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Note on extended hypergeometric function

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Abstract:

In this paper, we present an extension of the classical hypergeometric functions using extended gamma function due to Jumarie and obtained some properties.

Keywords: Gamma function; Pochhammer symbols; Hypergeometric functions; Integral transforms; Fractional calculus.

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1. Introduction

In 2006, Jumarie [4] gave an interesting extension of the gamma function as

$$\Gamma_\alpha(x) = (\alpha!)^{-1} \int_0^\infty E_\alpha(-t^\alpha) t^{(x-1)\alpha} (dt)^\alpha, \quad 0 < \alpha < 1, \tag{1.1}$$

where $E_\alpha(u)$ is the Mittag-Leffler function (1.2)

$$E_\alpha(u) = \sum_{k=0}^\infty \frac{u^k}{(\alpha k)!}, \quad \alpha \in \mathbf{C}. \tag{1.2}$$

Here $(\alpha!)$ denotes $\Gamma(\alpha+1)$. We use this notation throughout the paper.

The integral involved in (1.1) is the fractional integral with respect to $(dt)^\alpha$ given by Jumarie (see, for details [3], [4], [5], [7], [8], [9], [10], [11]). This modified integral has the following connection via fractional calculus, with the usual integral [5]

$$\int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi. \tag{1.3}$$

We define the extension of Gauss hypergeometric function as follows

$${}_2^{\alpha}F_1(a,b;c;z) = \sum_{n=0}^\infty \frac{(a)_{n,\alpha}\,(b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.4}$$

where $0 < \alpha \leq 1,\ a, b, c \in \mathbf{C}$ with c is never zero or negative integer.

Here $(\lambda)_{n,\alpha}$ is the extended Pochhammer symbol defined by

$$(\lambda)_{n,\alpha} = \frac{\Gamma_\alpha(\lambda+v)}{\Gamma_\alpha(\lambda)}, \tag{1.5}$$

where $\lambda \in \mathbf{C},\ n \in \mathbf{N} \cup \{0\},\ 0 < \alpha \leq 1.$

In the definition (1.5), $\Gamma_\alpha(\lambda)$ is the extended gamma function given by (1.1).

2. Generalized integral transforms

In 2012, Virchenko [16] introduced the following generalizations of classical integral transforms.

Definition: The generalized Laplace integral transform

$$L_{\gamma_1,\gamma_2}\{f(x);y\} = \int_0^\infty x^{\gamma_2}\, e^{-(xy)^{\gamma_1}}\, f(x)\,dx \tag{2.1}$$

and

$$\tilde{L}_{\gamma_1,\gamma_2,\gamma}\{f(x);y\} = \int_0^\infty x^{\gamma_2}\, e^{-(xy)^{\gamma_1}}\, {}_1\Phi_1^{\tau,\beta}\,(\alpha;c;-b(xy)^{\gamma\gamma_1})\, f(x)\,dx, \tag{2.2}$$

where $x > 0$, $\gamma \in C$, $\gamma_1 > 0$, $\gamma_2 > 0$, $b \geq 0$, ${}_1\Phi_1^{\tau,\beta}$ is the (τ,β)-generalized hypergeometric function [15]

$$\begin{gathered}{}_1\Phi_1^{\tau,\beta}\,(\alpha;c;-b(xy)^{\gamma\gamma_1}) = \frac{1}{B(a,c-a)}\int_0^1 t^{a-1}\,(1-t)^{c-a-1}\,{}_1\Psi_1\left[\begin{array}{c|c}(c;\tau) & \\ (c;\beta) & zt^{\tau}\end{array}\right]dt \\ (\text{Re (c)} > \text{Re (a)} > 0; \{\,\tau,\,\beta\} \subset R; \tau > 0; \tau-\beta < 1)\end{gathered} \tag{2.3}$$

and ${}_p\Psi_q$ is the generalized hypergeometric Wright function

$${}_p\Psi_q\left[\begin{array}{c|c}(a_1;\alpha_1)\ldots(a_p;\alpha_p) & \\ (b_1;\beta_1)\ldots(b_q;\beta_q) & z\end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=0}^{p}\Gamma(a_i+n\,\alpha_i)}{\prod_{j=0}^{q}\Gamma(b_j+n\,\beta_j)}\frac{z^n}{n!}, \tag{2.4}$$

$$(z \in C;\, a_i, b_j \in C;\, \{\alpha_i,\,\beta_j\} \subset R;\, a_i, b_j \neq 0;\, 1 + \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i \geq 0.$$

