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Existence of coincidence points for Feng-Liu type multivalued contractions with a singlevalued mapping

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Abstract

In this paper we establish coincidence point results for multivalued Feng-Liu type contractions with a singlevalued mapping. There is a supporting example. Several other existing results are contained in our theorems.

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1. Introduction and Mathematical Background

Multivalued nonlinear contractions appeared first in fixed point theory in the work of Nadler [15]. This work was followed by a development of the branch of fixed point theory in the domain of setvalued analysis through works like [5, 7, 9, 10, 13, 16, 18]. The book of Almezel et al. [1] gives a comprehensive account of this development. In a paper published in 2009 [8], Ćirić proved two multivalued results supported with illustrations where several other fixed point results of nonlinear multivalued contractions could be combined into a single result. Our aim in this paper is to extend these results to coincidence point theorems involving a singlevalued and a multivalued mapping.

The coincidence point problem is a generalization of fixed point problem where two mappings are involved. Considering two singlevalued mappings $f,\ g : X \to X$, where X is a nonempty set, a point $z \in X$ is a coincidence point of f and g if $fz = gz$. It is naturally suggested that for the existence of a coincidence point of two self-mappings defined on a metric space there should be additional relations connecting the two mappings. Through a broad classification there are two categories of such relations, one is the class of commuting relations and the other is the category of compatible relations. Two self mappings f and g of a metric space (X, d) are commuting if $fgx = gfx$ for all x. Compatible mappings are commuting mappings in the asymptotic sense, that is, two self mappings f and g of a metric space (X, d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n,\ gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$, for some $x \in X$ is satisfied. The concept of compatibility was introduced by Jungck [11]. There are different generalizations of these two concepts like, weakly commuting, weakly compatible etc. which have been used in both coincidence point problems and common fixed point problems of various types. Some of the works in this area are noted in [2, 3, 4, 12, 17]. For our purpose in this paper which is to prove a generalization of the work of Ćirić [8], and hence of the works of Feng et al. [9], Klim et al. [13] and Mizoguchi et al. [14], which are generalized by the result of [8] to a coincidence point result of a singlevalued and a multivalued mapping. We use compatibility conditions between these two mappings. The results of Ćirić which we generalize are developments over the fixed point results of the type of contractions introduced by Feng and Liu in 2006 [9] which has come to be known as Feng-Liu contractions in literatures. We use this nomenclature through out the present paper.

Let $(X,\ d)$ be a metric space. We use following notations and definitions:

$$\begin{gathered} N(X)=\{A: A \text{ is a nonempty subset of } X\}, \\ Cl(X)=\{A: A \text{ is a nonempty closed subset of } X\}, \\ CB(X)=\{A: A \text{ is a nonempty closed and bounded subset of } X\}, \\ C(X)=\{A: A \text{ is a nonempty compact subset of } X\}, \\ D(x,B)=\ \inf\{d(x,y): y\in B\},\ \text{where } x\in X \text{ and } B\in Cl(X), \\ D(A,B)=\inf\{d(x,y): x\in A,\ y\in B\},\ \text{where} A,B\in Cl(X), \\ H(A,B)=\ \max\{\underset{x\in A}{sup}\ D(x,B),\ \underset{y\in B}{sup}\ D(y,A)\},\ \text{where } A,B\in CB(X). \end{gathered}$$

H is known as the Hausdorff metric on $CB(X)$ [15]. Further, if (X,d) is complete then $(CB(X),H)$ is also complete.

Definition 1.1 ([6]). Let $T: X\to N(X)$ be a multivalued mapping. An element $x\in X$ is called a fixed point of T if $x\in Tx$.

Definition 1.2 ([6]). Let $T: X\to N(X)$ be a multivalued mapping and $g: X\to X$. An element $x\in X$ is called a coincidence point of g and T if $gx\in Tx$.

The set of coincidence points of g and T is denoted by $C(g,\ T)$.

