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Fuzzy *(b, θ)*-separation axioms

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Abstract:

Dutta and Tripathy recently introduced fuzzy (b, θ)-open set in fuzzy topology. The aim of this paper is to introduce fuzzy (b, θ)-separation axioms with the help of fuzzy (b, θ)-open set and to establish some properties by defining fuzzy (b, θ)-neighbourhood and fuzzy (b, θ)-quasi neighbourhood of a fuzzy point.

Keywords: Fuzzy topological spaces; Fuzzy b-open set; Fuzzy (b, θ)-open set; Fuzzy (b, θ)-quasi neighbourhood.

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1. Introduction and preliminaries

Fuzzy set theory was introduced by Zadeh [6]. Since then several mathematicians are applying this notion to develop some new concepts in different fields. Chang [2] introduced the notion of fuzzy topological space and after that several nearly open sets of general topological space have been introduced in fuzzy topological spaces. As a result, Salleh and Waheb [5], Benchalli and Karnel [1] introduced the notion of fuzzy θ-open sets and fuzzy b-open sets respectively. Recently, Dutta and Tripathy [3] defined fuzzy (b, θ)-open sets and established many properties.

In this paper by X, we denote the fuzzy topological space [in short, FTS] (X, τ).

Let X be a non-empty fuzzy set. According to Ming and Ming [4], a fuzzy point is a fuzzy set in X which is zero everywhere except at the point 'a' (say), where it takes the value $0 < \alpha \leq 1$ and it is denoted by a_α. If for a fuzzy set P in X, $a_\alpha \in P$ means $\alpha \leq P(a)$, where $0 < \alpha \leq 1$. Also, if we have $\alpha + P(a) > 1$, then it means that a_α is quasi-coincident with P and can be written as $(a_\alpha)_q P$. Similarly, if for $a \in X$, we have $P(a) + Q(a) > 1$ for the fuzzy sets P and Q of X, then we can write it as $P_q Q$, which means that P is quasi-coincident with Q. Negation of these kind of statements is denoted by $(a_\alpha)_{\overline{q}} P$ or $P_{\overline{q}} Q$.

Again, the fuzzy sets which take the value 0 and 1 are denoted by 0_X and 1_X respectively, where $\lambda_\phi(a) = 0$ and $\lambda_X(a) = 1$, for all $a \in X$ where $\lambda_P : X \to [0, 1]$ is the membership function of a fuzzy set P. Here, $\lambda_P(a)$ is the membership grade of a in P. A fuzzy set P is contained in a fuzzy set Q written as $P \leq Q$ if $\lambda_P \leq \lambda_Q$. Complement of a fuzzy set P is defined by $\lambda'_P = 1 - \lambda_P$. Union and intersection of a collection $\{P_i : i \in I\}$ of fuzzy sets in X can be written as $\bigvee_{i=1} P_i$ and $\bigwedge_{i=1} P_i$ respectively and are denoted by $\lambda_{\bigvee_{i=1} P_i}(a) = \text{Sup } \{\lambda_{P_i}(a) : i \in I\}$ and $\lambda_{\bigwedge_{i=1} P_i}(a) = \text{Inf } \{\lambda_{P_i}(a) : i \in I\}$, for all $a \in X$.

A fuzzy set P in X is said to be fuzzy b-open [1] if $P \leq int(cl(P)) \vee cl(int(P))$. The complement of fuzzy b-open set is called fuzzy b-closed.

According to Dutta and Tripathy [3], a fuzzy point a_α in X is said to be fuzzy (b, θ) -cluster point of a fuzzy set P in X if $Fbcl(Q) \wedge P \neq 0_X$, for every fuzzy b-open set Q of X containing a_α, where $Fbcl(Q)$ denotes the fuzzy b-closure of Q. The set of all fuzzy (b, θ)-cluster points of P is said to be fuzzy (b, θ)-closure of P and it is denoted by $Fbcl_\theta(P)$. A fuzzy P is said to be fuzzy (b, θ)-closed if $P = Fbcl_\theta(P)$. The complement of fuzzy (b, θ)-closed set is fuzzy (b, θ)-open.

Here, we can denote the set of all fuzzy (b, θ)-open (respectively, fuzzy (b, θ)-closed) sets of X by FB_θ-$O(X)$ (respectively FB_θ-$C(X)$).