Definition: The generalized Stieltjes integral transform

$$\begin{aligned}&P_1^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{f(u);x\} = \tilde{P}_1\{f(u);x\} \\ &= \frac{\Gamma(c)}{\Gamma(a_1)\,\Gamma(a_2)}\int_0^\infty \frac{u^{\gamma_2}\,f(u)}{(x^{\gamma_1}+u^{\gamma_1})^{\gamma_3}}\,{}_2\Psi_1\left[\begin{array}{c|c}(a_1;\tau);(a_2;\gamma) & \\ (c;\,\beta) & -b\Big(\frac{u^{\gamma_1}}{x^{\gamma_1}+u^{\gamma_1}}\Big)^{\gamma_4}\end{array}\right]du\end{aligned} \tag{2.5}$$

and

$$P_2^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{f(u);x\} = \tilde{P}_2\{f(u);x\}$$

(2.6)
$$= \frac{\Gamma(c)}{\Gamma(a_1)\,\Gamma(a_2)} \int_0^\infty \frac{u^{\gamma_2}\, f(u)}{(x^{\gamma_1}+u^{\gamma_1})^{\gamma_3}}\, {}_2\Psi_1\left[\begin{array}{c}(a_1;\tau);(a_2;\gamma)\\ (c;\,\beta)\end{array}\middle| -b\Big(\frac{x^{\gamma_1}}{x^{\gamma_1}+u^{\gamma_1}}\Big)^{\gamma_4}\right] du,$$

where $Re(a_1) > 0$, $Re(a_2) > 0$, $Re(c) > 0$, $\gamma_i > 0$; $\{\tau, \beta\} \subset R$; $\tau > 0$; $\tau - \beta < 1$; $b \geq 0$ and ${}_2\Psi_1$ is defined by (2.3).

Theorem 1. *If* $0 < x < 1$, $\gamma \in C$, $\gamma_1 > 0$, $\gamma_2 > 0$, $b \geq 0$; $\alpha_1, \alpha_2, \rho \in C; \rho \notin \{0, -1, -2, ...\}; 0 < \alpha \leq 1$, *then*

(2.7)
$$L_{\gamma_1,\gamma_2}\{{}_2^{\alpha}F_1(\alpha_1,\alpha_2;\rho;x);y\} = \frac{1}{\gamma_1} y^{-\gamma_2-1}\,\sigma_1(y),$$

where

$$\sigma_1(y) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,\alpha}\,(\alpha_2)_n}{(\rho)_n} \Gamma\left(\frac{\gamma_2+n+1}{\gamma_1}\right) \frac{y^{-n}}{n!}.$$

In particular when $\gamma_1 = 1$, *then*

$$L_{1,\gamma_2}\{{}_2^{\alpha}F_1(\alpha_1,\alpha_2;\rho;x);y\} = y^{-\gamma_2-1}\Gamma(\gamma_2+1)\,{}_2^{\alpha}F_1\left(\alpha_1,\alpha_2,\gamma_2+1;\rho;\frac{1}{y}\right).$$

(2.8)

and

(2.9)
$$\tilde{L}_{\gamma_1,\gamma_2,\gamma}\{{}_2^{\alpha}F_1(\alpha_1,\alpha_2;\rho;x);y\} = \frac{1}{\gamma_1}\frac{\Gamma(c)}{\Gamma(a)} y^{-\gamma_2-1}\,\sigma_2(b,x),$$

where

$$\sigma_2(b,x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\,(\alpha_2)_n}{(\rho)_n}\, {}_2\Psi_1\left[\begin{array}{c}(a;\tau);\left(\frac{\gamma_2+n+1}{\gamma_1};\gamma\right)\\ (c;\,\beta)\end{array}\middle| -b\right] \frac{y^{-n}}{n!}.$$