Definition 1.3 ([18]). Let $T: X\to CB(X)$ be a multivalued mapping and $g: X\to X$. The pair of mappings $(g,\ T)$ is said to compatible if $\lim\limits_{n\to\infty} D(gy_{n+1},Tgx_n)=0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X with $y_{n+1}\in Tx_n$, $n\in \mathbf{N}$ such that $\lim\limits_{n\to\infty} gx_n=\lim\limits_{n\to\infty} y_n=t$ for some t in X.

Definition 1.4. Let $T: X\to CB(Y)$ be a multivalued mapping, where (X,d), (Y,ρ) are two metric spaces and H_ρ is the Hausdorff metric on $CB(Y)$. The mapping T is said to be continuous at $x\in X$ if for any sequence $\{x_n\}$ in X, $H_\rho(Tx,Tx_n)\to 0$ whenever $d(x_n,x)\to 0$ as $n\to\infty$.

Definition 1.5. A function $f: X\to \mathbf{R}$ is said to be lower semi-continuous if for any sequence $\{x_n\}\subseteq X$ and $x\in X,\ x_n\to x\Rightarrow fx\le \liminf_{n\to\infty} fx_n$.

We use the following class of functions in our theorems.

By Φ we denote the collection of all functions $\phi:[0,\infty)\to[a,1)$, where $0<a<1$, such that $\limsup\limits_{r\to t}\phi(r)<1$ for each $t\in[0,\ \infty)$.

2. Generalized Feng -Liu contraction of first type :

Theorem 2.1. Let $(X,\ d)$ be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping and $g : X \to X$. Suppose that (i) $Tx \subseteq g(X)$ for every $x \in X$, (ii) the pair of mappings $(g,\ T)$ is compatible, (iii) g and T are continuous, (iv) for every $x \in X$ there corresponds a $y \in X$ such that

$$\sqrt{\phi(fx)}\ d(gx,\ gy) \le fx \quad \text{and} \quad fy \le \phi(fx)\ d(gx,\ gy),$$

where $f : X \to [0,\ \infty)$ is defined as $fx = D(gx,\ Tx)$, for $x \in X$ and $\phi \in \Phi$. Then $C(g,\ T)$ is nonempty.

Proof. By the definition of ϕ, we have $\phi(fx) < 1$, for each $x \in X$. Then it follows that for any $x \in X$, there exists a $v \in Tx$ such that

$$\sqrt{\phi(fx)}\ d(gx,\ v) \le D(gx,\ Tx) = fx.$$

By assumption (i), for any $v \in Tx$, we have a $y \in X$ such that $v = gy$. Then the above inequality becomes

$$\sqrt{\phi(fx)}\ d(gx,\ gy) \le D(gx,\ Tx) = fx. \tag{2.1}$$

Let $x_0 \in X$. By (2.1) and the assumption (iv), there exists $x_1 \in X$ with $y_1 = gx_1 \in Tx_0$ such that

$$\sqrt{\phi(fx_0)}\ d(gx_0,\ gx_1) \le fx_0$$

and

$$fx_1 \le \phi(fx_0)\ d(gx_0,\ gx_1).$$

Again, by (2.1) and the assumption (iv), there exists $x_2 \in X$ with $y_2 = gx_2 \in Tx_1$ such that

$$\sqrt{\phi(fx_1)}\ d(gx_1,\ gx_2) \le fx_1$$

and

$$fx_2 \le \phi(fx_1)\ d(gx_1,\ gx_2).$$

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \ge 0$,

$$y_{n+1} = gx_{n+1} \in Tx_n, \tag{2.2}$$

$$\sqrt{\phi(fx_n)}\ d(gx_n,\ gx_{n+1}) \leq fx_n \tag{2.3}$$

and

$$fx_{n+1} \leq \phi(fx_n)\ d(gx_n,\ gx_{n+1}), \tag{2.4}$$

where $fx_n = D(gx_n,\ Tx_n)$. Using (2.3) and (2.4), we have

$$\begin{aligned} fx_{n+1} \leq \phi(fx_n)\ d(gx_n, gx_{n+1}) &= \sqrt{\phi(fx_n)}\ [\sqrt{\phi(fx_n)} d(gx_n, gx_{n+1})] \\ &\leq \sqrt{\phi(fx_n)}\ fx_n. \end{aligned} \tag{2.5}$$