It was shown in Theorem 3.1 of [3], that $Fbcl_\theta(P) = \bigwedge\{Q : P < Q$ and $Q \in FB_\theta$-$C(X)\}$.

Similarly, the fuzzy (b, θ)-interior can be defined as $Fbint_\theta(P) = \bigvee\{Q : Q < P$ and $Q \in FB_\theta$-$O(X)\}$.

The notion of fuzzy quasi-neighbourhood of a fuzzy point was defined by Ming and Ming [4] in the following manner:

A fuzzy subset P of a FTS X is called fuzzy-quasi neighbourhood of a fuzzy point a_α if there exists a fuzzy open set Q in X such that $(a_\alpha)_q Q \leq P$.

Analogously, we can define the notion of fuzzy (b, θ)-neighbourhood and fuzzy (b, θ)-quasi neighbourhood as given below:

A fuzzy subset P in a FTS X is called

(i) fuzzy (b, θ)-neighbourhood (in short $F(b, \theta)$-nbd) of a fuzzy point a_α if there exists a fuzzy (b, θ)-open set Q in X such that $a_\alpha \in Q \leq P$.

(ii) fuzzy (b, θ)-quasi neighbourhood (in short, $F(b, \theta)$-q-nbd) of a fuzzy point a_α if there exists a fuzzy (b, θ)-open set Q in X such that $(a_\alpha)_q Q \leq P$.

Here, we can denote the set of all $F(b, \theta)$-nbds (respectively, $F(b, \theta)$-q-nbds) of a_α by $FB_\theta N(a_\alpha)$ (respectively, $FB_\theta QN(a_\alpha)$.

One can easily verify that

(i) A fuzzy set P is fuzzy (b, θ)-open if and only if for every fuzzy point a_α such that $(a_\alpha)_q P$, $P \in FB_\theta QN(a_\alpha)$. This is because of $(a_\alpha)_q P \leq P$.

(ii) For a fuzzy set P and for a fuzzy point a_α in X, $a_\alpha \in Fbcl_\theta(P)$ if and only if for every $Q \in FB_\theta QN(a_\alpha)$, we have $Q_q P$.

2. Fuzzy (b,θ)-separation axioms

In this section, we introduce a few definitions and related theorems.

Definition 2.1. A FTS X is said to be fuzzy (b,θ)-T_0 (in short $F(b,\theta)$-T_0) if for every pair (a_α, b_β) of fuzzy points in X such that $a_\alpha \neq b_\beta$, the following conditions are satisfied:

(i) when $a \neq b$, then either $P \in FB_\theta N(a_\alpha)$ such that $P_{\overline{q}}\ b_\beta$ or $Q \in FB_\theta N(b_\beta)$ such that $Q_{\overline{q}}\ a_\alpha$,

(ii) when $a = b$ and $\alpha < \beta$ (say), then there exists $Q \in FB_\theta QN(b_\beta)$ such that $Q_{\overline{q}} a_\alpha$.

Theorem 2.1. A FTS X is $F(b,\theta)$-T_0 if and only if for every pair (a_α, b_β) of fuzzy points such that $a_\alpha \neq b_\beta$, either $a_\alpha \notin Fbcl_\theta(b_\beta)$ or $b_\beta \notin Fbcl_\theta(a_\alpha)$.

Proof. Let X is $F(b,\theta)$-T_0-space and (a_α, b_β) be the pair of fuzzy points in X such that $a_\alpha \neq b_\beta$.

If $a \neq b$, then there exists a $P \in FB_\theta N(a_\alpha)$ such that $P_{\overline{q}}\ b_\beta$ or there exists a $Q \in FB_\theta N(b_\beta)$ such that $Q_{\overline{q}} a_\alpha$. Suppose that there exists a $P \in FB_\theta N(a_\alpha)$ such that $P_{\overline{q}} b_\beta$. Then, $P \in FB_\theta QN(a_\alpha)$ such that $P_{\overline{q}} b_\beta$. Hence, $a_\alpha \notin Fbcl_\theta(b_\beta)$.