Proof. From the linearity property of generalized Laplace transform (2.1) (see, [16]), one can write the following

$$L_{\gamma_1,\gamma_2}\{{}_2^{\alpha}F_1(\alpha_1,\alpha_2;\rho;x);y\} = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,\alpha}\,(\alpha_2)_n}{(\rho)_n} \left[\int_0^\infty e^{-(xy)^{\gamma_1}} x^{\gamma_2+n}\, dx\right] \frac{1}{n!}.$$

(2.10)

Let us evaluate the integral involved in (2.10). By taking the substitutions $xy = t$ and $t^{\gamma_1} = u$, we can get

$$\int_0^\infty e^{-(xy)^{\gamma_1}} x^{\gamma_2+n}\, dx$$
$$= y^{-\gamma_2-k-1} \int_0^\infty e^{-t^{\gamma_1}} t^{\gamma_2+k}\, dt$$
$$= \frac{1}{\gamma_1} y^{-\gamma_2-n-1} \int_0^\infty e^{-u}\, u^{\left(\frac{\gamma_2+n+1}{\gamma_1}-1\right)}\, du = \frac{1}{\gamma_1} y^{-\gamma_2-n-1}\, \Gamma\left(\frac{\gamma_2+n+1}{\gamma_1}\right),$$

using this in (2.10), one can get the formula (2.7).

Assertion (2.9) can be derived by using the formula [16]

$$\tilde{L}_{\gamma_1,\gamma_2,\gamma}\{x^k; y\} = \frac{1}{\gamma_1} \frac{\Gamma(c)}{\Gamma(a)} y^{-\gamma_2-k-1} {}_2\Psi_1 \left[\begin{array}{c} (a;\, \tau); \left(\frac{\gamma_2+n+1}{\gamma_1}; \gamma\right) \\ (c;\, \beta) \end{array} \middle| -b \right], \tag{2.11}$$

and applying similar method leads to the proof of (2.7).

Theorem 2. *If* $Re(a_1) > 0$, $Re(a_2) > 0$, $Re(c) > 0$, $\gamma_i > 0$; $\{\tau, \beta\} \subset R$; $\tau > 0$; $\tau - \beta < 1$; $b \geq 0$; $\alpha_1, \alpha_2,\, \rho \in C; \rho \notin \{0, -1, -2, ...\}; 0 < \alpha \leq 1$, *then*

$$P_1^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{{}_2^{\alpha}F_1(\alpha_1, \alpha_2; \rho; x); y\} = \frac{1}{\gamma_1} \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)} y^{-\gamma_1\,\gamma_3+\gamma_2+1}\, \sigma_3(b, x), \tag{2.12}$$

where

$$\sigma_3(b,x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\, (\alpha_2)_n\, \Gamma\left(\gamma_3 - \frac{\gamma_2+n+1}{\gamma_1}\right)}{(\rho)_n}\, {}_3\Psi_2 \left[\begin{array}{l} (a_1; \tau); (a_2; \gamma) \left(\frac{\gamma_2+n+1}{\gamma_1}; \gamma_4\right) \\ (c;\, \beta); (\gamma_3; \gamma_4) \end{array} \middle| -b \right] \frac{y^n}{n!};$$

and

$$P_1^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\{{}_2^{\alpha}F_1(\alpha_1, \alpha_2; \rho; x); y\} = \frac{1}{\gamma_1} \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)} y^{-\gamma_1\,\gamma_3+\gamma_2+1}\, \sigma_4(b, x), \tag{2.13}$$

where

$$\sigma_4(b,x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\, (\alpha_2)_n\, \Gamma\left(\frac{\gamma_2+n+1}{\gamma_1}\right)}{(\rho)_n}\, {}_3\Psi_2 \left[\begin{array}{l} (a_1; \tau); (a_2; \gamma) \left(\gamma_3 - \frac{\gamma_2+n+1}{\gamma_1}; \gamma_4\right) \\ (c;\, \beta); (\gamma_3; \gamma_4) \end{array} \middle| -b \right] \frac{y^n}{n!}.$$