We shall show that $fx_n \to 0$ as $n \to \infty$. If possible, suppose that $fx_n = D(gx_n,\ Tx_n) = 0$, for some n. It follows that $gx_n \in \overline{Tx_n} = Tx_n$, where $\overline{Tx_n}$ is the closure of Tx_n. Therefore, x_n is a coincidence point of g and T, that is $x_n \in C(g,\ T)$ and hence the proof is completed. So, we assume that $fx_n > 0$ for all n. Then using the property of ϕ, we have from (2.5) that

$$fx_{n+1} \leq \sqrt{\phi(fx_n)}\ fx_n < fx_n, \quad \text{for all } n.$$

Therefore, $\{fx_n\}$ is a strictly decreasing sequence of positive real numbers and consequently there exists a $\delta \geq 0$ such that $\lim_{n\to\infty} fx_n = \delta$. We shall show that $\delta = 0$. If possible, suppose that $\delta > 0$. Then taking limit supremum on both sides of (2.5) and using the property of ϕ, we have

$$\delta \leq \limsup_{n\to\infty} \sqrt{\phi(fx_n)}\ \delta < \delta,$$

which is a contradiction. Therefore, $\delta = 0$ and hence

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} D(gx_n,\ Tx_n) = 0. \tag{2.6}$$

Now we shall show that $\{gx_n\}$ is a Cauchy sequence. Let $\alpha = \limsup_{n\to\infty} \sqrt{\phi(fx_n)}$. By the definition of ϕ, we have $\alpha < 1$. Let β be a real number such that $\alpha < \beta < 1$. Using the definition of limit supremum, there exists $n_0 \in \mathbf{N}$ such that

$$\sqrt{\phi(fx_n)} < \beta,\ \text{for all}\ \ n \geq n_0.$$

Then we have from (2.5) that

$$fx_{n+1} < \beta\ fx_n, \text{ for all } \ n \geq n_0. \tag{2.7}$$

Hence by repeated application of the above inequality, we have

$$fx_{n+1} < \beta^{n+1-n_0}\ fx_{n_0}, \text{ for all } \ n \geq n_0. \tag{2.8}$$

As $\phi(t) \geq a > 0$ for all $t \geq 0$, from (2.3), we get

$$d(gx_n,\ gx_{n+1}) \leq \frac{1}{\sqrt{\phi(fx_n)}}\ fx_n\ \leq \frac{1}{\sqrt{a}}\ fx_n, \text{ for all } n. \tag{2.9}$$

Using (2.8) and (2.9), we obtain

$$d(gx_n,\ gx_{n+1}) < \frac{1}{\sqrt{a}}\ \beta^{n-n_0}\ fx_{n_0}, \quad \text{for all } \ n \geq n_0. \tag{2.10}$$

Let $m, n \in \mathbf{N}$ with $m > n \geq n_0$. Then By (2.10) and the triangle inequality, we have

$$d(gx_n,\ gx_m) \leq \sum_{k=n}^{m-1} d(gx_k,\ gx_{k+1}) < \sum_{k=n}^{m-1} \frac{1}{\sqrt{a}}\ \beta^{k-n_0}\ fx_{n_0} \leq \frac{1}{(1-\beta)}\ \frac{1}{\sqrt{a}}\ \beta^{n-n_0}\ fx_{n_0}.$$