Again, if $a = b$ and $\alpha < \beta$ (say), then there exists $Q \in FB_\theta QN(b_\beta)$ such that $Q_{\overline{q}} a_\alpha$ and so, we have $b_\beta \notin Fbcl_\theta(a_\alpha)$.

Conversely, let (a_α, b_β) be the pair of fuzzy points in X such that $a_\alpha \neq b_\beta$. Suppose if possible, $a_\alpha \notin Fbcl_\theta(b_\beta)$. If $a \neq b$, since $a_\alpha \notin Fbcl_\theta(b_\beta)$, so $a_\alpha \in (Fbcl_\theta(b_\beta))'$ and hence $(Fbcl_\theta(b_\beta))'(a) = \alpha$. Then, $(Fbcl_\theta(b_\beta))' \in FB_\theta N(a_\alpha)$ such that $(Fbcl_\theta(b_\beta))'_{\overline{q}}\ b_\beta$ since $b_\beta \notin (Fbcl_\theta(b_\beta))'$.

Again, if $a = b$ and $\alpha > \beta$, then there exists a $P \in FB_\theta QN(a_\alpha)$ such that $P\overline{q}\, b_\beta$.

Definition 2.2. A FTS X is said to be fuzzy (b,θ)-T_1 (in short $F(b,\theta)$-T_1) if for every pair (a_α, b_β) of fuzzy points in X such that $a_\alpha \neq b_\beta$, the following conditions are satisfied:

(i) when $a \neq b$, then there exists $P \in FB_\theta N(a_\alpha)$ and $Q \in FB_\theta N(b_\beta)$ such that $P\overline{q}\, b_\beta$ and $Q\overline{q}\, a_\alpha$,

(ii) when $a = b$ and $\alpha < \beta$ (say), then there exists $Q \in FB_\theta QN(b_\beta)$ such that $Q\overline{q}a_\alpha$.

Theorem 2.2. A FTS X is $F(b,\theta)$-T_1 if and only if every fuzzy point a_α is fuzzy (b,θ)-closed in X.

Proof. Let X be $F(b,\theta)$-T_1-space and (a_α, b_β) be the pair of fuzzy points in X such $a_\alpha \neq b_\beta$.

If $a \neq b$, then there exists $P, Q \in FB_\theta$-$O(X)$ and $P \in FB_\theta N(a_\alpha)$, $Q \in FB_\theta N(b_\beta)$, such that $P\overline{q}\, b_\beta$ and $Q\overline{q}\, a_\alpha$. Then $a_\alpha \in Q'$. Since $Q' \in FB_\theta$-$C(X)$, so $Fbcl_\theta(a_\alpha) \leq Q'$, which is equivalent to $Q\overline{q}(Fbcl_\theta(a_\alpha))$. Thus, we have $Fbcl_\theta(a_\alpha) \leq a_\alpha$. Consequently, $a_\alpha = Fbcl_\theta(a_\alpha)$. Hence, every fuzzy point a_α is fuzzy (b,θ)-closed in X.

Again, if $a = b$, then proof is analogous to the above part.

Conversely, let a_α and b_β are two fuzzy points in X such that $a_\alpha \neq b_\beta$. If $a \neq b$, since a_α and b_β are fuzzy (b,θ)-closed in X, so $(a_\alpha)'$ and $(b_\beta)'$ are fuzzy (b,θ)-open. Then, we have $(a_\alpha)' \in FB_\theta N(b_\beta)$ and $(b_\beta)' \in FB_\theta N(a_\alpha)$ such that $(a_\alpha)'_{\overline{q}}\, (a_\alpha)$ and $(b_\beta)'_{\overline{q}}\, (b_\beta)$.
Again, if $a = b$ and $\alpha < \beta$ (say), it is obvious that $(a_\alpha)' \in FB_\theta QN(b_\beta)$ such that $(a_\alpha)'_{\overline{q}}\, (a_\alpha)$.

Definition 2.3. A FTS X is said to be fuzzy (b,θ)-T_2 (in short $F(b,\theta)$-T_2) if for every pair (a_α, b_β) of fuzzy points in X such that $a_\alpha \neq b_\beta$, the following conditions are satisfied:

(i) when $a \neq b$, then there exists $P \in FB_\theta N(a_\alpha)$ and $Q \in FB_\theta N(b_\beta)$ such that $P\overline{q}\, Q$.