Proof. From the definition (2.5), we have

$$P_1^{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\quad\{{}_2^{\alpha}F_1(\alpha_1,\alpha_2;\rho;x);y\} = \frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)}\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n}{(\rho)_n}$$
$$\times\int_0^{\infty}\frac{x^{\gamma_2+n}}{(y^{\gamma_1}+x^{\gamma_1})^{\gamma_3}}\ {}_2\Psi_1\left[\begin{array}{c}(a_1;\tau);(a_2;\gamma)\\(c;\beta)\end{array}\middle|-b\Big(\frac{x^{\gamma_1}}{y^{\gamma_1}+x^{\gamma_1}}\Big)^{\gamma_4}\right]dx\frac{1}{n!}. \tag{2.14}$$

From (2.3) and changing the variable from $\frac{x}{y}=t$ and then $t^{\gamma_1}=u$, the integral appeared in (2.14) becomes

$$\int_0^{\infty}\frac{x^{\gamma_2+n}}{(y^{\gamma_1}+x^{\gamma_1})^{\gamma_3}}\ {}_2\Psi_1\left[\begin{array}{c}(a_1;\tau);(a_2;\gamma)\\(c;\beta)\end{array}\middle|-b\Big(\frac{x^{\gamma_1}}{y^{\gamma_1}+x^{\gamma_1}}\Big)^{\gamma_4}\right]dx$$
$$=\frac{y^{-\gamma_1\gamma_3+\gamma_2+n+1}}{\gamma_1}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\tau k)\Gamma(a_2+\gamma k)}{\Gamma(c+\beta k)}\int_0^{\infty}u^{\left(\frac{\gamma_2+n+1}{\gamma_1}+\gamma_4k-1\right)}(1+u)^{-\gamma_3-k\gamma_4}du\,\frac{(-b)^k}{k!}$$
$$=\frac{y^{-\gamma_1\gamma_3+\gamma_2+n+1}}{\gamma_1}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+\tau k)\Gamma(a_2+\gamma k)\,\Gamma\Big(\frac{\gamma_2+n+1}{\gamma_1}+\gamma_4k-1\Big)\,\Gamma\Big(\gamma_3-\frac{\gamma_2+n+1}{\gamma_1}\Big)}{\Gamma(c+\beta k)\,\Gamma(\gamma_3+\gamma_4k)}\frac{(-b)^k}{k!}.$$

Putting this in (2.14), we can reach the assertion (2.12).
Using the similar method, one can derive formula (2.13).

3. Generalized fractional operators and ${}_p^{\alpha}F_q(a,b;c;z)$

The generalized fractional integral operators involving the Appell's function due to Saigo and Maeda [13] are defined by

$$\left(I_{0+}^{\delta,\delta',\beta,\beta',\gamma}f\right)(x)$$
$$=\frac{x^{-\delta}}{\Gamma(\gamma)}\int_0^{x}(x-t)^{\gamma-1}t^{-\delta'}F_3\left(\delta,\delta',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\,dt \tag{3.1}$$
$$(\delta,\delta',\beta,\beta',\gamma\in\mathbf{C}\ \mathit{and}\ x>0;\ Re(\gamma)>0)$$

and

$$\left(I_{0-}^{\delta,\delta',\beta,\beta',\gamma}f\right)(x)$$
$$=\frac{x^{-\delta}}{\Gamma(\gamma)}\int_x^{\infty}(t-x)^{\gamma-1}t^{-\delta'}F_3\left(\delta,\delta',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,dt \tag{3.2}$$
$$(\delta,\delta',\beta,\beta',\gamma\in\mathbf{C}\ \mathit{and}\ x>0;\ Re(\gamma)>0).$$

and fractional derivative operators are defined by

$$\left(D_{0+}^{\delta,\delta',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\delta,-\delta',-\beta,-\beta',\gamma} f\right)(x) \tag{3.3}$$

and

$$\left(D_{0-}^{\delta,\delta',\beta,\beta',\gamma} f\right)(x) = \left(I_{0-}^{-\delta,-\delta',-\beta,-\beta',\gamma} f\right)(x), \tag{3.4}$$

where F_3 is the third Appell function (also known as the Horn function) [14]

$$F_3\left(\delta,\delta',\beta,\beta';\gamma;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(\delta)_m\,(\delta')_n\,(\beta)_m\,(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \quad \max\{|x|,|y|\} < 1. \tag{3.5}$$