Then it follows that $\lim\limits_{n,m\to\infty} d(gx_n,\ gx_m) = 0$, which implies that $\{gx_n\}$ is a Cauchy sequence. Since the metric space $(X,\ d)$ is complete, there exists a point $z \in X$ such that

$$gx_n \to z \ \text{ as } \ n \to \infty. \tag{2.11}$$

By (2.2) and (2.11), we have

$$y_n \to z \ \text{ as } \ n \to \infty. \tag{2.12}$$

Here $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $y_{n+1} \in Tx_n$ for all $n \in \mathbf{N}$ and $\lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} y_n = z$, where $z \in X$. Applying the assumptions (ii) and (iii) and using (2.11), (2.12), we have

$$\lim_{n\to\infty} D(gy_{n+1},\ Tgx_n) = 0, \text{ that is, } D(gz,\ Tz) = 0,$$

which implies that $gz \in \overline{Tz} = Tz$, where $\overline{Tz}$ denotes the closure of Tz. Therefore, z is a coincidence point of the pair (g, T), that is, $z \in C(g,\ T)$ and hence $C(g,\ T)$ is nonempty.

In the next theorem we consider $g(X)$ to be closed in the metric space $(X,\ d)$ and the function $fx = D(gx,\ Tx)$ is lower semi-continuous. Due to these considerations we need not require the assumptions (ii) and (iii) which we consider in Theorem 2.1.

Theorem 2.2. Let $(X,\ d)$ be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping and $g : X \to X$. Suppose that (i) $Tx \subseteq g(X)$ for every $x \in X$, (ii) $g(X)$ is a closed subset of X, (iii) for every $x \in X$ there corresponds a $y \in X$ satisfying

$$\sqrt{\phi(fx)}\ d(gx,\ gy) \le fx \quad \text{and} \quad fy \le \phi(fx)\ d(gx,\ gy),$$

where the function f defined as in Theorem 2.1 is lower semi-continuous and $\phi \in \Phi$. Then $C(g,\ T)$ is nonempty.

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. Arguing similarly as in the proof of Theorem 2.1, we prove that (2.6) is satisfied and also $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is a closed subset of the complete metric space X, there exists $z \in g(X)$ such that $gx_n \to z$ as $n \to \infty$. Now $z \in g(X)$ implies that there exists $u \in X$ such that $gu = z$. Then $gx_n \to gu$ as $n \to \infty$.

Now we shall show that u is a coincidence point of T and g. As $fx = D(gx,\ Tx)$ is lower semi-continuous at $x = u$, using (2.6), we have

$$0 \le D(gu,\ Tu) = fu \le \liminf_{n\to\infty} fx_n = \liminf_{n\to\infty} D(gx_n,\ Tx_n) = 0,$$

which implies that $D(gu,\ Tu) = 0$. Then arguing similarly as in the proof of Theorem 2.1, $u \in C(g,\ T)$, that is, $C(g,\ T)$ is nonempty.

Remark 2.1. Taking g to be the identity function on X, we have Theorem 2.1 in [8].

3. Generalized Feng-Liu contraction of second type :

Theorem 3.1. Let $(X,\ d)$ be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping and $g : X \to X$. Suppose that (i) $Tx \subseteq g(X)$ for every $x \in X$, (ii) the pair of mappings $(g,\ T)$ is compatible, (iii) g and T are continuous, (iv) for every $x \in X$ there corresponds a $y \in X$ satisfying

$$\sqrt{\phi(d(gx,\ gy))}\ d(gx,\ gy) \le fx \ \text{ and }\ fy \le \phi(d(gx,\ gy))\ d(gx,\ gy),$$

where the function f and ϕ are same as in Theorem 2.1. Then $C(g,\ T)$ is nonempty.

