for every compact operator* $K \in L(X)$.

Recall that given a compact subset σ of $\mathbf{C}$, we say that a hole of σ is any bounded component of the complement $\mathbf{C} \setminus \sigma$. If σ has no holes, then $\mathbf{C} \setminus \sigma$ is connected, since $\mathbf{C} \setminus \sigma$ has always an unbounded component. Next, we give some applications of the previous results obtained.

(1) Let ASC denote the class of Banach space operators $T \in L(X)$ which satisfy the abstract shift condition $T^{\infty}(X) = \{0\}$, where $T^{\infty}(X)$ is the hyper-range of T defined as $T^{\infty}(X) := \bigcap_{n=1}^{\infty} T^n(X)$. An interesting subclass of the class ASC is that of weighted right shift operators T, denoted by $T \in$ WRS, in $L(\ell^p(\mathbf{N}))$, $1 \leq p < \infty$. For an operator $T \in$ ASC the following properties are well known. (See [1, Sections 2.5 and 3.10] and [14, Section 1.6] for more details.)
(i) For an operator $T \in L(X)$, the lower bound of T is defined by

$$k(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

Now, let

$$i(T) = \lim_{n \to \infty} k(T^n)^{1/n} = \sup_{n \in \mathbf{N}} k(T^n)^{1/n}.$$

If $r(T)$ denotes the spectral radius of T then $i(T) \leq r(T)$. Moreover, if $T \in$ WRS and $\mathbf{D}(0, r(T))$ denotes the closed disc centered at 0 of radius $r(T)$, then $\sigma(T) = \mathbf{D}(0, r(T))$ and $\sigma_{\mathrm{a}}(T) = \{\lambda \in \mathbf{C} : i(T) \leq |\lambda| \leq r(T)\} = \sigma_{\mathrm{sf}}(T) = \sigma_{\mathrm{f}}(T)$, where $\sigma_{\mathrm{f}}(T)$ is the Fredholm spectrum of T.
(ii) If $T \in$ ASC and $i(T) = r(T)$, then $\sigma(T) = \mathbf{D}(0, r(T))$ and $\sigma_{\mathrm{a}}(T) = \partial\mathbf{D}(0, r(T))$.
(iii) Put ASCI $= \{T \in \text{ASC} : i(T) = r(T)\}$. If either $T \in$ WRS or $T \in$ ASCI, then $\sigma_{\mathrm{w}}(T)$ has no holes, $\rho_{\mathrm{w}}(T) = \mathbf{C} \setminus \mathbf{D}(0, r(T))$ is connected and $\operatorname{iso}\sigma_{\mathrm{w}}(T) = \emptyset$, which implies by Theorem 4.15 that $T + K$ is finitely