The formula of generalized fractional integrals (3.1) and (3.2), and derivatives (3.3) and (3.4) of the power function are given below

$$\begin{aligned}&\left(I_{0+}^{\delta,\delta',\beta,\beta',\gamma} z^{\rho-1}\right)(x)\\ &= \frac{\Gamma(\rho)\,\Gamma(\rho+\gamma-\delta-\delta'-\beta)\,\Gamma(\rho+\beta'-\delta')}{\Gamma(\rho+\beta')\,\Gamma(\rho+\gamma-\delta-\delta')\,\Gamma(\rho+\gamma-\delta'-\beta)}\, x^{\rho+\gamma-\delta-\delta'-1}\\ &(Re(\gamma) > 0,\ Re(\rho) > \max\{0, Re(\delta+\delta'+\beta-\gamma),\ (\delta'-\beta')\},\ x > 0)\end{aligned} \tag{3.6}$$

$$\begin{aligned}&\left(I_{0-}^{\delta,\delta',\beta,\beta',\gamma} z^{\rho-1}\right)(x)\\ &= \frac{\Gamma(1-\rho-\gamma+\delta+\delta')\,\Gamma(1-\rho+\delta+\beta'-\gamma)\,\Gamma(1-\rho+\beta)}{\Gamma(1-\rho)\,\Gamma(1-\rho+\delta+\delta'+\beta+\beta'-\gamma)\,\Gamma(1-\rho+\delta-\beta)}\, x^{\rho+\gamma-\delta-\delta'-1}\\ &(Re(\gamma) > 0,\ Re(\rho) < 1+\max\{0, Re(-\beta),\ Re(\delta+\delta'-\gamma),\ Re(\delta+\beta'-\gamma)\},\ x > 0)\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\left(D_{0+}^{\delta,\delta',\beta,\beta',\gamma} z^{\rho-1}\right)(x)\\ &= \frac{\Gamma(\rho)\,\Gamma(\rho+\delta-\beta)\,\Gamma(\rho+\delta+\delta'+\beta'-\gamma)}{\Gamma(\rho-\beta)\,\Gamma(\rho+\delta-\delta'-\gamma)\,\Gamma(\rho+\delta+\beta'-\gamma)} x^{\rho+\delta+\delta'-\gamma-1}\\ &(Re(\gamma) > 0,\ Re(\rho) > \max\{0, Re(-\delta+\beta),\ Re(-\delta-\delta'-\beta'+\gamma)\},\ x > 0)\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\left(D_{0-}^{\delta,\delta',\beta,\beta',\gamma} z^{-\rho}\right)(x)\\ &= \frac{\Gamma(-\delta-\delta'+\gamma+\rho)\,\Gamma(-\delta'-\beta+\gamma+\rho)\,\Gamma(\beta'+\rho)}{\Gamma(\rho)\,\Gamma(-\delta-\delta'-\beta+\gamma+\rho)\,\Gamma(-\delta'+\beta'+\rho)} x^{\delta+\delta'-\gamma-\rho}\end{aligned} \tag{3.9}$$

$$(Re(\gamma) > 0,\ Re(\rho) > \max\{Re(-\beta'),\ Re(\delta'+\beta-\gamma)+[Re(\gamma)]+1\},\ x > 0) \tag{3.10}$$

Theorem 3. *If* $x > 0$, $0 < \alpha \le 1$, $\{\delta, \delta', \beta, \beta', \gamma, a_i, b_j\} \subset C$, $i = 1, ..., p$, $j = 1, ..., q$, $Re(\gamma) > 0$, $1 + \min\{Re(\gamma - \delta - \delta' - \beta), Re(\gamma - \delta - \delta'), Re(\beta')\} > 0$, *then*