Proof. Replacing $\phi(fx)$ by $\phi(d(gx,\ gy))$ and following the arguments as in the proof of Theorem 2.1, we construct two iterative sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{n+1} = gx_{n+1} \in Tx_n$ satisfying

$$\sqrt{\phi(d(gx_n,\ gx_{n+1})}\ d(gx_n,\ gx_{n+1}) \le fx_n \tag{3.1}$$

and

$$fx_{n+1} \le \phi(d(gx_n,\ gx_{n+1}))\ d(gx_n,\ gx_{n+1}). \tag{3.2}$$

From (3.1) and (3.2), we have

$$\begin{aligned} fx_{n+1} &\le \sqrt{\phi(d(gx_n,\ gx_{n+1}))}\ [\sqrt{\phi(d(gx_n,\ gx_{n+1}))}d(gx_n, gx_{n+1})] \\ &\le \sqrt{\phi(d(gx_n,\ gx_{n+1}))}\ fx_n. \end{aligned} \tag{3.3}$$

Due to the facts as described in the proof of Theorem 2.1, we may assume that $fx_n > 0$ for all n. Using the property of ϕ, we have from (3.3) that $fx_{n+1} \le \sqrt{\phi(d(gx_n,\ gx_{n+1}))}\ fx_n < fx_n$. So, $\{fx_n\}$ is a strictly decreasing sequence of positive real numbers. Therefore, there exists some $\delta \ge 0$ such that

$$\lim_{n \longrightarrow \infty} fx_n = \delta. \tag{3.4}$$

As $\phi(t) \ge a$ for all t, from (3.1), we get

$$d(gx_n,\ gx_{n+1}) \le \frac{fx_n}{\sqrt{\phi(d(gx_n,\ gx_{n+1})}} \le \frac{fx_n}{\sqrt{a}}. \tag{3.5}$$

From (3.4) and (3.5), we conclude that the sequence $\{d(gx_n,\ gx_{n+1})\}$ is bounded. Therefore, there exists a $\gamma \ge 0$ such that

$$\liminf_{n \longrightarrow \infty} d(gx_n,\ gx_{n+1}) = \gamma. \tag{3.6}$$

Since $gx_{n+1} \in Tx_n$, it follows that $d(gx_n,\ gx_{n+1}) \geq D(gx_n,\ Tx_n) = fx_n$, for each $n \geq 0$. This implies that $\gamma \geq \delta$. Now we shall prove that $\gamma = \delta$. At first suppose that $\delta = 0$. Then from (3.5) and (3.6) we have $0 \leq \gamma = \liminf_{n \longrightarrow \infty} d(gx_n,\ gx_{n+1}) \leq \lim_{n \longrightarrow \infty} \dfrac{fx_n}{\sqrt{a}} = \dfrac{0}{\sqrt{a}} = 0$, which implies that $\gamma = 0$. Thus, if $\delta = 0$, then $\gamma = \delta$.

Now we take $\delta > 0$ and suppose that $\gamma > \delta$. Then $\gamma - \delta > 0$ and so from (3.4), (3.6) and the definition of limit infimum we can find a positive integer n_0 such that

$$fx_n < \delta + \frac{\gamma - \delta}{4}, \text{ for all } \ n \geq n_0 \tag{3.7}$$

and

$$\gamma - \frac{\gamma - \delta}{4} < d(gx_n,\ gx_{n+1}), \text{ for all } \ n \geq n_0. \tag{3.8}$$

Then from (3.1), (3.7) and (3.8), we have

$$\begin{aligned} \sqrt{\phi(d(gx_n, gx_{n+1})}\ [\gamma - \frac{\gamma - \delta}{4}] \ &< \ \sqrt{\phi(d(gx_n, gx_{n+1})}\ d(gx_n, gx_{n+1}) \\ &\leq \ fx_n < \delta + \frac{\gamma - \delta}{4}, \text{ for all } \ n \geq n_0. \end{aligned}$$