polaroid for every compact operator K. Also, $T+K$ verifies property (V_{Π}) by Corollary 4.16.
(2) Consider an operator $T \in L(X)$ such that T^* has SVEP, $\sigma(T)$ has no holes and $\operatorname{iso}\sigma(T)=\emptyset$. Then, T verifies Browder's theorem, $\sigma(T)=\sigma_{\mathrm{a}}(T)$, $\rho(T)=\rho_{\mathrm{a}}(T)$ and $\operatorname{iso}\sigma(T)=\operatorname{iso}\sigma_{\mathrm{a}}(T)=\emptyset$. Thus, we have $\sigma(T)\setminus\sigma_{\mathrm{w}}(T)=p_{00}(T)\subseteq\operatorname{iso}\sigma(T)=\emptyset$ and $\sigma_{\mathrm{a}}(T)\setminus\sigma_{\mathrm{uw}}(T)\subseteq p_{00}^{a}(T)\subseteq\operatorname{iso}\sigma_{\mathrm{a}}(T)=\emptyset$, which implies that $\sigma(T)=\sigma_{\mathrm{a}}(T)=\sigma_{\mathrm{w}}(T)=\sigma_{\mathrm{uw}}(T)$ and hence $\operatorname{iso}\sigma_{\mathrm{uw}}(T)=\operatorname{iso}\sigma_{\mathrm{w}}(T)=\emptyset$ and $\rho(T)=\rho_{\mathrm{a}}(T)=\rho_{\mathrm{uw}}(T)=\rho_{\mathrm{w}}(T)$ is a connected subset of $\mathbf{C}$. In view of Theorem 4.15 and Corollary 4.16, for every compact operator $K\in L(X)$ we have $T+K$ is finite polaroid and verifies property (V_{Π}), respectively. An example of this type of operators are the normal operators on Hilbert spaces for which the resolvent is connected and the spectrum has no isolated points. A more general example, are the generalized scalar operators, since these operators are decomposable, and hence, their adjoint operators are also decomposable and have SVEP, see [14]. This class of operators contains the invertible isometries on Banach spaces (see [14, Theorem 1.5.13]), and the composition operator $T_\gamma : C(\Omega)\to C(\Omega)$, defined by $T_\gamma f = f\circ\gamma$, where $C(\Omega)$ is the Banach space of all continuous complex-valued functions on a compact Hausdorff space Ω and γ is an arbitrary homomorphism.
(3) Let $H^2(\mathbf{T})$ denote the Hardy space of the unit circle $\mathbf{T}$ in the complex plane. Given $\phi\in L^\infty(\mathbf{T})$, the Toeplitz operator with symbol ϕ is the operator on $H^2(\mathbf{T})$ defined by

$$T_\phi : f \longrightarrow P(\phi f),$$

where $f\in H^2(\mathbf{T})$ and P is the orthogonal projection of $L^\infty(\mathbf{T})$ onto $H^2(\mathbf{T})$. We denote by $C(\mathbf{T})$ the algebra of all complex-valued continuous functions on $\mathbf{T}$. Consider $T\in C(\mathbf{T})$ and denote $\Gamma=\phi(\mathbf{T})$. Recall that the winding number of a curve in the plane around a given point is an integer representing the total number of times that curve travels counterclockwise around the point. Given $\lambda\notin\Gamma$, denotes by $wn(\phi,\lambda)$ the winding number of Γ determined by ϕ with respect λ. It is shown in [3, Corollary 4.98] that $\sigma(T_\phi)=\sigma_{\mathrm{w}}(T_\phi)=\sigma_{\mathrm{b}}(T_\phi)=\Gamma\cup\{\lambda\in\mathbf{C}: wn(\phi,\lambda)=0\}$, Therefore, $\rho_{\mathrm{w}}(T)$ is connected if and only if the winding number of ϕ with respect to each hole of Γ is nonzero. By [3, Theorem 4.99], ϕ is non-constant if and only if $\operatorname{iso}\sigma_{\mathrm{w}}(T)=\emptyset$. Thus, Theorem 4.15 and Corollary 4.16 apply in the case of Toeplitz operators with continuous symbol ϕ non-constant.

According to Rashid and Prasad [17], we say that an operator $T\in L(X)$ verifies property (Sb) if $\sigma(T)\setminus\sigma_{\mathrm{ubw}}(T)=p_{00}(T)$. Following Sanabria et al.

[18], we say that T verifies property (Sab) if $\sigma(T) \setminus \sigma_{\text{ubw}}(T) = p_{00}^a(T)$. In [18, Corollary 2.9], it is shown that property (Sab) is equivalent to property (Sb). Recall [20] that an operator $T \in L(X)$ verifies property (V_{Π_a}) if $\sigma(T) \setminus \sigma_{\text{uw}}(T) = \Pi_a(T)$. In [20, Corollary 2.21], it is shown that property (V_Π) is equivalent to properties (V_{Π_a}), (Sb) and (Sab). Consequently, all the results of this work are also valid if we replace property (V_Π) by any of the properties (V_{Π_a}), (Sb) and (Sab).

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