$$\left(I_{0+}^{\delta,\delta',\beta,\beta',\gamma}\left[{}_p^{\alpha}F_q(z)\right]\right)(x) = x^{\gamma-\delta-\delta'}\frac{\Gamma(\gamma-\delta'-\beta+1)\,\Gamma(\gamma-\delta-\delta'+1)\,\Gamma(\beta'+1)}{\Gamma(\gamma-\delta-\delta'-\beta+1)\,\Gamma(\beta'-\delta'+1)}$$
$$ {}_{p+3}^{\ \alpha}F_{q+3}\left[\begin{array}{c} 1, \gamma-\delta-\delta'-\beta+1, \beta'-\delta'+1, a_1, ..., a_p; \\ \beta'+1, \gamma-\delta'-\beta+1, \gamma-\delta-\delta'+1, b_1, ..., b_q; \end{array} x\right], \tag{3.11}$$

and if $x > 0$, $0 < \alpha \le 1$, $\{\delta, \delta', \beta, \beta', \gamma, a_i, b_j\} \subset C$, $i = 1, ..., p$, $j = 1, ..., q$, $Re(\gamma) > 0$, $1 + \min\{Re(\delta - \delta' - \gamma), Re(\delta + \beta' - \gamma), Re(-\beta)\} > 0$, *then*

$$\left(I_{0-}^{\delta,\delta',\beta,\beta',\gamma}\left[\frac{1}{z}{}_p^{\alpha}F_q\left(\frac{1}{z}\right)\right]\right)(x) = \frac{\Gamma(1-\gamma+\delta+\delta')\,\Gamma(1+\delta+\beta'-\gamma)\,\Gamma(1-\beta)}{\Gamma(1+\delta+\delta'+\beta+\beta'-\gamma)\,\Gamma(1+\delta-\beta)}$$
$$x^{\gamma-\delta-\delta'-1}\,{}_{p+3}^{\ \alpha}F_{q+3}\left[\begin{array}{c} 1-\gamma+\delta+\delta', 1+\delta+\beta'-\gamma, 1-\beta, a_1, ..., a_p; \\ 1+\delta+\delta'+\beta+\beta'-\gamma, 1+\delta-\beta, 1, b_1, ..., b_q; \end{array} \frac{1}{x}\right]. \tag{3.12}$$

Proof. The hypergeometric function with p numerator and q denominator parameters is defined by [1]

$$ {}_pF_q\left[\begin{array}{c} a_1, ...a_p; \\ b_1, ...b_q; \end{array} z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}\frac{z^n}{n!}. \tag{3.13}$$

From (3.13) and (3.6), we can write

$$\left(I_{0-}^{\delta,\delta',\beta,\beta',\gamma}\left[{}_p^{\alpha}F_q(z)\right]\right)(x) = \sum_{n=0}^{\infty}\frac{(a_1)_{n,\alpha}(a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n\, n!}$$
$$\cdot\left(\frac{\Gamma(\gamma-\delta-\delta'-\beta+n+1)\,\Gamma(\beta'-\delta'+n+1)\Gamma(n+1)}{\Gamma(\gamma-\delta-\delta'+n+1)\,\Gamma(\gamma-\delta'-\beta+n+1)\,\Gamma(\beta'+n+1)}x^{\gamma-\delta-\delta'+n}\right)$$
$$= x^{\gamma-\delta-\delta'}\frac{\Gamma(\gamma-\delta'-\beta+1)\,\Gamma(\gamma-\delta-\delta'+1)\,\Gamma(\beta'+1)}{\Gamma(\gamma-\delta-\delta'-\beta+1)\,\Gamma(\beta'-\delta'+1)}$$
$$\sum_{n=0}^{\infty}\frac{(1)_n(\gamma-\delta-\delta'-\beta+1)_n(\beta'-\delta'+1)_n}{(\beta'+1)_n(\gamma-\delta-\delta'+1)_n(\gamma-\delta'-\beta+1)_n}\frac{(a_1)_{n,\alpha}(a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}\frac{x^n}{n!},$$

which in view of definition (3.13), is the right hand side of assertion (3.12).

In the same way, using (3.13) and applying the formula (3.7), we can derive the assertion (3.12).