Hence, we get

$$\sqrt{\phi(d(gx_n, gx_{n+1})} < \frac{\gamma + 3\ \delta}{3\ \gamma + \delta}, \quad \text{for all } \ n \geq n_0. \tag{3.9}$$

$$\text{Take } s = \frac{\gamma + 3\ \delta}{3\ \gamma + \delta}, \text{ that is, } s = 1 - \frac{2\ (\gamma - \delta)}{3\ \gamma + \delta} < 1, \ \Big[\text{as } \gamma > \delta\Big]. \tag{3.10}$$

Now from (3.3), (3.9) and (3.10), we have

$$fx_{n+1} \leq \sqrt{\phi(d(gx_n,\ gx_{n+1})}\ fx_n < \left[\frac{\gamma + 3\ \delta}{3\ \gamma + \delta}\right] fx_n = s\, fx_n, \text{ for all } n \geq n_0. \tag{3.11}$$

By repeated application of the above inequality, we have

$$f x_{n_0+k} < s^k\ f x_{n_0}, \quad \text{for any} \quad k \geq 1. \tag{3.12}$$

Since $0 < s < 1$, $s^k\ f x_{n_0} \to 0$ as $k \to \infty$. As $\delta > 0$, there exists a positive integer k_0 such that

$$s^{k_0}\ f x_{n_0} < \delta. \tag{3.13}$$

As $\{f x_n\}$ is a strictly decreasing sequence of positive real numbers converging to $\delta > 0$, we obtain

$$\delta \leq f x_n, \quad \text{for all} \quad n. \tag{3.14}$$

Using (3.12), (3.13) and (3.14), we have

$$\delta \leq f x_{n_0+k_0} < s^{k_0}\ f x_{n_0} < \delta,$$

which is a contradiction. Therefore, the assumption $\gamma > \delta$ is not true. Thus, we have $\gamma = \delta$.

Next we shall show that $\gamma = 0$. If possible, suppose that $\gamma > 0$. As $\liminf_{n \longrightarrow \infty}\ d(gx_n,\ gx_{n+1}) = \gamma$ (by (3.6)), there exists a subsequence $\{d(gx_{n_k},\ gx_{n_k+1})\}_{k=0}^{\infty}$ of $\{d(gx_n,\ gx_{n+1})\}_{n=0}^{\infty}$ such that

$$\lim_{k \longrightarrow \infty} d(gx_{n_k},\ gx_{n_k+1}) = \gamma, \tag{3.15}$$

which, by the property of ϕ, implies that

$$\overline{\lim} \sqrt{\phi(d(gx_{n_k},\ gx_{n_k+1}))} = \limsup_{k \to \infty} \sqrt{\phi(d(gx_{n_k},\ gx_{n_k+1}))} < 1. \tag{3.16}$$

Now, from (3.3), we have

$$f x_{n_k+1} \leq \sqrt{\phi(d(gx_{n_k},\ gx_{n_k+1}))}\ f x_{n_k}.$$

Taking limit supremum on both sides and using (3.4) and (3.16), we have

$$\gamma = \delta \leq \delta\ \limsup_{k \to \infty} \sqrt{\phi(d(gx_{n_k},\ gx_{n_k+1}))} < \delta = \gamma,$$

which is contradiction. Therefore, $\gamma = \delta = 0$ and hence from (3.4) and (3.15), we get

$$\lim_{k\to\infty} fx_{n_k} = 0 \text{ and } \lim_{k\to\infty} d(gx_{n_k},\ gx_{n_k+1}) = 0. \tag{3.17}$$

Let

$$\alpha = \limsup_{k\to\infty} \sqrt{\phi(d(gx_{n_k},\ gx_{n_k+1}))}.$$

By (3.16), we have $\alpha < 1$. Let β be such that $\alpha < \beta < 1$. Using the definition of limit supremum, there exists $k_0 \in \mathbf{N}$ such that

$$\sqrt{\phi(d(gx_{n_k},\ gx_{n_k+1}))} < \beta, \text{ for each } \quad k \geq k_0. \tag{3.18}$$