(ii) when $a = b$ and $\alpha < \beta$ (say), then there exists $P \in FB_\theta N(a_\alpha)$ and $Q \in FB_\theta QN(b_\beta)$ such that $P \overline{q}\, Q$.

Theorem 2.3. A FTS X is $F(b,\theta)$-T_2 if and only if every fuzzy point a_α in X, $a_\alpha = \bigwedge\{Fbcl_\theta(Q) : Q \in FB_\theta N(a_\alpha)\}$ and for any $a, b \in X$ with $a \neq b$, there exists a $P \in FB_\theta N(a_1)$ such that $b \notin (Fbcl_\theta(P))_0$, where $(Fbcl_\theta(P))_0$ is support of $Fbcl_\theta(P)$.

Proof. Let X is $F(b,\theta)$-T_2-space and a_α, b_β are the fuzzy points in X such that $b_\beta \neq a_\alpha$.

If $a \neq b$, then there exists $P, Q \in FB_\theta$-$O(X)$ and $b_\beta \in P$, $a_\alpha \in Q$ such that $P \overline{q}\, Q$. Then, we have $Q \in FB_\theta N(a_\alpha)$ and $P \in FB_\theta QN(b_\beta)$ such that $P \overline{q}\, Q$. Hence, $b_\beta \notin Fbcl_\theta(Q)$.

Again, if $a = b$ and $\alpha < \beta$, then there exists $P \in FB_\theta QN(b_\beta)$ and $Q \in FB_\theta N(a_\alpha)$ such that $P \overline{q}\, Q$. Then $b_\beta \notin Fbcl_\theta(Q)$.

Finally, for any two distinct points $a_1, b_1 \in X$, since X is $F(b,\theta)$-T_2, there exists $P, Q \in FB_\theta$-$O(X)$ such that $a_1 \in P, b_1 \in Q$ and $P \overline{q}\, Q$. Then, we have $Q'(b) = 0$ and $P \leq Q'$. Since, $Q' \in FB_\theta$-$C(X)$, $Fbcl_\theta(P) \leq Q'$. Thus, we have $(Fbcl_\theta(P))(b) = 0$. That is $b \notin (Fbcl_\theta(P))_0$.

Conversely, let a_α and b_β be two fuzzy points in X such that $a_\alpha \neq b_\beta$.

If $a \neq b$, first we suppose that atleast one of α and β is less than 1, say $0 < \alpha < 1$. Then, there exists a positive real number μ with $0 < \alpha + \mu < 1$. By given hypothesis, there exists $P \in FB_\theta N(b_\beta)$ such that $a_\mu \notin Fbcl_\theta(P)$. Then, there exists $Q \in FB_\theta$-$O(X)$ such that $(a_\mu)_q Q$ and $Q \bar{q} P$. Since, $(a_\mu)_q Q$, $Q(a) > 1 - \mu > \alpha$ and hence, $Q \in FB_\theta N(a_\alpha)$ such that $P \bar{q} Q$.

Again, if $\alpha = \beta = 1$, by hypothesis, there exists $P \in FB_\theta N(a_1)$ such that $(Fbcl_\theta(P))(b_1) = 0$. Then, we have $Q = (Fbcl_\theta(P))' \in FB_\theta N(b_1)$ such that $P \bar{q} Q$.

If $a \neq b$ and $\alpha < \beta$ (say), then there exists $P \in FB_\theta N(a_\alpha)$ such that $b_\beta \notin Fbcl_\theta(P)$. Hence, there exists $Q \in FB_\theta QN(b_\beta)$ such that $P \bar{q} Q$. Hence, X is $F(b,\theta)$-T_2.

Lemma 2.1. Let $P, Q \in FB_\theta\text{-}O(X)$ in a FTS X. If $P_{\bar{q}}Q$, then $Fbcl_\theta(P)_{\bar{q}}Q$.

Definition 2.4. A FTS X is said to be fuzzy (b,θ)-regular (in short, $F(b,\theta)$-regular) if for every fuzzy point a_α in X and for every $P \in FB_\theta\text{-}O(X)$ with $P \in FB_\theta QN(a_\alpha)$, there exists $Q \in FB_\theta\text{-}O(X)$ with $Q \in FB_\theta QN(a_\alpha)$ such that $Fbcl_\theta(Q) \leq P$.