Theorem 4. *If* $x>0$, $0<\alpha\leq 1$, $\{\delta,\delta',\beta,\beta',\gamma,a_i,b_j\}\subset C$, $i=1,...,p$, $j=1,...,q$, $Re(\gamma)>0$, $1+\min\{Re(\delta-\beta),\ Re(\delta+\delta'+\beta'-\gamma)\}>0$, *then*

$$\left(D_{0+}^{\delta,\delta',\beta,\beta',\gamma}\left[{}_{p}^{\alpha}F_q(z)\right]\right)(x)=x^{\delta+\delta'-\gamma}\frac{\Gamma(1+\delta-\beta)\,\Gamma(1+\delta+\delta'+\beta'-\gamma)}{\Gamma(1-\beta)\,\Gamma(1+\delta-\delta'-\gamma)\,\Gamma(1+\delta+\beta'-\gamma)}$$
$${}_{p+3}^{\ \ \alpha}F_{q+3}\left[\begin{matrix}1,1+\delta-\beta,1+\delta+\delta'+\beta'-\gamma,a_1,...,a_p;\\ 1-\beta,1+\delta-\delta'-\gamma,1+\delta+\beta'-\gamma,b_1,...,b_q;\end{matrix}\ x\right],\tag{3.14}$$

and if $x>0$, $0<\alpha\leq 1$, $\{\delta,\delta',\beta,\beta',\gamma,a_i,b_j\}\subset C$, $i=1,...,p$, $j=1,...,q$, $Re(\gamma)>0$, $1+\min\{Re(-\delta-\delta'+\gamma),\ Re(-\delta'-\beta+\gamma),Re(\beta')\}>0$, then

$$\left(D_{0-}^{\delta,\delta',\beta,\beta',\gamma}\left[\frac{1}{z}{}_{p}^{\alpha}F_q\left(\frac{1}{z}\right)\right]\right)(x)=\frac{\Gamma(-\delta-\delta'+\gamma+1)\,\Gamma(-\delta'-\beta+\gamma+1)\,\Gamma(\beta'+1)}{\Gamma(-\delta-\delta'-\beta+\gamma+1)\,\Gamma(-\delta'+\beta'+1)}$$
$$x^{\delta+\delta'-\gamma-1}\,{}_{p+3}^{\ \ \alpha}F_{q+3}\left[\begin{matrix}-\delta-\delta'+\gamma+1,-\delta'-\beta+\gamma+1,\beta'+1,a_1,...,a_p;\\ -\delta-\delta'-\beta+\gamma+1,-\delta'+\beta'+1,\ 1,b_1,...,b_q;\end{matrix}\ \frac{1}{x}\right].\tag{3.15}$$

Proof. From (3.13) and (3.8), we get

$$\left(D_{0+}^{\delta,\delta',\beta,\beta',\gamma}\left[{}_{p}^{\alpha}F_q(z)\right]\right)(x)=\sum_{n=0}^{\infty}\frac{(a_1)_{n,\alpha}\,(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n\,n!}$$
$$\cdot\left(\frac{\Gamma(n+1)\,\Gamma(\delta-\beta+n+1)\,\Gamma(\delta+\delta'+\beta'-\gamma+n+1)}{\Gamma(-\beta+n+1)\,\Gamma(\delta-\delta'-\gamma+n+1)\,\Gamma(\delta+\beta'-\gamma+n+1)}x^{\delta+\delta'-\gamma+n}\right)$$
$$=x^{\delta+\delta'-\gamma}\frac{\Gamma(1+\delta-\beta)\,\Gamma(1+\delta+\delta'+\beta'-\gamma)}{\Gamma(1-\beta)\,\Gamma(1+\delta-\delta'-\gamma)\,\Gamma(1+\delta+\beta'-\gamma)}$$
$$\sum_{n=0}^{\infty}\frac{(1)_n(1+\delta-\beta)_n(1+\delta+\delta'+\beta'-\gamma)_n}{(1-\beta)_n(1+\delta-\delta'-\gamma)_n(1+\delta+\beta'-\gamma)_n}\frac{(a_1)_{n,\alpha}\,(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!},$$

By using (3.13) again, we can reach to (3.14).

Similarly, using formula (3.9), assertion (3.15) can be derived.

Remark 1. *If we take* $\alpha=1$ *in the above mentioned theorems, we get the corresponding results for the classical hypergeometric functions* ${}_2F_1$ *and* ${}_pF_q$.

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