Now from (3.3) and (3.18), we can say that

$$fx_{n_k+1} < \beta\ fx_{n_k}, \quad \text{for each} \quad k \geq k_0, \tag{3.19}$$

which is same as (2.7) of Theorem 2.1. Then arguing similarly as in proof of Theorem 2.1, we prove that $\{gx_{n_k}\}$ is a Cauchy sequence in X. Since the metric space $(X,\ d)$ is complete, there exists a point $z \in X$ such that

$$gx_{n_k} \to z \text{ as } k \to \infty, \tag{3.20}$$

that is, we have

$$y_{n_k} \to z \text{ as } k \to \infty. \tag{3.21}$$

Here $\{x_{n_k}\}$ and $\{y_{n_k}\}$ are two sequences in X such that $y_{n_k+1} \in Tx_{n_k}$ for all $k \in \mathbf{N}$ and $\lim_{k\to\infty} gx_{n_k} = \lim_{k\to\infty} y_{n_k} = z$, where $z \in X$. Applying the assumptions (ii) and (iii) and using (3.20), (3.21), we have

$$\lim_{k\to\infty} D(gy_{n_k+1},\ Tgx_{n_k}) = 0, \text{ that is, } D(gz,\ Tz) = 0,$$

which implies that $gz \in \overline{Tz} = Tz$, where $\overline{Tz}$ denotes the closure of Tz. Therefore, z is a coincidence point of the pair (g, T), that is, $z \in C(g,\ T)$ and hence $C(g,\ T)$ is nonempty.

In the next theorem we consider $g(X)$ to be closed in the metric space $(X,\ d)$ and the function $fx = D(gx,\ Tx)$ is lower semi-continuous. Due to these considerations we need not require the assumptions (ii) and (iii) which we consider in Theorem 3.1.

Theorem 3.2. Let $(X,\ d)$ be a complete metric space, $T : X \to CB(X)$ be a multivalued mapping and $g : X \to X$. Suppose that (i) $Tx \subseteq g(X)$ for every $x \in X$, (ii) $g(X)$ is a closed subset of X, (iii) for every $x \in X$ there corresponds a $y \in X$ satisfying

$$\sqrt{\phi(d(gx,\ gy))}\, d(gx,\ gy) \leq fx \quad \text{and} \quad fy \leq \phi(d(gx,\ gy))\, d(gx,\ gy),$$

where the function f defined as in Theorem 2.1 is lower semi-continuous and $\phi \in \Phi$. Then $C(g,\ T)$ is nonempty.

Proof. Arguing similarly as in the proof of Theorems 2.2 and 3.1, we have the required proof.

Remark 3.1. Taking g to be the identity function on X, we have Theorem 2.2 in [8].

Example 3.1. Let $X = [0,\ \infty)$ and "d" be the usual metric on X. Let $T : X \to CB(X)$, $g : X \to X$ and $\phi : [0,\ \infty) \to [\frac{25}{144},\ 1)$ be defined as follows:

$$\mathrm{Tx} = \begin{cases} [x + \frac{1}{x+1} - \frac{1}{2},\ 1], & \text{if } 0 \leq x \leq 1 \\ \{x^2\}, & \text{if } \ x > 1, \end{cases} \qquad \mathrm{gx} = \mathrm{x}^2 \text{ and } \phi(\mathrm{t}) = \begin{cases} \frac{25}{144}, & \text{if } \ 0 \leq t \leq 3, \\ \frac{25}{144} + \frac{1}{t}, & \text{if } \ t > 3. \end{cases}$$

Then all the conditions of Theorems 2.1, 2.2, 3.1 and 3.2 are satisfied and here $C(g,\ T) = [1,\ \infty)$.

Conclusion. Example 3.1 shows that our theorems are effective generalizations of the corresponding theorems appearing in [8]. Since [8] is a generalization of the results in [9, 13, 14], the present work also generalizes these works as well. It is to be investigated whether these results are also valid for weaker compatibility conditions. This may be treated as an open problem for future considerations.

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