Theorem 2.4. Let X be a FTS. Then, the following statements are equivalent:

(i) X is $F(b,\theta)$-regular,

(ii) for every fuzzy point $a_\alpha \in X$ and for every $A \in FB_\theta\text{-}C(X)$ with $a_\alpha \notin A$, there exists a $P \in FB_\theta\text{-}O(X)$ such that $a_\alpha \notin Fbcl_\theta(P)$ and $A \leq P$,

(iii) for every fuzzy point $a_\alpha \in X$ and for every $A \in FB_\theta\text{-}C(X)$ with $a_\alpha \notin A$, there exist $P, Q \in FB_\theta\text{-}O(X)$ such that $P \in FB_\theta QN(a_\alpha)$, $A \leq Q$ and $P_{\bar{q}}Q$,

(iv) for a fuzzy subset U in X and for $A \in FB_\theta\text{-}C(X)$ with $U \not\leq A$, there exist $P, Q \in FB_\theta\text{-}O(X)$ such that U_qP, $A \leq Q$ and $P_{\bar{q}}Q$,

(v) for a fuzzy subset U in X and for $P \in FB_\theta\text{-}O(X)$ with U_qP, there exist $Q \in FB_\theta\text{-}O(X)$ such that $U_qQ \leq Fbcl_\theta(Q) \leq P$.

Proof. (i) $\Rightarrow$ (ii) Let a_α be a fuzzy point in X and let $A \in FB_\theta\text{-}C(X)$ such that $a_\alpha \notin A$. Then we have $A' \in FB_\theta QN(a_\alpha)$ and $A' \in FB_\theta\text{-}O(X)$. Since X is $F(b,\theta)$-regular, so there exists $Q \in FB_\theta\text{-}O(X)$ with $Q \in FB_\theta QN(a_\alpha)$ such that $Fbcl_\theta(Q) \leq A'$.

Let $P = (Fbcl_\theta(Q))'$. Then, we have $P \in FB_\theta\text{-}O(X)$ and $Fbint_\theta(Fbcl_\theta(Q)) \in FB_\theta QN(a_\alpha)$.
Consequently, $a_\alpha \notin (Fbint_\theta(Fbcl_\theta(Q)))' = (Fbcl_\theta(Fbcl_\theta(Q))') = Fbcl_\theta(P)$.
Also, $A \leq (Fbcl_\theta(Q))' = P$.

(ii) $\Rightarrow$ (iii) Let a_α be fuzzy point in X and let $A \in FB_\theta\text{-}C(X)$ such that $a_\alpha \notin A$. By (ii), there exists $P \in FB_\theta\text{-}O(X)$ such that $a_\alpha \notin Fbcl_\theta(P)$ and $A \leq P$. Hence, $(Fbcl_\theta(P))' \in FB_\theta QN(a_\alpha)$ and $(Fbcl_\theta(P))'_{\bar{q}}P$, where

$(Fbcl_\theta(P))' \in FB_\theta\text{-}O(X)$.

(iii) $\Rightarrow$ (iv) Let U be any fuzzy set in X and let $A \in FB_\theta\text{-}C(X)$ such that $U \not\leq A$. Then, we have at least one fuzzy point $a_\alpha \in U$ such that $a_\alpha \notin A$. By (iii), there exist $P, Q \in FB_\theta\text{-}O(X)$ such that $P \in FB_\theta QN(a_\alpha)$, $A \leq Q$ and $P_{\bar{q}}Q$. Since $a_\alpha \in U$, thus we have U_qP

(iv) $\Rightarrow$ (v) Let U be any fuzzy set in X and $P \in FB_\theta\text{-}O(X)$. Then, $P' \in FB_\theta\text{-}C(X)$. Now, $UqP \Rightarrow U \not\leq P'$. By (iv), there exist $Q, R \in FB_\theta\text{-}O(X)$ such that U_qQ, $P' \leq R$ and $Q_{\bar{q}}R$. Then, by Lemma 2.1., we have $Fbcl_\theta(Q)_{\bar{q}}R$. Hence, $U_qQ \leq Fbcl_\theta(Q) \leq R' \leq P$.

(v) $\Rightarrow$ (i) It is obvious.

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